FREE PRODUCTS WITH AMALGAMATION AND HNN-EXTENSIONS WHICH ARE OF UNIFORMLY EXPONENTIAL GROWTH

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Given a group Γ generated by a finite set S, we have the *word length* $\ell_S : \Gamma \longrightarrow \mathbb{N}$, for which $\ell_S(\gamma)$ is the smallest integer n such that there exist $s_1, \dots, s_n \in S \cup S^{-1}$ with $\gamma = s_1 \dots s_n$, the *growth function* $\beta_S : \mathbb{N} \longrightarrow \mathbb{N}$, for which $\beta_S(k)$ is the number of $\gamma \in \Gamma$ with $\ell_S(\gamma) \leq k$, and the *exponential growth rate* $\omega(\Gamma, S) = \lim_{k\to\infty} \sqrt[k]{\beta_S(k)}$ (the limit exists by submultiplicativity : $\beta_S(k_1 + k_2) \leq \beta_S(k_1)\beta(k_2)$). It is easy to check that $\omega(\Gamma, S) > 1$ if and only if $\omega(\Gamma, S') > 1$ for *any* finite set S' of generators of Γ. We set

$$\omega(\Gamma) \; = \; \inf \left\{ \; \omega(\Gamma, S) \; | \; \; S \text{ is a finite set of generators of } \Gamma \; \right\}.$$

The group is of *exponential growth* if $\omega(\Gamma, S) > 1$ for some (hence for any) finite generating set S, and of *uniformly exponential growth* if $\omega(\Gamma) > 1$. M. Gromov has asked for an example of a finitely-generated group of exponential growth which is not of uniformly exponential growth (Remark 5.12 of [GLP-81]), but the question is still open [Gro-99]; we refer to [GrH] and Chapter VII of [Har] for more discussion on this problem.

The purpose of the present short note is to show that we have uniformly exponential growth for appropriate free products with amalgamation $A *_C B$, where C is a common subgroup of two groups A and B, and for HNN-extensions $G*_H^\theta = \langle G, t \mid \theta(h) = tht^{-1} \; \forall h \in H\rangle$, where H is a subgroup of a group G and θ an isomorphism from H onto a subgroup of G. More precisely, we show the following result, where $[A : C] \in \{1, 2, \dots, \infty\}$ denotes the index of a subgroup C of a group A.

Theorem. *(i) Let A, B be two finitely-generated groups and let C be a group given as a subgroup of both A and B. Assume moreover that $([A : C] - 1)\,([B : C] - 1) \geq 2$. Then*

$$\omega\,(A *_C B) \; \geq \; \sqrt[4]{2}.$$

In particular, the free product with amalgamation $A *_C B$ is of uniformly exponential growth.

(ii) Let H, K be two isomorphic subgroups of a finitely-generated group G and let $\theta : H \longrightarrow K$ be an isomorphism. Assume moreover that $[G : H] + [G : K] \geq 3$. Then

$$\omega\big(G *_H^\theta\,\big) \; \geq \; \sqrt[4]{2}.$$

In particular, the HNN-extension $G*_H^\theta$ is of uniformly exponential growth.

Recall that $\Gamma = A *_C B$ and $\Delta = G*_H^\theta$ act respectively on trees X and Y defined as follows.

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(i) The vertex set X^0 of X is the disjoint union of the coset spaces Γ/A and Γ/B. The edge set $X^1 = X^1_+ \sqcup \overline{X^1_+}$ has an orientation $X^1_+ = \Gamma/C$, and a positive edge $\gamma C \in X^1_+$ has origin γA and extremity γB. See n° I.4.1 in [Ser-77]. The tree X is bipartite, vertices in Γ/A (respectively in Γ/B) have degree $[A : C]$ (resp. $[B : C]$) and any $\gamma \in \Gamma$ defines an automorphism of X which is *even,* namely such that the combinatorial distance $d(x, \gamma(x))$ is even for any vertex $x \in X^0$.

(ii) The vertex set of Y is $Y^0 = \Gamma/G$. The edge set $Y^1 = Y^1_+ \sqcup \overline{Y^1_+}$ has an orientation $Y^1_+ = \Gamma/H$, and a positive edge $\delta H \in Y^1_+$ has origin δG and extremity $\delta t G$. See n° I.5.3 in [Ser-77]. Observe that Y is a regular tree of degree $[G : H] + [G : K]$.

Corollary. *Let G be a finitely-generated group. If there exists a finite generating set S of G such that $\omega(G, S) < \sqrt[4]{2}$, then G is neither a free product with amalgamation of the form $A *_C B$ with $([A : C] - 1)([B : C] - 1) \geq 2$, nor an HNN-extension of the form $G*^{\theta}_{H}$ with $[G : H] + [G : \theta(H)] \geq 3$.*

The previous theorem is a straightforward consequence of the following proposition.

Proposition. *Let X be a tree and let Γ be a finitely-generated group acting by automorphisms on X. Assume either that Γ is transitive on X^0; or that X is bipartite, that Γ acts by even automorphisms on X, and has exactly two orbits on X^0. Assume moreover that X is infinite and not isomorphic to a line.*

Then $\omega(\Gamma, S) \geq \sqrt[4]{2}$ for any finite set S of generators of Γ.

Recall that an automorphism s of X is *hyperbolic* if it has no fixed vertex and no fixed geometric edge. For such an $s \in Aut(X)$, there exist a geodesic L_s in X, called the *axis* of s, with vertex set $\left\{x \in X^0 \mid d(x, s(x)) = \min_{y \in X^0} d(y, s(y))\right\}$.

Lemma. *Consider two hyperbolic automorphisms s, t of a tree X; let Γ_0 denote the group they generate, and set $S_0 = \{s, t\}$. If $L_s \neq L_t$, then $\omega(\Gamma_0, S_0) \geq 2$.*

Proof of the lemma. To show this, we enumerate the vertices $(x_n)_{n\in\mathbb{Z}}$ of L_s and $(y_n)_{n\in\mathbb{Z}}$ of L_t in such a way that the following conditions hold. (i) $d(x_m, x_n) = d(y_m, y_n) = |m - n|$ for all $m, n \in \mathbb{Z}$. (ii) If L_s and L_t are disjoint, the geodesic segment in X joining L_s to L_t has x_0 and y_0 for its two ends. (iii) If L_s and L_t are not disjoint, then $x_0 = y_0$ and $x_m \neq y_n$ for all $m, n > 0$. Upon replacing s by s^{-1} and t by t^{-1}, we may furthermore assume that there exist integers $a, b \geq 1$ such that $s(x_n) = x_{n+a}$ and $t(y_n) = y_{n+b}$ for all $n \in \mathbb{Z}$. Let P (respectively Q) be the subset of X^0 of those vertices z such that $d(z, x_1) = d(z, x_0) - 1$ (resp. $d(z, y_1) = d(z, y_0) - 1$). Then $P \cap Q = \emptyset$ and

$$s^k(P \cup Q) \subset P \qquad \text{and} \qquad t^k(P \cup Q) \subset Q$$

for all $k \geq 1$. A standard and straightforward *Ping-Pong argument* (see e.g. Proposition VII.2 in [Har]) shows that S_0 generates a free semi-group M_0 on two generators. The notion of exponential growth rate extends readily to semi-groups, and we have $\omega(M_0, S_0) = 2$. Consequently, we have also

$$\omega(\Gamma_0, S_0) \geq \omega(M_0, S_0) = 2$$

and the lemma is proved. □

Proof of the proposition. Suppose first that S contains an element s which is hyperbolic on X, with axis L_s. If one had $u(L_s) = L_s$ for all $u \in S$, one would have $\gamma(L_s) = L_s$ for all $\gamma \in \Gamma$, and indeed $L_s = X$ since Γ has at most two orbits on X^0, in contradiction with the hypotheses. Thus there exists $t \in S$ with $t(L_s) \neq L_s$. If $S_0 = \{s, tst^{-1}\}$ and if Γ_0 is the subgroup of Γ generated by S_0, we have $\omega(\Gamma_0, S_0) \geq 2$ by the lemma.

Let $B(k)$ denote the ball of radius k about e in Γ relative to the word length ℓ_S, so that $S_0 \subset B(3)$; observe that $\omega(\Gamma, B(k)) = \omega(\Gamma, S)^k$ for all $k \geq 1$. We have

$$\omega(\Gamma, S)^3 \;=\; \omega(\Gamma, B(3)) \;\geq\; \omega(\Gamma_0, S_0) \;\geq\; 2$$

and this ends the proof in this case.

Suppose now that any $u \in S$ has some fixed vertex in X^0. Then there exists $s \in S \cup S^2$ which is hyperbolic on X, by n° I.6.5 in [Ser-77]. The argument used for the first case shows now that

$$\omega(\Gamma, S)^4 \;=\; \omega(\Gamma, B(4)) \;\geq\; 2$$

and the proof is complete. $\square$

Remarks. (i) For a free product $\Gamma = A * B$ of two groups satisfying $(|A|-1)\,(|B|-1) \geq 2$, the theorem can be improved to $\omega(\Gamma) \geq \sqrt{2}$. This bound is sharp because, if $A = \{1, \alpha\}$, $B = \{1, \beta, \beta^2\}$ and $S = \{\alpha, \beta\}$, then, setting $\sigma_S(k) = \beta_S(k) - \beta_S(k-1)$, we have $\sigma_S(k+2) = 2\sigma_S(k)$ for all $k \geq 1$, so that

$$\omega(A * B, S) \;=\; \lim_{k\to\infty} \frac{\sigma_S(k+2)}{\sigma_S(k+1)} \;=\; 2 \lim_{k\to\infty} \frac{\sigma_S(k)}{\sigma_S(k+1)} \;=\; \frac{2}{\omega(A * B, S)}$$

and consequently $\omega(A * B, S) = \sqrt{2}$.

For all the groups as in our theorem for which we know explicit computations, the uniform exponential growth rate satisfies also $\omega(-) \geq \sqrt{2}$.

(ii) The behaviour of a group $\Gamma = A *_C B$ with $([A:C]-1)\,([B:C]-1) = 2$ is not so simple.

There is an example due to M. Bridson, where $C \approx \mathbb{Z}^2$ and $[A:C] = [B:C] = 2$, for which C is not and $\Gamma = A *_C B$ is of exponential growth (see Complement VII.10 of [Har]); in Bridson's example, Γ has a subgroup of index 2 of the form $\mathbb{Z}^2 \rtimes_\theta \mathbb{Z}$ (notation being as below) with $\theta = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$, so that Γ is of uniformly exponential growth. (A finitely-generated group and a subgroup of finite index are together of uniformly exponential growth or not, by Proposition 3.3 of [ShW-92].)

If $C = 1$ and both A, B are of order 2, then $A * B$ is an infinite dihedral group, and is consequently not of exponential growth.

(iii) Similarly, if $H = K = G$, an HNN-extension $G*_H^\theta$ is a semi-direct product $\Delta = G \rtimes_\theta \mathbb{Z}$; if G is not of exponential growth, Δ is or is not of exponential growth depending on properties of θ. For example, for $G = \mathbb{Z}^d$ and $\theta \in GL(d, \mathbb{Z}) = Aut(\mathbb{Z}^d)$, the group Δ is of polynomial growth if θ is unipotent and of uniformly exponential growth if θ is as in the example below.

Example. *Let d be an integer, $d \geq 2$, and $\theta \in GL(d, \mathbb{Z})$ be a matrix which has an eigenvalue λ such that $|\lambda| > 1$. Then the corresponding semi-direct product $\Delta = \mathbb{Z}^d \rtimes_\theta \mathbb{Z}$ has uniformly exponential growth.*

In particular, the group of the example is of exponential growth, a result which goes back to [Šva-55].

Proof. Let S be a finite generating set of Δ. If $\pi : \Delta \longrightarrow \mathbb{Z}$ denotes the canonical projection, there exists $s \in S$ such that $\pi(s) = m \neq 0$; upon changing s to s^{-1}, we may assume that $m \geq 1$. As Δ is not abelian, there exists $v \in S$ such that $sv \neq vs$. Set $u = svs^{-1}v^{-1}$; as $u \in \mathbb{Z}^d$, we may identify sus^{-1} with $\theta^m(u)$. Set $S_0 = \{s, u\}$ and let Δ_0 be the subgroup of Δ generated by S_0; as S_0 is inside the ball of radius 4 about e in Δ relative to the word length ℓ_S, we have $\omega(\Delta, S)^4 \geq \omega(\Delta_0, S_0)$, and it is enough to show that $\omega(\Delta_0, S_0) \geq c$ for some constant $c > 1$ independent of the choice of S.

Let L be a linear form on $\mathbb{R}^d$ such that $L \circ \theta = \lambda L$; we have $L(u) \neq 0$ since $u \in \mathbb{Z}^d$ and $\lambda \notin \mathbb{Q}$. We assume here that λ is real, and leave the case of two complex conjugate eigenvalues $\lambda, \overline{\lambda}$ as an exercise to the reader. Let $n \geq 1$ be an integer such that $|\lambda|^n \geq 2$; this depends on θ, but not on S. Observe that $|\lambda|^{mn} \geq 2$. For $k \geq 1$ and $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$, we have

$$L\left(\epsilon_1 u + \epsilon_2 s^n u s^{-n} \ldots + \epsilon_k s^{n(k-1)} u s^{-n(k-1)}\right) = \left(\sum_{j=0}^{k-1} \epsilon_j \lambda^{jmn}\right) L(u)$$

and the numbers $\sum_{j=0}^{k-1} \epsilon_j \left(\lambda^{mn}\right)^j$ are pairwise distinct. Now in $\Delta = \mathbb{Z}^d \rtimes_\theta \mathbb{Z}$, we have

$$(\epsilon_1 u, mn)\,(\epsilon_2 u, mn) \ldots (\epsilon_k u, mn) = \left(\epsilon_1 u + \epsilon_2 s^n u s^{-n} + \ldots + \epsilon_k s^{n(k-1)} u s^{-n(k-1)}\,,\, kmn\right)$$

so that the ℓ_{S_0}-word-length of this element is at most $(n+1)k$. Thus the growth function of (Δ_0, S_0) satisfies $\beta_{S_0}((n+1)k) \geq 2^k$, and we have finally $\omega(\Delta_0, S_0) \geq 2^{\frac{1}{n+1}}$. (We have essentially followed the proof of Lemma 6.2 in [Tit-81].) $\square$

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