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On Quasiconvex Subgroups of Word Hyperbolic Groups⋆

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Abstract. We prove that a quasiconvex subgroup H of infinite index of a torsion free word hyperbolic group can be embedded in a larger quasiconvex subgroup which is the free product of H and an infinite cyclic group. Some properties of quasiconvex subgroups of word hyperbolic group are also discussed.

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1. Introduction

Word hyperbolic groups were introduced by M. Gromov as a geometric generalization of certain properties of discrete groups of isometries of hyperbolic spaces $\mathbb{H}^n$. Finite groups, finitely generated free groups, classical small cancellation groups and groups acting discretely and cocompactly on hyperbolic spaces are basic examples of word hyperbolic groups. Any word hyperbolic group is finitely presented. Finite extensions and free products of finitely many word hyperbolic groups are also word hyperbolic. A large number of results on word hyperbolic groups as well as conjectures and research problems are contained in the original article [9].

In this paper, we study properties of *quasiconvex* subgroups of word hyperbolic groups (see the next section for the definition). Our main result in fact gives a method for constructing quasiconvex subgroups of word hyperbolic groups.

THEOREM 1. *Let G be a non-elementary torsion-free word hyperbolic group and H be a quasiconvex subgroup of G of infinite index. Then there exists a non-trivial element $g \in G$ such that the subgroup $sgp\langle H, g\rangle$ generated by H and g is the free product $H * \langle g\rangle$ and is quasiconvex in G.*

The statement of the theorem was formulated by M. Gromov in [9] 5.3.C, with a very general sketch of a proof. We follow in part Gromov's approach.

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There are two parts in our proof of Theorem 1: first we find an element $g \in G$ such that the subgroup $sgp\langle H, g\rangle$ is a free product and then we prove that this subgroup is quasiconvex in G. For the first part, we choose a double coset HxH whose shortest representative x is sufficiently long, as a word in the generators of G. This is possible as we prove that the number of double cosets of a word hyperbolic group G modulo a quasiconvex subgroup H of infinite index is also infinite (Proposition 1). For g, we take x^M for M large enough.

The fact that, for the chosen g, the subgroup $sgp\langle H, g\rangle$ is quasiconvex is not trivial even if we know that it decomposes into the free product $H * \langle g\rangle$ of a quasiconvex subgroup H and a cyclic subgroup $\langle g\rangle$ and any cyclic subgroup of a word hyperbolic group is quasiconvex. In general, a subgroup of a word hyperbolic group, which is a free product of two quasiconvex, even cyclic, subgroups need not be quasiconvex. For example, let $G = \langle a, t \mid at^{-1}ata^2t^{-2}a^{-1}t^2 = 1\rangle$ and let M be the so-called Moldavansky subgroup, that is, $M = sgp\langle a, t^{-1}at, t^{-2}at^2\rangle$. It is known [10] that G is a torsion-free non-elementary word hyperbolic group and M is a non-quasiconvex free subgroup of rank 2.

Note that, under the assumptions of Theorem 1, we can construct an infinite sequence $H = F_0 < F_1 < \cdots < F_k < \cdots$ of subgroups of G starting with H where F_i is the free product of H and a free group of rank i. To do this, we only need to notice that the subgroup $H * \langle g\rangle$ in Theorem 1 will have infinite index in G if we replace g by any proper power of it. In particular, taking $H = 1$, we get an ascending sequence of quasiconvex free subgroups of G of ascending rank. As an immediate consequence of Theorem 1, we also get that, for given G and H, there are infinitely many elements g satisfying the conclusion of the theorem.

Theorem 1 is proved in Sections 1–4. In Section 5, we give a short proof of the result due to I. Kapovich and H. Short that an infinite quasiconvex subgroup of a word hyperbolic group has a finite index in its commensurator. We give also some corollaries to this result.

2. Preliminary Information

2.1. HYPERBOLIC SPACES AND GROUPS

Let X be a metric space. The *Gromov inner product* of points x and y of X with respect to a point $z \in X$ is defined to be

$$(x, y)_z = \tfrac{1}{2}(|x - z| + |y - z| - |y - x|),$$

where $|x - y|$ denotes the distance between x and y.

By a *geodesic segment* between points $x, y \in X$, we mean an isometry (and also its image) $[0, |x - y|] \to X$ such that $0 \mapsto x$ and $|x - y| \mapsto y$. We use the notation $[x, y]$ for some fixed geodesic segment between x and y.

A metric space is called *geodesic* if any two of its points can be joined by a geodesic segment. For $n \geqslant 2$, by a *geodesic n-gon* $[x_1, \ldots, x_n]$ in a geodesic metric space we

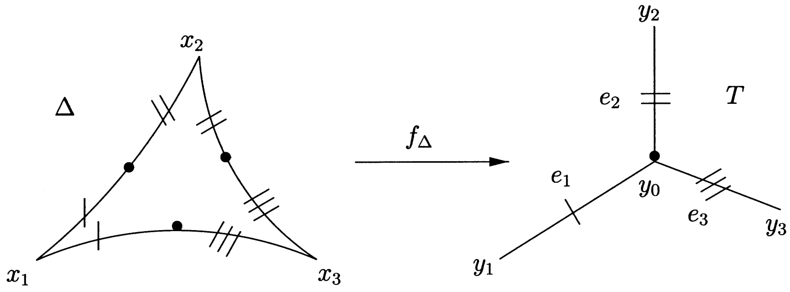

Figure 1.

mean a sequence of geodesic segments $[x_1, x_2], \ldots, [x_n, x_1]$ which we call the *sides* of $[x_1, \ldots, x_n]$.

A map f defined on a metric space is called *ϵ-thin* if $f(x) = f(y)$ implies $|x - y| \leqslant \epsilon$ for all x and y.

Let $\Delta = [x_1, x_2, x_3]$ be a geodesic triangle in a geodesic metric space, and let T be a metric tree with three extremal vertices y_1, y_2 and y_3 so that $|y_i - y_j| = |x_i - x_j|$, see Figure 1. It is easy to see that the length of the edge $e_i = [y_0, y_i]$ is equal to $(x_j, x_k)_{x_i}$, where $\{i, j, k\} = \{1, 2, 3\}$. The triangle Δ is called *ϵ-thin* if the map $f_\Delta : \Delta \to T$ which sends x_i to y_i and which is an isometry on the sides of Δ, is an ϵ-thin map.

A geodesic metric space X is called *δ-hyperbolic*, for $\delta \geqslant 0$, if any geodesic triangle in X is δ-thin. The following lemma in fact gives several equivalent definitions of a hyperbolic space but we formulate and use the equivalence only in one direction.

LEMMA 1 ([9, 6.3.B], [6, 2.21]). *Let X be a δ-hyperbolic metric space. Then the following assertions are true:*

(H1) $(x, y)_w \geqslant \min\{(x, z)_w, (z, y)_w\} - 2\delta$ *for any* $x, y, z, w \in X$;
(H2) $|x - y| + |z - w| \leqslant \max\{|x - z| + |y - w|, |x - w| + |y - z|\} + 4\delta$ *for any* $x, y, z, w \in X$;
(H3) *any side of a geodesic triangle in X belongs to the δ-neighbourhood of the union of the other two sides.*

Let G be a group with a fixed set $\mathcal{A}$ of generators. The *Cayley graph* $C(G)$ of G is a directed graph whose set of vertices is G and whose set of edges is $G \times (\mathcal{A} \cup \mathcal{A}^{-1})$. An edge (g, a) starts at the vertex g and ends at the vertex ga. We consider an edge (g, a) of $C(G)$ as labelled by the letter a. The label $\varphi(\rho)$ of a path $\rho = e_1 e_2 \ldots e_n$ in $C(G)$ is the word $\varphi(e_1)\varphi(e_2)\ldots\varphi(e_n)$ where $\varphi(e_i)$ is the label of the edge e_i. We regard $\varphi(\rho)$ as an element of G. We endow $C(G)$ with a metric by assigning to each edge the metric of the unit segment $[0, 1]$ and then defining the distance $|x - y|$ to be the length of a shortest path between x and y. Thus, $C(G)$ becomes a geodesic metric space. Notice that the metric is invariant under the natural left action of G.

For any $g \in G$, we define the length $|g|$ of g as the length of a shortest word in $\mathcal{A} \cup \mathcal{A}^{-1}$ representing g. It is clear that $|g| = |\rho|$ where ρ is any geodesic path in $C(G)$ with $\varphi(\rho) = g$.

Let G be a finitely generated group. It is called *δ-hyperbolic* with respect to a finite generating set $\mathcal{A}$ if the Cayley graph of G with respect to $\mathcal{A}$ is a δ-hyperbolic space. A group G is called *word hyperbolic* if it is δ-hyperbolic for some $\delta \geqslant 0$ and $\mathcal{A}$. It turns out that the word hyperbolicity of a group is independent of the finite generating set chosen [9, 2.3.E].

Below we shall use properties of a δ-hyperbolic space given in Lemma1, for the Cayley graph $C(G)$ of a given δ-hyperbolic group G. We refer to them as to (H1)–(H3).

A word hyperbolic group is called *elementary* if it has a cyclic subgroup of finite index. So it is finite or virtually infinite cyclic.

LEMMA 2 ([9, 8.5.M], [6, p. 156]). *Let G be a non-elementary torsion-free word hyperbolic group. Then the centralizer $C_G(g)$ of any element $g \in G$ is a cyclic subgroup.*

LEMMA 3 ([15, Lemma 12]). *If $ag^k a^{-1} = g^l$ in a torsion-free word hyperbolic group then $k = l$ or $g = 1$.*

2.2. QUASIGEODESICS

Let ρ be a path in a geodesic metric space X. We assume ρ has the natural parametrization by arc length. Let $\lambda > 0$ and $c \geqslant 0$. The path ρ is called *(λ, c)-quasigeodesic* if $|\rho(s) - \rho(t)| \geqslant \lambda|s - t| - c$ for any points $\rho(s)$ and $\rho(t)$ on ρ.

LEMMA 4 ([9, 7.2.A], [6, p. 87]). *For any $\lambda > 0$ and $c, \delta \geqslant 0$, there exists a number $R = R(\delta, \lambda, c)$ such that any (λ, c)-quasigeodesic path ρ in a δ-hyperbolic space and any geodesic path τ with the same endpoints as ρ are in the R-neighbourhood of each other.*

It is known that paths labelled by elements of infinite order of a hyperbolic group are quasigeodesic. More precisely, we have

LEMMA 5 ([9], [16, Lemma 1.11]). *For any word W representing an element of infinite order in a hyperbolic group G, there exist constants $\lambda > 0$ and $c \geqslant 0$ such that any path with the label W^m in the Cayley graph of G is (λ, c)-quasigeodesic for any integer m.*

A word W is called *cyclically minimal in the group G* if it is a shortest representative of its conjugacy class in G. For cyclically minimal words in torsion-free groups, the statement of the previous lemma can be strengthened in the following way by choosing λ and c independent on W.

LEMMA 6 ([15, Lemma 27]). *For any torsion-free hyperbolic group G, there are constants $\lambda > 0$ and $c \geqslant 0$ such that for any cyclically minimal word W in G and any $m \in \mathbb{Z}$, any path with the label W^m in the Cayley graph of G is (λ, c)-quasigeodesic.*

LEMMA 7 ([15, Lemma 21]). *Let $c \geqslant 7\delta$ and $c_1 > 12(c+\delta)$, and suppose that a geodesic n-gon $[x_1, \ldots, x_n]$ in a δ-hyperbolic metric space satisfies the conditions $|x_{i-1} - x_i| > c_1$ for $i = 2, \ldots, n$ and $(x_{i-2}, x_i)_{x_{i-1}} < c$ for $i = 3, \ldots, n$. Then the polygonal line $\rho = [x_1, x_2] \cup [x_2, x_3] \cup \ldots \cup [x_{n-1}, x_n]$ is contained in the $2c$-neighbourhood of the side $[x_n, x_1]$, and the side $[x_n, x_1]$ is contained in the 7δ-neighbourhood of ρ.*

2.3. QUASICONVEX SUBSETS AND SUBGROUPS

A subset Y of a geodesic metric space X is called *quasiconvex* (or K-quasiconvex) if any geodesic path in X with endpoints in Y lies in the K-neighbourhood of Y for some $K \geqslant 0$. It is clear that any finite, bounded or cobounded subset of a geodesic metric space is quasiconvex. Lemma 4 implies in particular that any quasigeodesic path in a hyperbolic space is quasiconvex.

If we regard a subgroup of a group as a set of vertices in the Cayley graph of the group, we get a definition of *quasiconvex subgroup*. It is obvious that finite subgroups and subgroups of finite index are quasiconvex. In a finitely generated group, any quasiconvex subgroup is finitely generated and the intersection of any two quasiconvex subgroups is quasiconvex [18]. It follows from Lemma 5 that any cyclic subgroup of a word hyperbolic group is quasiconvex. This is true also for virtually cyclic subgroups [4, Pr.1.4, Ch.10]. A quasiconvex subgroup of a word hyperbolic group is word hyperbolic [4, Pr.4.2, Ch.10]. But this is not true in general for a finitely generated subgroup of a word hyperbolic group (see [17], [1] and [3]).

Below we shall need the following lemma.

LEMMA 8 ([7, Lemma 1.2]). *Let H be a K-quasiconvex subgroup of a δ-hyperbolic group G. If a shortest representative of the double coset HgH has length greater than $2K + 2\delta$, then the intersection $H \cap g^{-1}Hg$ consists of elements shorter than $2K + 8\delta + 2$ and, hence, is finite.*

3. Double Cosets of Quasiconvex Subgroups

The aim of this section is to prove the following proposition.

PROPOSITION 1. *Let G be a word hyperbolic group and H a quasiconvex subgroup of G of infinite index. Then the number of double cosets of G modulo H is infinite.*

As the following examples show, the statement is not true if the group is not hyperbolic or the subgroup is not quasiconvex.

EXAMPLE 1 ([2, Ch.IV, § 2, n° 2.3]). Let $G = GL(n, \mathbb{Q})$ and let H be the subgroup of G of all upper triangle matrices. Then H is of infinite index but the number of double cosets of G modulo H is finite.

The following example is due to P. de la Harpe.

EXAMPLE 2. Let $G = \langle a, b | \; b^2 = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}_2$. This group is hyperbolic since it is a free product of two hyperbolic groups. We define an action of G on the disjoint union $\mathbb{Z} \amalg \{\infty\}$ as follows: $a(n) = n + 1$ for all $n \in \mathbb{Z}$, $a(\infty) = \infty$ and $b(0) = \infty$, $b(\infty) = 0$, and $b(n) = n$ for all $n \in \mathbb{Z} \setminus \{0\}$. Let H be the stabilizer of $\{\infty\}$. As the action is transitive, H is of infinite index in G. However, the number of double cosets of G modulo H is finite. Namely, $G = H \amalg HbH$. The subgroup H is not quasiconvex because it is not finitely generated.

The proof of the proposition relies on the following lemmas.

LEMMA 9. *Let G be a word hyperbolic group. Let H be a K-quasiconvex set in G, stable with respect to the inversion, and x be a shortest element of the set Hx. Then for every* $h \in H$ *we have* $(x, h)_e \leqslant K + \delta$.

Proof. Let $h \in H$. Let α and β be geodesic paths in $C(G)$ starting at e and ending at x and h, respectively. We take points p on α and q on β with $|p - e| = |q - e| = (x, h)_e$. By δ-hyperbolicity of G, $|p - q| \leqslant \delta$. By K-quasiconvexity of H, there is $g \in H$ such that $|q - g| \leqslant K$. Then

$$|g^{-1}x| = |x - g| \leqslant |x - p| + |p - q| + |q - g| \leqslant |x| - (x, h)_e + \delta + K.$$

Since x is a shortest representative of Hx, we have $|g^{-1}x| \geqslant |x|$ which implies $(x, h)_e \leqslant K + \delta$. □

LEMMA 10. *For any integer* $m \geqslant 1$ *and numbers* $\delta, K, C \geqslant 0$, *there exists* $A = A(m, \delta, K, C) \geqslant 0$ *with the following property.*

Let G be a δ-hyperbolic group with a generating set containing at most m elements and H a K-quasiconvex subgroup of G. Let $g_1, \ldots, g_n, s$ *be elements of G such that*

(i) *cosets* Hg_i *and* Hg_j *are different for* $i \neq j$;
(ii) g_n *is a shortest representative of* Hg_n;
(iii) $|g_i| \leqslant |g_n|$ *for* $1 \leqslant i < n$;
(iv) *for* $i \neq n$, *all the products* $g_i g_n^{-1}$ *belong to the same double coset HsH with* $|s| \leqslant C$.

Then $n \leqslant A = A(m, \delta, K, C)$.

Proof. Let $d = \max\{3K + 8\delta + 1, C\}$. For each $i < n$, we choose a factorization $g_i g_n^{-1} = h_i s_i k_i$ where $h_i, k_i \in H$ and $|s_i| \leqslant d$, with the minimal possible $|h_i| + |k_i|$. This can be done due to (iv).

Let $A = A(m, \delta, K, C)$ be greater than the number of elements of G of length less or equal to $2d + 3K + 4\delta$. We prove that $|k_i| \leqslant d + 3K + 4\delta$ for all $i < n$. This will suffice for proving the lemma. Indeed, this implies $|s_i k_i| \leqslant 2d + 3K + 4\delta$. By the choice of A, if $n > A$ then for some pair of indices $1 \leqslant i < j < n$, the elements $s_i k_i$ and $s_j k_j$ coincide. But then we get $g_i g_n^{-1} = h_i s_i k_i$, $g_j g_n^{-1} = h_j s_i k_i$ and, hence, $Hg_i = Hg_j$ contradicting (i).

Assume the converse, i.e. $|k_i| > d + 3K + 4\delta$ for some $i < n$. Without loss of generality, we suppose $i = 1$. Let α be a geodesic path in $C(G)$ labelled with g_1 which begins at g_1^{-1} and ends at e (e denotes the trivial element of G), and let ω be a geodesic path in $C(G)$ labelled with g_n which begins at g_n^{-1} and ends at e. By $\overline{\alpha}$ we denote the path inverse to α. Let η, σ and κ be geodesic paths in $C(G)$ labelled with h_1, s_1 and k_1, respectively, such that $\eta\sigma\kappa\omega\overline{\alpha}$ is a closed path starting and ending at g_1^{-1}, see Figure 2.

First we prove that

$$(g_n^{-1}k_1^{-1}, e)_{g_n^{-1}} = \tfrac{1}{2}(|g_n| + |k_1| - |g_n^{-1}k_1^{-1}|) \leqslant K + \delta. \tag{1}$$

Indeed, $(g_n^{-1}k_1^{-1}, e)_{g_n^{-1}} = (g_n, k_1^{-1})_e$. So, (1) follows from the previous lemma and the fact that g_n is a shortest representative of Hg_n.

Now let ξ be a geodesic path in $C(G)$ joining e and $g_n^{-1}k_1^{-1}$. By (1) and δ-hyperbolicity of G, for some point t on ξ we have $|g_n^{-1} - t| \leqslant K + 2\delta$. Using

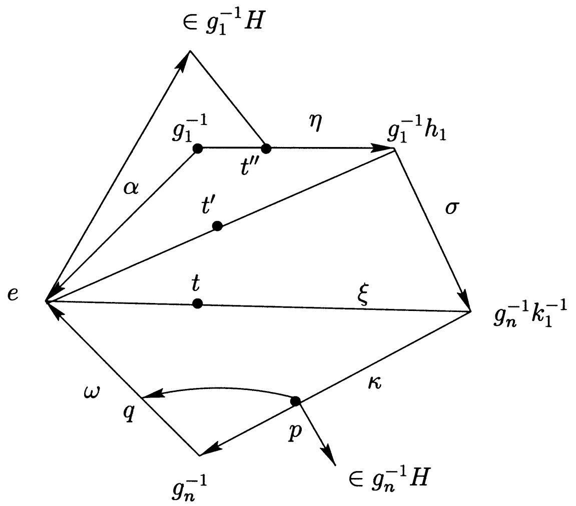

Figure 2.

(1) again and the assumption we see that

$$(g_n^{-1}k_1^{-1}, g_1^{-1}h_1)_e$$
$$\geqslant |g_n^{-1}k_1^{-1}| - |s_1| \geqslant |g_n| + |k_1| - 2(K+\delta) - d > |g_n^{-1}| + K + 2\delta.$$

Hence, by δ-hyperbolicity, for some point t' lying on a geodesic path joining e and $g_1^{-1}h_1$, we have $|t - t'| \leqslant \delta$. Using δ-hyperbolicity of G once more, we find a point t'' on $\overline{\alpha}\eta$ with $|t' - t''| \leqslant \delta$. Thus we get $|g_n^{-1} - t''| \leqslant K + 4\delta$. If t'' lies on $\overline{\alpha}$ then using (iii) we obtain

$$|t'' - g_1^{-1}| = |g_1| - |t'' - e| \leqslant |g_n| - (|g_n| - K - 4\delta) = K + 4\delta.$$

Taking g_1^{-1} instead of t'' in this case, we may assume that t'' always lies on η and $|g_n^{-1} - t''| \leqslant 2K + 8\delta$. By K-quasiconvexity of H, there is $g \in g_1^{-1}H$ such that $|t'' - g| \leqslant K$ and, hence, $|g_n^{-1} - g| \leqslant 3K + 8\delta$. Then $g_1g_n^{-1}$ may be represented as $h's'$ where $h' = g_1 g \in H$, $s' = g^{-1}g_n^{-1}$ and $|s'| = |g_n^{-1} - g| < d$. But since $|h'| = |g_1^{-1} - g| \leqslant |h_1| + K < |h_1| + |k_1|$ we get a contradiction with the choice of h_1, s_1 and k_1. This finishes the proof. □

Proof of Proposition 1. Let G be a δ-hyperbolic group and H a K-quasiconvex subgroup of G of infinite index. Assume that the number of double cosets of G modulo H is finite, say N. Then the length of a shortest representative of any double coset is bounded by a number C. Take any $n > AN + 1$ with $A = A(G, \delta, K, C)$ from the previous lemma. Since H is of infinite index, there exist n elements $g_1, \dots, g_n \in G$ satisfying conditions (i)–(iii) of Lemma10. Then by the choice of n, there exists a double coset HsH containing at least $A + 1$ elements of the form $g_ig_n^{-1}$ for $i < n$. But this contradicts Lemma 10. □

4. Proof of Theorem 1

DEFINITION 1. We call a word of the form $u^{-1}wu$ (formally, the pair of words u and w) a *reduced transform* if the following conditions are satisfied:

(i) Among all words $u^{-1}wu$ representing the same element of G, w has the minimal possible length.
(ii) For a fixed length of w, among all words $u^{-1}wu$ representing the same element of G, u has the minimal possible length.

The following lemma is also of independent interest.

LEMMA 11. *For any $m \geqslant 1$ and $\delta \geqslant 0$ there is a number $L = L(m, \delta) > 0$ with the following property.*

Let G be a δ-hyperbolic group with a generating set containing at most m elements. Then for any reduced transform $u^{-1}wu$, any path in $C(G)$ labelled with $u^{-1}w^ku$,

$k \in \mathbb{Z}$, and any geodesic path with the same endpoints lie in the L-neighbourhood of each other.

Proof. By Lemmas 6 and 4, there is a number $T > 0$ such that for any cyclically minimal word w, any path in $C(G)$ labelled with w^k and any geodesic path with the same endpoints lie in the T-neighbourhood of each other.

Let $\eta\kappa\theta$ be a path in $C(G)$ labelled with $u^{-1}w^k u$ where the labels of η, κ and θ are u^{-1}, w^k and u, respectively. Let α be a geodesic path with the same endpoints as of $\eta\kappa\theta$. Using δ-hyperbolicity of G we see that α lies in the $(T + 2\delta)$-neighbourhood of $\eta\kappa\theta$. We shall now prove that $\eta\kappa\theta$ lies in the L-neighbourhood of α for some $L > 0$ independent on the number k and the reduced transform $u^{-1}wu$. Without loss of generality we assume that $\eta\kappa\theta$ starts at the vertex e. Then κ starts at u^{-1} and θ starts at $u^{-1}w^k$ and ends at $u^{-1}w^k u$.

Denote $T_1 = T + \delta + 1$. First we prove that

$$(e, u^{-1}w^k)_{u^{-1}} \leqslant T_1. \tag{2}$$

Assume that (2) does not hold. Let β be a geodesic path with the same endpoints as of κ. Choose points p and p' on η and β respectively, with $|p - u^{-1}| = |p' - u^{-1}| = T_1$. By the assumption and δ-hyperbolicity of G, $|p - p'| \leqslant \delta$. There is a point q on κ with $|q - p'| \leqslant T$. We may assume that q is a vertex of $C(G)$. We have

$$|q - e| \leqslant |q - p'| + |p - p'| + |p - e| \leqslant T + \delta + |u| - T_1 < |u|.$$

This means that $|u^{-1}v| < |u|$ for some initial segment v of the word w^k. But then

$$u^{-1}wu = (v^{-1}u)^{-1} \cdot v^{-1}wv \cdot v^{-1}u$$

where $v^{-1}u$ may be represented by a word shorter than u and $v^{-1}wv$ is equal to a cyclic shift of w. This contradicts condition (ii) of Definition 1, thus finishing the proof of (2).

Similarly to (2), with η replaced by θ we obtain

$$(e, w^k u)_{w^k} \leqslant T_1. \tag{3}$$

Now we show that

$$(e, u^{-1}w^k u)_{u^{-1}w^k} \leqslant L_1, \tag{4}$$

where $L_1 = (2m)^{4T_1+6\delta} + 3T_1 + 3\delta$ and m is the number of generators of G. We consider two cases.

Case 1. $|w^k| > 2T_1 + 2\delta$. By (H1),

$$(u^{-1}, u^{-1}w^k u)_{u^{-1}w^k} \geqslant \min\{(e, u^{-1})_{u^{-1}w^k},\ (e, u^{-1}w^k u)_{u^{-1}w^k}\} - 2\delta. \tag{5}$$

By (3),

$$(u^{-1}, u^{-1}w^k u)_{u^{-1}w^k} = (e, w^k u)_{w^k} \leqslant T_1$$

and by (2),

$$(e, u^{-1})_{u^{-1}w^k} = |w^k| - (e, u^{-1}w^k)_{u^{-1}} > T_1 + 2\delta.$$

Since

$$(u^{-1}, u^{-1}w^k u)_{u^{-1}w^k} < (e, u^{-1})_{u^{-1}w^k} - 2\delta,$$

we get from (5)

$$(e, u^{-1}w^k u)_{u^{-1}w^k} \leqslant (u^{-1}, u^{-1}w^k u)_{u^{-1}w^k} + 2\delta \leqslant T_1 + 2\delta.$$

Case 2. $|w^k| \leqslant 2T_1 + 2\delta$. Assume that (4) is false. Let γ be a geodesic path joining e and $u^{-1}w^k$. Let p be any vertex of $C(G)$ lying on η. By (2) and δ-hyperbolicity of G, there is a point p' on γ with $|p' - p| \leqslant T_1 + \delta$. By the assumption and δ-hyperbolicity, if $|p' - u^{-1}w^k| \leqslant L_1$, then there is a point p'' on θ with $|p' - p''| \leqslant \delta$. We have

$$|p' - u^{-1}w^k| \leqslant |p' - p| + |p - u^{-1}| + |u^{-1} - u^{-1}w^k| \leqslant |p - u^{-1}| + 3T_1 + 3\delta.$$

Hence, we have proved that if $|p - u^{-1}| \leqslant L_1 - 3T_1 - 3\delta$ then there is a point p'' on θ such that $|p - p''| \leqslant T_1 + 2\delta$.

The vertex p divides η into two paths labelled with u_2^{-1} and u_1^{-1} where $u = u_1u_2$. Let q be a vertex that divides θ into paths labelled with u_1 and u_2. From $|p - p''| \leqslant T_1 + 2\delta$, $|p - u^{-1}| = |q - u^{-1}w^k|$ and $|u^{-1} - u^{-1}w^k| \leqslant 2T_1 + 2\delta$ it easily follows that $|p - q| \leqslant 4T_1 + 6\delta$. Thus, we have proved that for any initial segment u_1 of the word u with $|u_1| \leqslant L_1 - 3T_1 - 3\delta$, we have

$$|u_1^{-1}w^k u_1| \leqslant 4T_1 + 6\delta.$$

Since $L_1 - 3T_1 - 3\delta$ is greater than the number of elements of G of length at most $4T_1 + 6\delta$, there are two different initial segments x and xy of the word u such that

$$x^{-1}w^k x = y^{-1}x^{-1}w^k xy.$$

By Lemma 2, y and $x^{-1}wx$ lie in a cyclic subgroup of G and hence commute. Then $u^{-1}wu = (xz)^{-1}wxz$ where $u = xyz$. But $|xz| < |u|$ contrary to condition (ii) of Definition 1. This finishes the proof of (4).

Now by δ-hyperbolicity and (2), the path $\eta\kappa$ lies in the $(T + T_1 + \delta)$-neighbourhood of γ, and by δ-hyperbolicity and (4), $\gamma\theta$ lies in the $(L_1 + \delta)$-neighbourhood of α. Hence, $\eta\kappa\theta$ lies in the L-neighbourhood ofα where $L = T + T_1 + L_1 + 2\delta$. □

LEMMA 12. *For any* $m \geqslant 1$ *and* $\delta \geqslant 0$ *there are constants* $E = E(m, \delta)$, $D = D(m, \delta) > 0$ *with the following property. Let G be a* δ*-hyperbolic group with a generating set containing at most m elements. Then for any* $x, y \in G$, *if* $(x, y)_e \leqslant \frac{1}{2}|x| - E$ *then for any* $k > 0$, $(x^k, y)_e \leqslant (x, y)_e + D$.

Proof. We take $E = 2L + \delta + 1$ and $D = E + L$ where L is given in Lemma 11. Let $u^{-1}wu$ be a reduced transform representing x. Let μ and ρ be the geodesic paths in

$C(G)$ starting at e and ending at x and at x^k, respectively. Let τ be the path starting at e and labelled with $u^{-1}wu$. We take a point p on τ with $\ell = |e-p| = (x,y)_e + E$. We have $|u^{-1}w| \geqslant |u|$, for otherwise $u^{-1}wu = (w^{-1}u)^{-1}w(w^{-1}u)$ contrary to condition(ii) of Definition 1. This implies $|x| \leqslant |u^{-1}w| + |u| \leqslant 2|u^{-1}w|$ and, since $\ell \leqslant \frac{1}{2}|x|$, we may assume that p lies on the initial segment of τ labelled with $u^{-1}w$. By Lemma 11, there are points p' on μ and p'' on ρ such that $|p-p'| \leqslant L$ and $|p-p''| \leqslant L$. In particular,

$$|e-p''| \leqslant \ell + L \quad \text{and} \quad |e-p'|, |e-p''| \geqslant \ell - L.$$

Assume that $(x^k, y)_e > (x,y)_e + D$. Then $|e-p''| < (x^k, y)_e$ and by δ-hyperbolicity of G, there is a point q on a geodesic path with the endpoints e and y, such that $|e-q| = |e-p''|$ and $|p''-q| \leqslant \delta$. Since

$$|x-p'| = |x| - |e-p'| \leqslant |x| - \ell + L$$

and

$$|q-y| = |y| - |e-q| \leqslant |y| - \ell + L,$$

we have

$$\begin{aligned} |x-y| &\leqslant |x-p'| + |p'-p''| + |p''-q| + |q-y| \\ &\leqslant |x| - \ell + L + 2L + \delta + |y| - \ell + L \\ &= |x-y| - \delta - 2 \end{aligned}$$

obtaining a contradiction. □

Now we prove a statement which will allow us to obtain that $\langle H, g\rangle = H * \langle g\rangle$ and $\langle H, g\rangle$ is quasiconvex in Theorem 1, under certain conditions on products of g and elements of H. The idea of this is given in Lemma 7. But we need a slightly more elaborate statement because the segments in Lemma 7 are required to be sufficiently long while an element of H may have a small length.

LEMMA 13. *Let $n \geqslant 1$, $r \geqslant 0$ and elements $y_i, z_i \in G$ $(1 \leqslant i \leqslant n)$ satisfy*

$$|z_i| > 3r + 5\delta \quad (1 \leqslant i \leqslant n), \tag{6}$$

$$|y_1 z_1| \geqslant |y_1| + |z_1| - 2r, \qquad |z_{i-1} y_i z_i| \geqslant |z_{i-1}| + |y_i| + |z_i| - 2r \qquad (1 < i \leqslant n). \tag{7}$$

Then the following assertions are true:

(i) *One has*

$$|y_1 z_1 y_2 z_2 \dots y_n z_n| \geqslant |y_1 z_1 y_2 z_2 \dots y_{n-1} z_{n-1}| + |y_n| + |z_n| - 4r - 4\delta.$$

In particular, if $|z_i| > 4r + 4\delta$ for all i then by induction, $y_1 z_1 y_2 z_2 \dots y_n z_n \neq 1$.

(ii) *Let ρ be a path in $C(G)$ labelled with $y_1z_1y_2z_2\dots y_nz_n$ and τ a geodesic path with the same endpoints as ρ. If $r \geqslant 4\delta$ and $|z_i| > 14r + 48\delta$ for all i then ρ is contained in the $(3r + 7\delta)$-neighbourhood of τ, and τ is contained in the 8δ-neighbourhood of ρ.*

Proof. (i) We use induction on n. If $n = 1$, the statement follows from the hypothesis. Let $n > 1$. Denote

$$a = y_1z_1y_2z_2\dots y_nz_n,$$
$$b = y_1z_1y_2z_2\dots y_{n-1}z_{n-1},$$
$$c = y_1z_1y_2z_2\dots y_{n-2}z_{n-2}y_{n-1},$$
$$d = z_{n-1}y_nz_n,$$
$$f = y_nz_n.$$

By the inductive assumption,

$$|b| \geqslant |y_1z_1y_2z_2\dots y_{n-2}z_{n-2}| + |y_{n-1}| + |z_{n-1}| - 4r - 4\delta \geqslant \geqslant |c| + |z_{n-1}| - 4r - 4\delta \tag{8}$$

By (7),

$$|d| \geqslant |z_{n-1}| + |y_n| + |z_n| - 2r \geqslant |z_{n-1}| + |f| - 2r.$$

Summing this with (8) and using (6), we get

$$|b| + |d| \geqslant |f| + |c| + 2|z_{n-1}| - 6r - 4\delta > |f| + |c| + 6\delta. \tag{9}$$

By (H2),

$$|b| + |d| \leqslant \max\{|a| + |z_{n-1}|, |c| + |f|\} + 4\delta.$$

If $|a| + |z_{n-1}| \leqslant |c| + |f|$, then

$$|b| + |d| \leqslant |c| + |f| + 4\delta < |b| + |d| - 6\delta + 4\delta,$$

obtaining a contradiction. Hence, $|a| + |z_{n-1}| > |c| + |f|$. Then

$$b| + |d| \leqslant |a| + |z_{n-1}| + 4\delta.$$

Using (7) we get

$$|a| \geqslant |b| + |d| - |z_{n-1}| - 4\delta$$
$$\geqslant |b| + |z_{n-1}| + |y_n| + |z_n| - 2r - |z_{n-1}| - 4\delta$$
$$\geqslant |b| + |y_n| + |z_n| - 4r - 4\delta$$

as desired.

(ii) By (7),

$$|z_{i-1}| + |y_iz_i| \geqslant |z_{i-1}y_iz_i| \geqslant |z_{i-1}| + |y_i| + |z_i| - 2r \geqslant |z_{i-1}| + |y_iz_i| - 2r.$$

Hence, $(y_iz_i, z_{i-1}^{-1})_e \leqslant r$ and

$$|y_iz_i| \geqslant |y_i| + |z_i| - 2r \quad (1 \leqslant i \leqslant n). \tag{10}$$

Using this with (6), we get

$$|z_i| + |y_iz_i| - |y_i| \geqslant 2|z_i| - 2r > 2(r + 2\delta),$$

i.e.

$$((y_{i-1}z_{i-1})^{-1}, z_{i-1}^{-1})_e > r + 2\delta.$$

By (H1),

$$r \geqslant (y_iz_i, z_{i-1}^{-1})_e \geqslant \min\{(y_iz_i, (y_{i-1}z_{i-1})^{-1})_e, ((y_{i-1}z_{i-1})^{-1}, z_{i-1}^{-1})_e\} - 2\delta$$

and, hence,

$$(y_iz_i, (y_{i-1}z_{i-1})^{-1})_e \leqslant r + 2\delta. \tag{11}$$

Let x_i be the initial point of the subpath of ρ labelled with y_i. Obviously, we have $|x_i - x_{i+1}| = |y_iz_i|$. By (10) and the condition on $|z_i|$,

$$|x_i - x_{i+1}| > 14r + 48\delta - 2r = 12(r + 4\delta).$$

By (11),

$$(x_{i-2}, x_i)_{x_{i-1}} < r + 3\delta \quad \text{for } i = 3, \dots, n.$$

So we can apply Lemma 7 to a geodesic n-gon $[x_1, \dots, x_n]$ with $[x_n, x_1] = \tau$. Thus, the polygonal line

$$\eta = [x_1, x_2] \cup [x_2, x_3] \cup \dots \cup [x_{n-1}, x_n]$$

is contained in the $2(r + 3\delta)$-neighbourhood of τ, and τ is contained in the 7δ-neighbourhood of η.

It follows from (10) and δ-hyperbolicity that for any i the subpath ρ_i of ρ labelled with y_iz_i lies in the $(r + \delta)$-neighbourhood of $[x_i, x_{i+1}]$ and $[x_i, x_{i+1}]$ lies in the δ-neighbourhood of ρ_i. Hence, ρ is contained in the $(3r + 7\delta)$-neighbourhood of τ, and τ is contained in the 8δ-neighbourhood of ρ.

Proof of Theorem 1. Let G be a non-elementary torsion-free δ-hyperbolic group and H a K-quasiconvex subgroup of G of infinite index. We want to find a non-trivial element $g \in G$ such that $sgp\langle H, g\rangle = H * \langle g\rangle$ and $sgp\langle H, g\rangle$ is quasiconvex in G.

Take $N = 2K + 2E + 2\delta$ where E is as in Lemma 12. By Proposition1, we find $x \in G$ such that $|x| > N$ and x is a shortest representative in its double coset HxH. It follows from Lemma 9 applied to Hx and Hx^{-1}, that $(x^{\pm 1}, h)_e \leqslant K + \delta$ for all $h \in H$.

For the required g, we take x^M for a sufficiently large M. By Lemma 13, to prove that $sgp\langle H, g\rangle = H * \langle g\rangle$ and $sgp\langle H, g\rangle$ is quasiconvex it suffices to verify the conditions of Lemma 13 for some r, where y_i's are any elements of H and z_i's are of the form g^t, $t \neq 0$. So we have to show that, for some r,

$$|x^{Mt}| > 14r + 48\delta, \text{ for any } t \neq 0, \tag{12}$$

$$|hx^{Mt}| \geqslant |h| + |x^{Mt}| - 2r, \text{ for any } t \neq 0 \text{ and } h \in H, \tag{13}$$

$$|x^{Ms}hx^{Mt}| \geqslant |x^{Ms}| + |h| + |x^{Mt}| - 2r, \text{ for any } s, t \neq 0 \text{ and } h \in H. \tag{14}$$

By Lemma 12, for any $h \in H$,

$$(x^t, h)_e \leqslant K + D + \delta. \tag{15}$$

In particular, this implies (13) for any $r \geqslant K + D + \delta$ and M.
The rest of the proof is divided into a number of steps.

CLAIM 1. *For any $h \in H$, if $|h| > 2K + 2D + 4\delta$ then*

$$|x^s hx^t| \geqslant |x^s| + |h| + |x^t| - 4K - 4D - 8\delta, \quad \textit{for any } s, t \neq 0.$$

By (H1),

$$(x^s, x^s hx^t)_{x^s h} \geqslant \min\{(e, x^s)_{x^s h}, (e, x^s hx^t)_{x^s h}\} - 2\delta. \tag{16}$$

By (15),

$$(x^s, x^s hx^t)_{x^s h} = (h^{-1}, x^t)_e \leqslant K + D + \delta.$$

Using $|h| \geqslant 2K + 2D + 4\delta$ and (15), again we get

$$(e, x^s)_{x^s h} = |h| - (x^{-s}, h)_e > K + D + 3\delta.$$

Since $(x^s, x^s hx^t)_{x^s h} < (e, x^s)_{x^s h} - 2\delta$, we obtain from (16) that

$$K + D + \delta \geqslant (x^s, x^s hx^t)_{x^s h} \geqslant (e, x^s hx^t)_{x^s h} - 2\delta,$$

which implies

$$|x^s hx^t| \geqslant |x^s h| + |x^t| - 2K - 2D - 6\delta \geqslant |x^s| + |h| + |x^t| - 4K - 4D - 8\delta$$

as required.

CLAIM 2. *For any $h \in H$, $\langle x \rangle \cap \langle h \rangle = 1$.*

Indeed, if $x^t = h^s$ for some $t, s \neq 0$, then $(x^{rt}, h^{rt})_e = |x^{rt}|$ for any $r \neq 0$ which contradicts to (15) and Lemma 5.

CLAIM 3. *There is a number* $B > 0$ *depending on* x *such that for every* $h \in H$ *with* $|h| \leqslant 2K + 2D + 4\delta$,

$$|x^s h x^t| \geqslant |x^s| + |h| + |x^t| - 2B, \quad \text{for any } s, t \neq 0.$$

Let $B > 0$ be any number. Assume that $|x^s h x^t| < |x^s| + |h| + |x^t| - 2B$ for some s and t. Without loss of generality, we assume $s > 0$.

By (15), $|hx^t| \geqslant |h| + |x^t| - 2(K + D + \delta)$. Hence

$$|x^s| + |hx^t| - |x^s h x^t| > 2B - 2(K + D + \delta). \tag{17}$$

Let μ be the path in $C(G)$ starting at e and labelled with x^s. Let ρ be the path in $C(G)$ starting at $x^s h$ and labelled with x^t. Let μ' and ρ' be the corresponding geodesic paths. By Lemmas 4 and 5, there is number $F > 0$ depending only on G and x such that μ and μ' are in the F-neighbourhood of each other, and the same is true for ρ and ρ'. In particular, for every point p on μ there is a point p' on μ' such that $|p - p'| \leqslant F$. By δ-hyperbolicity of G, for any point p' on μ' with $|p' - x^s| \leqslant (e, x^s h x^t)_{x^s}$ there is a point p'' on a geodesic path τ joining x^s and $x^s h x^t$, with $|p' - p''| \leqslant \delta$. Since $|x^s - x^s h| = |h|$, again by δ-hyperbolicity, for any p'' lying on τ there is a point q' on ρ' with $|p'' - q'| \leqslant |h| + \delta$. Since $(e, x^s h x^t)_{x^s} > B - K - D - \delta$ by (17), it follows that for any point p on μ with $|p - x^s| \leqslant B - K - D - F - \delta$ there is a point q on ρ with $|p - q| \leqslant Q$ where $Q = |h| + 2F + 2\delta$. By eventually adding $|x|$ to Q we may assume that q divides ρ into two paths labelled with x^j and x^{t-j}. We take p dividing μ into two paths labelled with x^{s-i} and x^i. Then, by what we have proved, for any i between 0 and s with $i|x| \leqslant B - K - D - F - \delta$, there exists j such that

$$|x^i h x^j| \leqslant |h| + |x| + 2F + 2\delta \leqslant |x| + 2K + 2D + 2F + 6\delta. \tag{18}$$

Now we take B such that the number of all i satisfying $i|x| \leqslant B - K - D - F - \delta$ is greater than the number of all elements of G of length at most $|x| + 2K + 2D + 2F + 6\delta$. Then by (18), for some i_1, i_2, j_1 and j_2 with $i_1 \neq i_2$ we get $x^{i_1} h x^{j_1} = x^{i_2} h x^{j_2}$. Denoting $k = i_1 - i_2$ and using Lemma3 we obtain $h^{-1} x^k h = x^k$. Then x^k belongs to the centralizer $C_G(h)$ of h in G. By Lemma 2, $\langle x \rangle \cap \langle h \rangle \neq 1$. But this contradicts to Claim 2. This finishes the proof of Claim 3.

Now using Claim 3 and Claim1, we see that there exists $r > 0$ such that (14) holds for all M. To finish the proof of the theorem, it remains to choose M satisfying (12). Such an M exists since x is of infinite order. □

5. Commensurators of Quasiconvex Subgroups

Recall that two subgroups H_1 and H_2 of a group G are *commensurable* if their intersection $H_1 \cap H_2$ is of finite index both in H_1 and in H_2. The set

$$Comm_G(H) = \{g \in G \mid H \text{ and } gHg^{-1} \text{ are commensurable}\}$$

is called the commensurator of a subgroup H in a group G. Obviously, $Comm_G(H)$ is a group and $Comm_G(H) \supset N_G(H)$, where $N_G(H)$ is the normalizer of H in G. We are going to prove

THEOREM 2 (see also [11]). *Let G be a word hyperbolic group and H an infinite quasiconvex subgroup of G. Then* $[Comm_G(H) : H] < \infty$.

To prove the theorem, we will use the following simple observation.

LEMMA 14. *Let H be a subgroup of a group G. Then the number of left cosets of G modulo H contained in a double coset HgH is equal to the index* $[H : H \cap gHg^{-1}]$.

Proof. Denote $K = H \cap gHg^{-1}$. To any left coset $hgH \subseteq HgH$, $h \in H$, there corresponds a left coset $hK \subseteq H$. For any $h, h' \in H$, the equality $hgH = h'gH$ is equivalent to $h = h'gh_1g^{-1}$ for some $h_1 \in H$ which holds if and only if $hK = h'K$. Hence the correspondence is one-to-one. □

Proof of Theorem 2. If $[G : H] < \infty$ the statement is obvious. Suppose that $[G : H] = \infty$.

Let $g \in Comm_G(H)$. Since H is infinite by the hypothesis of the theorem and $[H : H \cap gHg^{-1}] < \infty$, the intersection $H \cap g^{-1}Hg = g^{-1}(H \cap gHg^{-1})g$ is also infinite. Then by Lemma 8, the length of a shortest representative of the double coset HgH is at most $2K + 2\delta$ where K is the constant of quasiconvexity of$H$. Thus there are only finitely many double cosets HgH with $g \in Comm_G(H)$. By Lemma 14, any such coset HgH contains only finitely many left cosets of G modulo H. Hence the number of left cosets $gH \subseteq Comm_G(H)$ is finite. □

As an immediate consequence of Theorem 2 and the inclusion $Comm_G(H) \supset N_G(H)$ we get the following two corollaries.

COROLLARY 1 (see also [14]). *Let G be a word hyperbolic group and H an infinite quasiconvex subgroup of G. Then* $[N_G(H) : H] < \infty$.

COROLLARY 2 (see also [14]). *Any infinite quasiconvex normal subgroup of a word hyperbolic group is of finite index.*

COROLLARY 3. *Let G be a word hyperbolic group and H an infinite quasiconvex subgroup of G. Then the subgroup* $Comm_G(H)$ *is quasiconvex.*

Proof. It is known [4, Pr.1.4, Ch. 10] that if A and B are subgroups of a word hyperbolic group G, A is quasiconvex, $A \subseteq B$ and $[B : A] < \infty$ then B is quasiconvex as well. The statement follows now from Theorem 2. □

Corollary 3 implies in particular that under its assumptions, $Comm_G(H)$ is a word hyperbolic group, since any quasiconvex subgroup of a word hyperbolic group is itself word hyperbolic [4, Pr.4.2, Ch. 10].

The following result follows almost immediately from Theorem 2. A similar statement is true for irreducible lattices in semisimple Lie groups and for quasiconvex subgroups in geometrically finite groups (see for example [13] and [8]).

COROLLARY 4. *Let G be a word hyperbolic group, and let H_1 and H_2 be quasiconvex infinite subgroups of G. If $Comm_G(H_1) = Comm_G(H_2)$ then H_1 and H_2 are commensurable.*

Proof. By Theorem 2, both H_1 and H_2 are of finite index in their common comensurator $C = Comm_G(H_1) = Comm_G(H_2)$. Then $[C : H_1 \cap H_2] < \infty$ which implies $[H_1 : H_1 \cap H_2] < \infty$ and $[H_2 : H_1 \cap H_2] < \infty$. □

Recall that if G is a discrete group and H is a subgroup of G then the action of G on a Hilbert space $\ell^2(G/H)$ given by the left translation is called the quasi-regular representation of G in $\ell^2(G/H)$. It follows from work of Mackey [12] (see also [5]) that if H is of finite index in its commensurator $Comm_G(H)$ then the quasi-regular representation of G in $\ell^2(G/H)$ is a finite direct sum of irreducible representations. Thus, from Theorem 2 we immediately get the following corollary:

COROLLARY 5. *Let G be a word hyperbolic group and H an infinite quasiconvex subgroup of G. Then the quasi-regular representation of G in $\ell^2(G/H)$ is a finite direct sum of irreducible representations.*

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