ON THE CAYLEY GRAPH OF A GENERIC FINITELY PRESENTED GROUP.

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ABSTRACT. We prove that in a certain statistical sense the Cayley graph of almost every finitely presented group with $m \geq 2$ generators contains a subdivision of the complete graph on $l \leq 2m+1$ vertices. In particular, this Cayley graph is non planar. We also show that some group constructions preserve the planarity.

1. INTRODUCTION

To any finite presentation of a group in terms of generators and defining relations there is an associated Cayley graph. This graph depends on the choice of the group generating set. So, in general, the same group has completely different Cayley graphs (from the graph theory viewpoint). In particular, it is not hard to find a group and two different sets of generators such that the Cayley graph with respect to one generating set is planar and not planar with respect to the other. As an example, take the cyclic group $\mathbb{Z}/5\mathbb{Z}$ of order five and two generating sets. The first one consisting of a single non trivial element and the other one consisting of all non trivial elements. Then with respect to the first generating set, the Cayley graph is a cycle, so it is planar, but with respect to the second one it is not, as it is the complete graph on five vertices.

On the other hand, the existence of planar Cayley graph may give an information about algebraic structure of a group. In fact, groups having planar Cayley graphs are rather scarce. In 1896, applying Cayley's method of the graphical presentation of a group, in particular, to rotation groups of the regular three- and four-dimensional bodies, Maschke classified all finite groups with planar Cayley graphs [15]. These are the finite subgroups of the special orthogonal group $SO(3)$ (i.e. cyclic, dihedral, and the rotational symmetry groups of the regular solids) and their direct products with the group of order 2. It is worth to notice that for finite graphs the way given in [15] to define the planarity is very natural and it is not ambiguous. The situation for infinite graphs is more complicated. In this case there are two nonequivalent definitions of planarity.

Definition 1.

(I) *A graph is planar without accumulation points if there is an embedding of the graph in $\mathbb{R}^2$ such that there are no accumulation points for its set of vertices.*

(II) *A graph is planar if there is an embedding of the graph in $\mathbb{R}^2$.*

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Infinite groups admitting planar Cayley graphs without accumulation points of vertices were treated by Levinson. In [14], he and Rapaport find all "special planar" presentations. "Special planar" means that the Cayley graph can be chosen point-symmetric (with the same counterclockwise succession of the edges at each vertex) and locally finite (without accumulation points of vertices). They gave also some conditions on the set of defining relations which are necessary to make the presentation special planar. In [13], Levinson produces moreover an algorithm to decide whether or not a Cayley graph of a group with solvable word problem is planar without accumulation points of vertices.

A different geometrical approach of group planarity was initiated by Poincaré [17]. It gave rise to the following question : Which groups have a planar Cayley complex ? Now there exists a complete classification of such groups. These are Fuchsian groups and free products of countably many cyclic groups [11, Prop. III.5.4], [20, Ch. 4]. Planarity of the Cayley complex implies planarity of the Cayley graph, but the converse is not true. An example of a planar group without planar Cayley complex is the free product of $\mathbb{Z}^2$ by $\mathbb{Z}$, see Figure 1.

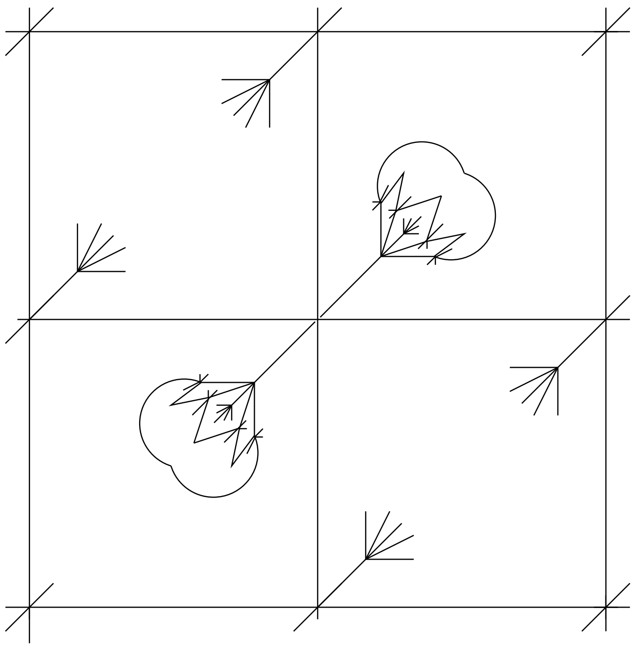

FIGURE 1. Part of ball of radius 3 in $\mathbb{Z}^2 * \mathbb{Z}$

To allow accumulation points in the definition of planarity gives more freedom, so we ask ourselves whether or not the planarity with accumulation points of vertices is frequent. The aim of the present paper is to show that for any $m \geq 2$ and $l \leq 2m+1$ one can find a subdivision (see Section 2.1 for the definition of subdivision) of the complete graph K_l (and hence of every finite graph on at most l vertices) in the Cayley graph of almost every finite presentation of a group with m generators and long enough defining relations. In particular, the Cayley graph of such a *generic* group is non planar (even with the above mentioned relaxation of the notion of planarity).

More precisely, for any fixed m and n, let $N = N(m, n, t)$ denote the number of all group presentations

$$G = \langle x_1, \dots, x_m \mid r_1 = 1, \dots, r_n = 1 \rangle, \tag{1}$$

where $\{r_1, \dots, r_n\}$ are cyclically reduced words in the alphabet $X_m = \{x_1^{\pm 1}, \dots, x_m^{\pm 1}\}$ of length $|r_i| \le t$. Let $N_l = N_l(m, n, t)$ denote the number of all such group presentations whose Cayley graphs contain a subdivision of the complete graph K_l on l vertices.

Theorem 1. *For any $m \ge 2, n > 0$ and $l \le 2m + 1$*

$$\lim_{t \to \infty} N_l / N = 1.$$

Moreover, there is a real number $c > 0$ depending on m and n such that

$$1 - N_l/N < \exp(-ct) \text{ for all } t > 0.$$

Let $N_{np} = N_{np}(m, n, t)$ denote the number of all group presentations (1) with non planar Cayley graphs. The previous theorem together with Theorem 5 below imply immediately the following result.

Theorem 2. *For any $m \ge 2$ and $n > 0$*

$$\lim_{t \to \infty} N_{np} / N = 1.$$

Moreover, there is a real number $c > 0$ depending on m and n such that

$$1 - N_{np}/N < \exp(-ct) \text{ for all } t > 0.$$

By a result of Levinson [12], the Cayley graph of an infinite finitely generated group has the genus either 0 (hence the group is planar) or the infinity. Thus the previous theorem implies that the genus of a generic finitely presented group (that is the minimum of genus of its Cayley graphs taken over all generating sets, see Section 2.1) is the infinity.

To prove Theorem 1 we use firstly that the metric small cancellation condition $C'(\lambda)$ with $\lambda > 0$ is verified for a generic (in the above defined statistical sense) finitely presented group, see, for example, [1, 2]. Then we apply the technique of small cancellation theory to find the complete graph K_l with $l \le 2m + 1$ in the Cayley graph of a generic group presentation with m generators.

A probabilistic point of view on the notion of a generic group was first considered by Gromov [5], see also [6, 7] and [9, Problem 11.75] for an independent definition by Ol'shanskii. In [5] Gromov announced that word hyperbolicity of a finitely generated group is a generic property, in a sense slightly stronger than the definition above. A proof of this result was given by Ol'shanskii [16], see also Champetier [2] for more results in case of two defining relations. A survey on the "random" viewpoint in geometric group theory is recently presented by Ghys in [4].

The model of a generic group (via the asymptotic density) defined above is closely related to the density model developed by Gromov in [6]. This model depends on a density parameter d with $0 \le d \le 1$. It consists in choosing at random roughly $(2m-1)^{d\ell}$ words of a given length ℓ in the alphabet of m letters (more precisely the number of chosen words of length ℓ is between $C_1(2m-1)^{d\ell}$ and $C_2(2m-1)^{d\ell}$ for two given constants $C_1 < C_2$), then defining a group presentation on m generators where these words are defining relations, and finally letting $\ell \to \infty$. Under our model, one can assume that in a generic presentation all defining relations have almost the same length t. Indeed, the proportion of reduced words of length at most $t(1-\varepsilon)$ among all reduced words of length at most t decrease exponentially as $(2m-1)^{-t\varepsilon}$. Hence the share of corresponding group presentations is exponentially small. Thus the model we are using in the paper is the case of the density model with the density parameter $d = 0$. It is worth noticing that our main result, Theorem 1, remains true for all groups of density $d < 1/12$ as such groups are known to satisfy the small cancellation condition $C'(\lambda)$ with $\lambda < 2d$ [6, Ch. 9.B]. Hence our arguments (see our proof below) work in this case as well.

We also describe some group constructions preserving planarity.

Theorem 3.

(i) *Planarity is preserved under free products of groups.*

(ii) *For $j = 1, 2$, let G_j be a group generated by a finite set X_j containing a generator s_j of order 2, assume that the Cayley graph $C(G_i, X_i)$ is planar. Let G be the amalgamated product of G_1 and G_2 along $\{1, s_1\} = \{1, s_2\} \equiv \mathbb{Z}/2\mathbb{Z}$. Then G is a planar group. More precisely, $C(G, X_1 \cup X_2)$ is a planar graph.*

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2. Preliminary information

2.1. **Graphs.** A *graph* Γ consists of two sets $E(\Gamma)$ and $V(\Gamma)$; if there is no ambiguity we will write E and V instead of $E(\Gamma)$ and $V(\Gamma)$. The elements of V are all vertices of Γ. The elements of E are unordered pairs of distinct vertices, called edges. We therefore assume that there are no multiple edges between two vertices. A graph is finite if the vertex set V is finite, it is infinite otherwise but we assume that an infinite graph has only countably many vertices.

If $e = \{u, v\} \in E$, for $u, v \in V$, we say that u and v are *adjacent* vertices, and that vertex u and edge e are *incident* with each other, as are v and e. The *degree* of a vertex v is the number of vertices to which v is adjacent. A graph is *regular* if the vertices have the same degree.

We recall that two graphs Γ and Γ' are *isomorphic* if there exists a one-to-one mapping ϕ from $V(\Gamma)$ onto $V(\Gamma')$ such that for every g_1, g_2 in $V(\Gamma)$, $\{g_1, g_2\}$ is an edge of Γ if and only if $\{\phi(g_1), \phi(g_2)\}$ is an edge of Γ'.

An *elementary subdivision* of a graph Γ is the replacement of one edge by two edges incident to a vertex of degree 2. Namely, a graph Γ_1 is obtained from Γ by an elementary subdivision if $V(\Gamma_1) = V(\Gamma) \cup \{v\}$ with $v \notin V(\Gamma)$ and $E(\Gamma_1) = (E(\Gamma) \setminus \{e\}) \cup \{e_1\} \cup \{e_2\}$, where $e = \{u_1, u_2\} \in E(\Gamma)$ and $e_1 = \{u_1, v\}, e_2 = \{v, u_2\}$. A *subdivision* is a finite sequence of elementary ones.

The *complete graph* K_n is the graph with n vertices and an edge for every pair of vertices. The complete bipartite graph $K_{n,m}$ is the graph such that $V(K_{n,m})$ is the disjoint union of two subsets V_1 and V_2 of cardinality n and m respectively such that for every $v \in V_1$ and $w \in V_2$ there exists one edge joining v and w, and these are the only edges of $K_{n,m}$. It is not hard to see that there exists a subdivision of K_n which is a subgraph of the bipartite graph $K_{n,n}$, see Figure 2 for $n = 3$.

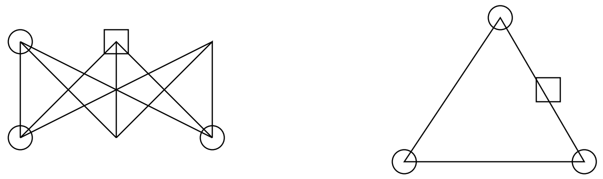


FIGURE 2. A subdivision of K_3 in $K_{3,3}$

Let Γ_1 and Γ_2 two graphs and let u be a vertex of Γ_1 and v be a vertex of Γ_2, we denote by $(\Gamma_1, u) * (\Gamma_2, v)$ the graph consisting of the union of Γ_1 and Γ_2 where u and v are identified. We call $(\Gamma_1, u) * (\Gamma_2, v)$ the *gluing* of (Γ_1, u) and (Γ_2, v) along u and v. Similarly, we define the gluing along two edges.

A geometric realization of a finite graph in $\mathbb{R}^3$ is a configuration in $\mathbb{R}^3$, where the vertices of the graph are represented by distinct points, and each edge e of the graph is a Jordan arc, i.e. the image of an injective continuous function $\psi : [0,1] \to \mathbb{R}^3$. Two arcs intersect only at a point representing common terminal vertices of two corresponding edges. It is clear that any finite graph may be realized in such a way in $\mathbb{R}^3$. If such a geometric realization of a finite graph exists in $\mathbb{R}^2$ instead of $\mathbb{R}^3$, it is natural to say that the graph is *planar*.

It is easy to see (and well known) that K_5 and $K_{3,3}$ are non planar.

Theorem 4 (Kuratovski [10]). *A finite graph Γ is planar if and only if Γ contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.*

Kuratovski's original proof of this theorem is topological, see [10]. For a readable combinatorial proof, see [19].

The notion of a geometric realization in $\mathbb{R}^3$ can be extended directly to infinite graphs. It is well-known that any graph with countably many vertices can be geometrically realized in $\mathbb{R}^3$. However, as seen in the introduction, the planarity for infinite graphs can be defined in different ways. If accumulation points of vertices are accepted, an analogous result to Kuratovski theorem is proved in [3].

Theorem 5 (Dirac and Schuster). *A graph Γ is planar if and only if it contains no subgraph homeomorphic with K_5 or $K_{3,3}$.*

Remark 1. In fact they prove that the extension of Kuratowski's theorem is equivalent to the following: If every finite subgraph of a countable infinite graph is planar, then the whole graph is planar.

We use the term "graph" for the abstract mathematical object or for a geometric realization of this object in $\mathbb{R}^3$. It is also interesting to consider embeddings of graphs in surfaces of positive genus [18, 12]. The minimum genus among all surfaces in which a graph can be realized in the above-mentioned way is called the *genus of the graph.*

2.2. **Cayley graphs.** Let $G = \langle X \mid \mathcal{R} \rangle$ be a finitely presented group, that is with a finite set of generators X and a finite set of defining relations $\mathcal{R}$. We assume that $x \neq_G 1$ for every $x \in X$. Given such a group presentation, there is associated the *Cayley graph.* This is a graph $C(G, X)$ whose set of vertices is G and the set of edges is $\{\{g_1, g_2\} \mid g_1, g_2 \in G \text{ and } \exists s \in X \text{ such that } g_2 = g_1 s\}$. With that definition, there exists two edges between g_1 and g_2 if the generator x is of order

2 or if x and x^{-1} are both contained in X. However we can glue these two edges together and this process does not change the planarity. We denote the Cayley graph by $C(G)$ whenever there is no ambiguity for the generating set.

Any non-directed edge $e = \{g_1, g_2\}$ can be viewed as two directed ones, one $e^+ = (g_1, g_2)$ and the other $e^- = (g_2, g_1)$. There exists a labelling function φ on the set of directed edges onto $X^{\pm 1}$ defined by $\varphi(e^+) = s$ for s such that $g_2 = g_1 s$, and $\varphi(e^-) = s^{-1}$. The label $\varphi(p)$ of a path $p = e_1 e_2 \dots e_n$ in $C(G)$ is the word $\varphi(e_1)\varphi(e_2)\dots\varphi(e_n)$ where $\varphi(e_i)$ is the label of the edge e_i according to the orientation. We regard $\varphi(p)$ as an element of G. It is clear that an element g equal to 1 in G if and only if any path labelled by g is closed in $C(G)$.

We endow $C(G)$ with a metric by assigning to each edge the metric of the unit segment $[0, 1]$ and defining the distance $|x - y|$ to be the length of a shortest path between x and y. Thus $C(G)$ becomes a geodesic metric space, that is, any two points can be connected by a geodesic.

Obviously, a Cayley graph is regular and connected. The converse is not true, for example the Petersen graph is not a Cayley graph, see [8, exercise IV.11, p.82]. It is clear that the Cayley graph depends on the choice of the group generating set. In particular, the planarity of the Cayley graph does depend on such a choice (as shown in the introduction).

A group G is said to be *planar* if there exists a generating set X such that the Cayley graph of G with respect to X is planar in sense II of Definition 1.

2.3. **Small cancellation groups.** Given a finite presentation $G = \langle X \,|\mathcal{R}\rangle$, let $\mathcal{R}^*$ denote the set containing all cyclic permutations of words $r_i \in \mathcal{R}$ and their inverses. Recall that a *piece* is a nontrivial word u in the alphabet $X^{\pm 1}$ such that there are two different defining relations $r_1, r_2 \in \mathcal{R}^*$ such that $r_1 = uv_1$ and $r_2 = uv_2$.

A group presentation satisfies the $C'(\lambda)$-*condition* with $\lambda > 0$ (so-called metric small cancellation condition) if for each piece u occurring in the relator r, $|u| < \lambda |r|$.

Example. The surface group of genus $g > 1$ has a presentation

$$S_g = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1\rangle$$

The set $\mathcal{R}^*$ contains $4g$ elements. A maximal piece consists of a single letter. So this presentation satisfies the condition $C'(\frac{1}{4g-1})$.

The following lemma is due to Greendlinger [11, Th. V.4.4].

Lemma 1. *Let a finite presentation $G = \langle X \,|\mathcal{R}\rangle$ satisfy the $C'(\lambda)$-condition with $\lambda \leq \frac{1}{6}$. Suppose that w is a reduced word in the alphabet $X^{\pm 1}$ representing the identity in G. Then w contains a subword v that is also a subword of a cyclic shift of some $r \in \mathcal{R}^*$ and satisfies $|v| > (1 - 3\lambda)|r|$.*

In fact, group presentations satisfying the $C'(\lambda)$-condition for $\lambda > 0$ are very frequent.

Lemma 2 ([1, 2])**.** *Let $\mathcal{R} = \{r_1, \dots, r_n\}$ be an n-tuple of cyclically reduced words in the alphabet $X^{\pm 1}$ of length $|r_i| \leq t$. Then the share of all n-tuples $\{r_1, \dots, r_n\}$ such that $\mathcal{R}^* = \mathcal{R}^*(r_1, \dots, r_n)$ does not satisfy the small cancellation condition $C'(\lambda)$ with $\lambda > 0$ decreases exponentially as $t \to \infty$.*

For more details and information about small cancellation groups we refer to [11].

3. Proof of Theorem 1

Let w be a reduced word in the alphabet $X_m = \{x_1^{\pm 1}, \ldots, x_m^{\pm 1}\}$, that is, it does not contain xx^{-1} with $x \in X_m$ as a subword. Recall that $N = N(m,n,t)$ is the number of all n-tuples $\{r_1, \ldots, r_n\}$ of cyclically reduced words in the alphabet X_m of length $|r_i| \leq t$. We denote by $N_w = N_w(m,n,t)$ the number of all n-tuples $\{r_1, \ldots, r_n\}$ such that a cyclic shift of some r_i or of its inverse r_i^{-1} contains w as a subword.

The following lemma is a technical tool in our proof of Theorem 1. It is intuitively clear.

Lemma 3. *For any $m, n > 0$ and any reduced word w in X_m*

$$\lim_{t \to \infty} N_w/N = 1.$$

Moreover, there is a real number $c > 0$ depending on m, n and the length $|w|$ of w such that $1 - N_w/N < \exp(-ct)$ for all $t > 0$.

Sketch of proof. Let w be a fixed word of length s. We will prove that the number of words of length t which does not contain $w^{\pm 1}$ is exponentially small compare to the number of words of length t as t tends to infinity. In particular this allows us to extend this result to n-tuples of cyclically reduced words of length t.

Let x in X_m be the first generator appearing in the writting of w and let y be an other generating element distinct from x and x^{-1}.

Let r be much longer than w and denote by t its length. We divide r into $t/(s+1)$ blocks of lenght $s+1$. For each of these blocks (except the first one), we have $(2m-1)^{s+1}$ choices (as we have to avoid the inverse of the last letter of the preceeding block). As xw and yw do not have the same first letter, at least one of the two words xw and yw belongs to these choices. If w is excluded, the number of choices drops to $(2m-1)^{s+1} - 1$ for each block of length $s+1$. Thus the number of choices is less than

$$(2m)^{s+1}((2m-1)^{s+1} - 1)^{(t/(s+1))-1}.$$

The first term $(2m)^{s+1}$ is there for the choice of the first block. This number of choices is exponentially small compare to $(2m-1)^t$ when t tends to infinity. $\square$

Lemma 4. *Let a finite presentation $G = \langle X \,|\mathcal{R}\rangle$ satisfy the $C'(\lambda)$-condition with $\lambda \leq \frac{1}{6}$. Then the intersection of two cycles in the Cayley graph of G is either empty or connected whenever labels of cycles are some words r_{i_1} and r_{i_2} from R^*.*

Proof of Lemma 4. Given two cycles in the Cayley graph of G satisfying the hypothesis above, assume that they have a non empty intersection. By contradiction, suppose that there are two connected components of this intersection. We denote labels of these paths by u and w, see Figure 3. Note that they can be empty words if the intersection is reduced to disjoint union of vertices. Then a cyclic shift of r_{i_1} is of the form $(r'_{i_1})ur''_{i_1}w$. We claim that either r'_{i_1} or r''_{i_1} is of length less than $(1-2\lambda)|r_{i_1}|$. Indeed, suppose by contradiction that $|r'_{i_1}|, |r''_{i_1}| \geq (1-2\lambda)|r_{i_1}|$. Then, $0 \leq |u| + |w| = |r_{i_1}| - |r'_{i_1}| - |r''_{i_1}| \leq (1 - 2(1-2\lambda))|r_{i_1}|$, hence $\lambda \geq 1/4$. This contradicts $\lambda \leq 1/6$. Similarly, a cyclic shift of r_{i_2} is of the form $r'_{i_2}u^{-1}r''_{i_2}w$ and either r'_{i_2} or r''_{i_2} is of length less than $(1-2\lambda)|r_{i_2}|$.

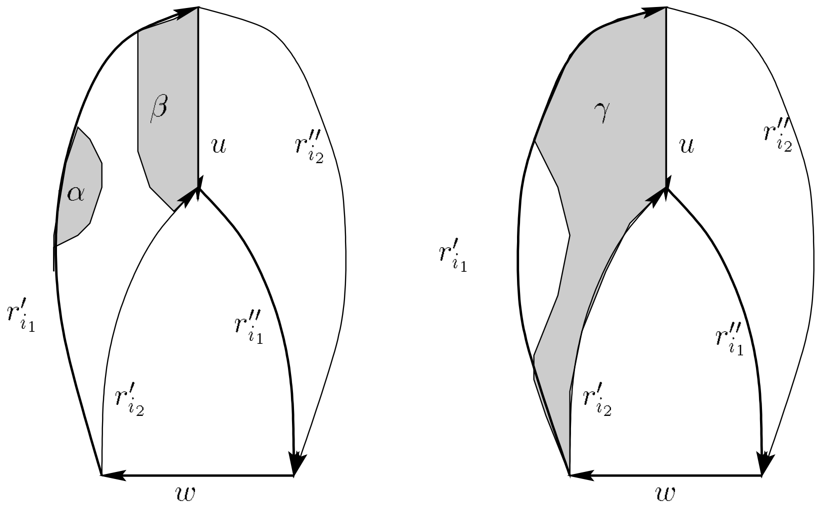


FIGURE 3

Among the four closed paths labelled by $r'_{i_1}u(r'_{i_2})^{-1}$, $r'_{i_1}r''_{i_2}w$, $r'_{i_2}r''_{i_1}w$, $(r''_{i_1})^{-1}u^{-1}r''_{i_2}$; let us take one whose label contains two "short" subwords of $r_{i_1}^{\pm 1}$ and $r_{i_2}^{\pm 1}$ (by short, we mean "of length less than $(1-2\lambda)|r_{i_1}|$ and $(1-2\lambda)|r_{i_2}|$ respectively").

Without loss of generality we assume that this path is labelled by $r'_{i_1}u(r'_{i_2})^{-1}$. Thus, $r'_{i_1}u(r'_{i_2})^{-1} = 1$ in G with $|r'_{i_1}| < (1-2\lambda)|r_{i_1}|$ and $|r'_{i_2}| < (1-2\lambda)|r_{i_2}|$.

By the assumption and the Greendlinger lemma for $C'(\lambda)$-groups with $\lambda \leq \frac{1}{6}$, see Lemma 1, the word $r'_{i_1}u(r'_{i_2})^{-1}$ contains a subword v that is also a subword of a cyclic shift of some $r_k^{\pm 1} \in R$ and satisfies $|v| > (1-3\lambda)|r_k|$ with $\lambda \leq \frac{1}{6}$.

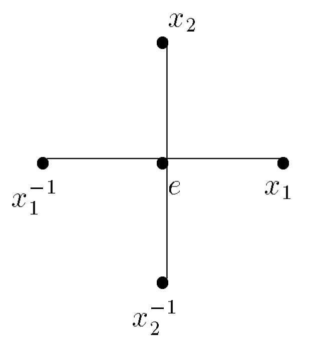
Let us show that $k \neq i_1, i_2$. By contradiction, suppose that $k = i_1$. (The case $k = i_2$ is similar.) Hence, $(1-3\lambda)|r_{i_1}| < |v| \leq |r'_{i_1}| + |u| \leq (1-2\lambda)|r_{i_1}| + \lambda|r_{i_1}|$. The second inequality holds as u is chosen to be maximal as connected component. This contradiction shows that $k \neq i_1$. The case $k = i_2$ is similar. Thus from now on $k \neq i_1, i_2$.

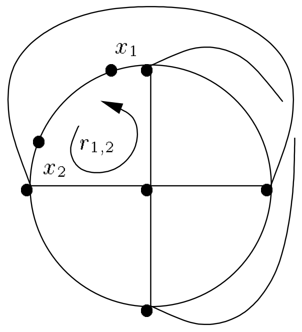
Suppose that v is a subword of r'_{i_1}, i.e. the face with the label r_k is in the position α, Figure 3. Then, v is a piece and by the $C'(\lambda)$-condition, $|v| \leq \lambda \min\{|r_{i_1}|, |r_k|\}$ contradicting $|v| > (1-3\lambda)|r_k|$ as $\lambda \leq \frac{1}{6}$. So, v is not a subword of r'_{i_1}. The same argument shows that v is not a subword of r'_{i_2}.

Another case is $v = v_1v_2$ with v_1 and v_2 are subwords of $r'_{i_1}u$ and $(r'_{i_2})^{-1}$ respectively (v_2 can be the empty word), position γ, Figure 3. As above, v_l is a piece and hence $|v_l| \leq \lambda \min\{|r_{i_l}|, |r_k|\}$, $l = 1, 2$. So, $|v| \leq 2\lambda|r_k|$ contradicting again $|v| > (1-3\lambda)|r_k|$ as $\lambda \leq \frac{1}{6}$.

The remaining case is $v = v_1v_2v_3$, where v_1, v_2, and v_3 are subwords of $r'_{i_1}u$, $(r'_{i_2})^{-1}$, and r'_{i_1} respectively. Since they are pieces, $|v_l| \leq \lambda|r_k|$, $l = 1, 2, 3$. This contradicts again $|v| > (1-3\lambda)|r_k|$. $\square$

Proof of Theorem 1. Let $G = \langle x_1, \ldots, x_m \mid \mathcal{R}\rangle$, where $\mathcal{R} = \{r_1, \ldots, r_n\}$ is an n-tuple of cyclically reduced words in the alphabet X_m of length $|r_i| \leq t$. We have to prove that generically the Cayley graph contains a subgraph which is a subdivision of a complete graph K_{2m+1} on $2m+1$ vertices. Let $B(e,1)$ the closed ball of radius 1 centered at the identity vertex e in the Cayley graph of G. It contains exactly $2m+1$ vertices, because, as the relations are generically long, it is a tree (see Figure 4 in case $m=2$). We take these $2m+1$ vertices as candidates for the vertices of the complete graph K_{2m+1} that we are looking for. The identity element e is already joined to all others. It remains to show that two arbitrary vertices on the sphere of radius 1 centered at e are joined by a path that is outside of the ball and that all these paths are disjoined except maybe at their endpoints. The elements of the





$r_{1,2} = x_2x_1x_2x_1$

FIGURE 4

sphere of radius 1 centered at e are indexed by X_m and for every pair of two distinct points x_i and x_j on this sphere with $x_i, x_j \in X_m$ there exists a geodesic path of length 2 in the ball joining x_i to x_j. The labelling of this path is $x_i^{-1}\, x_j$ (once we have chosen one directed path joining x_i to x_j, we don't take another joining x_j to x_i). For such a pair $\{x_i, x_j\}$ and the choosen path $x_i^{-1}x_j$, let define the word $w_{i,j} = x_j x_i^{-1}\, x_j x_i^{-1}$. By Lemma 3, we can assume that all the $w_{i,j}$ are subwords of cyclic shifts of defining relations $r_{i,j} \in R^*$ (which are not necessarily distinct). Each of these $r_{i,j}$ defines a cycle in the Cayley graph containing the vertices x_i, e and x_j. All these cycles have the vertex e in common. The intersection of any two of these cycles is contained in the ball of radius 1, by definition of the $w_{i,j}$'s and Lemma 4. Then the different cycles defined by $r_{i,j}$ and $r_{l,k}$ are disjoint outside of the ball of radius 1. Thus the subgraph consisting of the union of all $r_{i,j}$'s is a subdivision of K_{2m+1} (see Figure 4 in case $m=2$ and K_5). $\square$

Remark 2. In Theorem 1 it seems possible to drop the assumption on the number of vertices of the finite graph and replace it only by one on the maximal valency of the vertices.

4. Two constructions preserving planarity.

In this section we prove our third theorem. First we need the following result.

Lemma 5. *If Γ_1 and Γ_2 are two finite planar graphs, then for every v in Γ_1 and w in Γ_2, the gluing $(\Gamma_1, v) * (\Gamma_2, w)$ of Γ_1 and Γ_2 along v and w is planar.*

Proof. A geometric realization in $\mathbb{R}^2$ of Γ_1 gives a cellular decomposition of $\mathbb{R}^2$ and v is in the border of a cell, by an homeomorphism ϕ of the sphere sending a point of that cell at infinity, we obtain a geometric realization in $\mathbb{R}^2$ of Γ_1 where $\phi(v)$ is in the border of the unbounded cell. This can be done by an homeomorphism ψ for Γ_2 and w too. Then it is clear that the gluing of these two geometric realizations of Γ_1 and Γ_2 along $\phi(v)$ and $\psi(w)$ is planar. □

Remark 3. The proof of the preceding lemma can be extended to the gluing along two edges $e_1 \in E(\Gamma_1)$ and $e_2 \in E(\Gamma_2)$. The proof is exactly the same because it is possible to choose the homeomorphisms ϕ and ψ in such a way that $\phi(e_1)$ and $\psi(e_2)$ are in the border of the unbounded cell.

Proposition 1. *Let G_1 and G_2 be two planar groups, then the free product $G_1 * G_2$ is also planar.*

Proof. As the G_i is planar, $i = 1, 2$, there exists some generating set X_i such that the Cayley graph of G_i with respect to X_i is planar for $i = 1, 2$. We denote by X the generating set of $G_1 * G_2$ given by $X_1 \cup X_2$. We are going to prove that the Cayley graph Γ of $G_1 * G_2$ with respect to X is planar.

First we prove that the ball of radius n centered at the origin in Γ is planar for every $n \geq 0$. We denote by $B(G_i, n)$ the ball of radius n in G_i centered at the origin of G_i.

Any word of length at most n in $G_1 * G_2$ has a normal form $\alpha_1\beta_1\alpha_2\beta_2 \cdots \alpha_k\beta_k$ where the α_i are non trivial elements of G_1 except maybe α_1 and the β_i are non trivial elements of G_2 except maybe β_k. This writing is not necessarily unique because the writing of the α_i (respectively β_i) is not necessarily unique in G_1 (respectively G_2), but the k is. So the ball of radius n in the Cayley graph Γ of $G_1 * G_2$ can be described inductively by the following process.

First we construct the gluing of the graphs $(B(G_1, n), e_1) * (B(G_2, n), e_2)$ and denote by e the vertex on which the gluing is done. Then on every vertex at distance $0 < j < n$ of e labelled by a word in G_1, we glue $(B(\Gamma_2, n-j), e_2)$ and on vertices labelled in G_2 at distance $0 < j < n$, we glue $(B(\Gamma_1, n-j), e_1)$. The labelling of a minimal path from e to any vertex gives a word in $G_1 * G_2$ of the form $\alpha_1\beta_1$ or $\beta_1\alpha_1$. On every vertex ω at distance j of e which is not yet of degree $|X_1| + |X_2|$, we glue a ball $(B(G_1, n-j), e_1)$ if $\omega = \alpha_1\beta_1$ and we glue a ball $(B(G_2, n-j), e_2)$ otherwise. We continue this gluing process until every vertex at distance less than n is of degree $|X_1| + |X_2|$. This process is finite because the gluing of a ball $(B(G_i, n-j), e)$ to a vertex at distance j of e only add vertices at distance strictly bigger than j of e. The resulting graph is exactly the ball of radius n in Γ.

By Lemma 5, this gluing process preserves planarity. Hence by Theorem 5 (in [3], p.347) this ensures that the whole graph is planar. □

Remark 4. This proof can be extended to amalgamated products $G_1 *_A G_2$ with cyclic group $A = \langle z \rangle$ of order 2. Let G_1 and G_2 two planar groups and X_1 (respectively X_2) be a generating set of G_1 (respectively G_2) such that $C(G_i, X_i)$ is planar and X_i contains the generator z of A for $i = 1, 2$, by using the extension of Lemma 5 given in the Remark 3, we prove that the Cayley graph associated to $X_1 \cup X_2$ is planar.

Putting together Lemma 5, Proposition 1 and Remarks 3 and 4 gives a proof of Theorem 3.

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