A DIAGRAMMATIC COMPUTATION OF ABELIAN LINK INVARIANTS

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ABSTRACT. We show how the multivariable signature and Alexander polynomial of a colored link can be computed from a single symmetric matrix naturally defined from a colored link diagram. In the case of a single variable, it coincides with the matrix introduced by Kashaev [13], which was recently proven to compute the Levine-Tristram signature and the Alexander polynomial of oriented links [16, 3]. As a corollary, we obtain a multivariable extension of Kauffman's determinantal model of the Alexander polynomial [15], recovering a result of Zibrowius [18].

1. INTRODUCTION

As its title suggests, the aim of the present article is to give a way of computing several classical link invariants directly from a diagram. Before specifying these invariants, let us mention that this story is best told in the context of *colored links*, that we now recall.

Given an integer $\mu > 0$, a *μ-colored link* is an oriented link $L \subset S^3$ each of whose components is endowed with a *color* in $\{1, \ldots, \mu\}$ in such a way that all these colors are used. Two colored links are isotopic if they are related by an ambient isotopy which respects the orientation and color of all components. Clearly, a 1-colored link is just an oriented link, while a μ-component μ-colored link is an oriented ordered link. A μ-colored link can be described by an oriented link diagram D with colored components, an object which we will refer to as a *μ-colored diagram* (see Figure 1 for an example). As usual, a crossing v of D is naturally endowed with a sign that we denote by $\text{sgn}(v) = \pm 1$, see Figure 1. Finally, a crossing will be called *monochromatic* if the two corresponding strands are of the same color, and *bichromatic* otherwise.

The invariants we are interested in computing are the classical abelian invariants of a μ-colored link L, namely its multivariable *signature* and *nullity* $\sigma_L, \eta_L \colon (S^1 \setminus \{1\})^\mu \to \mathbb{Z}$, and its multivariable *Alexander polynomial* Δ_L in the normalized form given by the *Conway function* ∇_L. In the case $\mu = 1$, these are the well-known *Levine-Tristram signature* and *nullity* and *Alexander-Conway polynomial*, without doubt among the most studied of link invariants. We refer to Section 2 for the definition of these classical objects.
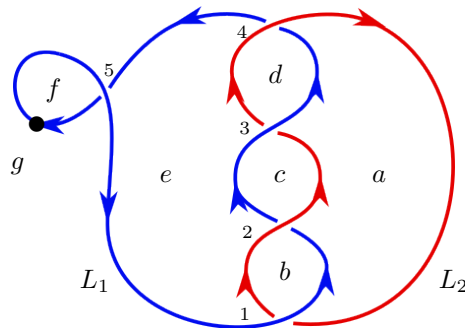
Figure 1. A 2-colored diagram D for a 2-colored link $L = L_1 \cup L_2$. The crossings are labelled 1 through 5 and the regions are labelled a through g. Crossings 1 to 4 are positive and bichromatic, while crossing 5 is negative and monochromatic. The marked point on L_1 will serve a further purpose.
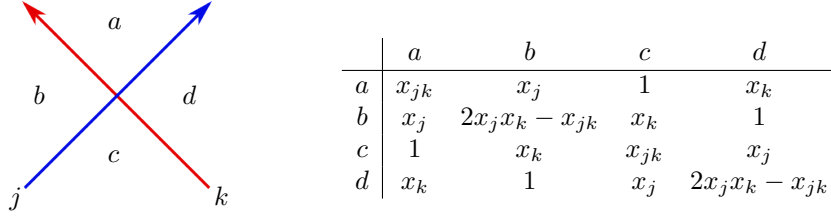
	a	b	c	d
a	x_{jk}	x_j	1	x_k
b	x_j	$2x_jx_k - x_{jk}$	x_k	1
c	1	x_k	x_{jk}	x_j
d	x_k	1	x_j	$2x_jx_k - x_{jk}$

Figure 2. A crossing v together with the corresponding 4×4 minor of $\tau_v(x)$. The incoming left strand is of color j, the incoming right strand of color k, and the four adjacent regions are $a, b, c,$ and d.

As our main result will show, these invariants can all be computed from a single symmetric matrix $\tau_D(x)$, whose coefficients are functions of formal variables $x = \{x_j, x_{jk} \mid 1 \leq j, k \leq \mu\}$ indexed by (unordered pairs of) colors. We now give its definition.

Definition 1. Given a μ-colored diagram D, let $\tau_D(x)$ be the symmetric matrix with rows and columns indexed by the regions of D defined by

$$\tau_D(x) = \sum_v \frac{\text{sgn}(v)}{\sqrt{1 - x_j^2}\sqrt{1 - x_k^2}} \tau_v(x),$$

where the sum is over all crossings of D, the indices $j, k \in \{1, \dots, \mu\}$ are the (possibly identical) colors of the two strands crossing at v, and the only non-vanishing coefficients of the matrix $\tau_v(x)$ are given in Figure 2.

Also, we shall denote by $\widetilde{\tau}_D(x)$ the matrix obtained by removing the two rows and columns corresponding to two adjacent regions of D determined by a marked point on D. We will assume without loss of generality that this point is on a strand of color 1.

Note that if the regions a, b, c, d around a crossing v are not all distinct, then one should add the corresponding rows and columns of $\tau_v(x)$. This happens in the following example.

Example 1. Consider the 2-colored diagram D illustrated in Figure 1. Ordering the regions alphabetically, the corresponding matrix is given by

$$\tau_D(x) = \frac{1}{\sqrt{1 - x_1^2}\sqrt{1 - x_2^2}} \begin{bmatrix} 4(2x_1x_2 - x_{12}) & 2x_2 & 2x_1 & 2x_2 & 4 & 0 & 2x_1 \\ 2x_2 & 2x_{12} & 1 & 0 & 2x_1 & 0 & 1 \\ 2x_1 & 1 & 2x_{12} & 1 & 2x_2 & 0 & 0 \\ 2x_2 & 0 & 1 & 2x_{12} & 2x_1 & 0 & 1 \\ 4 & 2x_1 & 2x_2 & 2x_1 & 4(2x_1x_2 - x_{12}) & 0 & 2x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2x_1 & 1 & 0 & 1 & 2x_2 & 0 & 2x_{12} \end{bmatrix}$$

$$- \frac{1}{1 - x_1^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2x_1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2x_1 \\ 0 & 0 & 0 & 0 & 2x_1 & 2x_1 & 4x_1^2 \end{bmatrix},$$

where the first summand contains the contributions from the four (positive) bichromatic crossings and the second summand is the contribution from the (negative) monochromatic crossing.

Here is our main result.

Theorem 1. *Let D be an arbitrary μ-colored diagram for a μ-colored link L.*

(i) For any $\omega = (\omega_1, \ldots, \omega_\mu) \in (S^1 \setminus \{1\})^\mu$, the signature and nullity of L are given by

$$\sigma_L(\omega) = \tfrac{1}{2}(\text{sign}(\widetilde{\tau}_D(\omega)) - w_{\mathrm{m}}(D)) \quad \text{and} \quad \eta_L(\omega) = \tfrac{1}{2}\,\text{null}(\widetilde{\tau}_D(\omega)),$$

where $w_{\mathrm{m}}(D)$ is the sum of the signs of all monochromatic crossings of D, and $\tau_D(\omega)$ stands for the evaluation of $\tau_D(x)$ at

$$x_j = \text{Re}(\omega_j^{1/2}), \quad x_{jk} = \text{Re}(\omega_j^{1/2}\omega_k^{1/2}).$$

(ii) If D is connected, then the Conway function of L satisfies

$$\nabla_L^2(t_1, \ldots, t_\mu) = \frac{1}{(t_1 - t_1^{-1})^2} \prod_v (-\text{sgn}(v)) \frac{t_j - t_j^{-1}}{2} \frac{t_k - t_k^{-1}}{2} \cdot \det(\widetilde{\tau}_D(t^2)),$$

where the product is over all crossings of D, the indices j, k are the (possibly identical) colors of the two strands crossing at v, and $\tau_D(t^2)$ stands for the evaluation of $\tau_D(x)$ at

$$x_j = \frac{t_j + t_j^{-1}}{2}, \quad x_{jk} = \frac{t_j t_k + t_j^{-1} t_k^{-1}}{2}.$$

Several remarks are in order.

Remarks. (i) We need to fix one square root of each coordinate $\omega_j \in S^1 \setminus \{1\}$ of ω: our choice is to take $\omega_j = e^{i\theta_j}$ with $\theta_j \in (0, 2\pi)$, and $\omega_j^{1/2} = e^{i\theta_j/2}$. In other words, $\omega_j^{1/2}$ denotes the unique square root such that $\text{Im}(\omega_j^{1/2})$ lies in $[0, 1)$. In particular, we have $(\bar{\omega}_j)^{1/2} = -(\overline{\omega_j^{1/2}})$. Note that $x_j^2 \neq 1$, so $\tau_D(\omega)$ is a well-defined symmetric *real* matrix.

(ii) In both points of Theorem 1, the evaluations of the formal variables satisfy $x_{jj} = 2x_j^2 - 1$ for all j. Therefore, if a crossing v is monochromatic, then the matrix $\tau_v(x)$ can be written in a simple form which only depends on the single variable x_j.

(iii) In particular, if $\mu = 1$, then $\tau_D(x)$ depends on a single variable. This matrix was first introduced by Kashaev in [13], see discussion below.

Example 2. Consider once again the 2-colored link illustrated in Figure 1. From the corresponding matrix $\tau_D(x)$ given in Example 1, we compute

$$\widetilde{\tau}_D(t^2) = \frac{-4}{(t_1 - t_1^{-1})(t_2 - t_2^{-1})} \begin{bmatrix} 2(t_1 t_2^{-1} + t_1^{-1} t_2) & t_2 + t_2^{-1} & t_1 + t_1^{-1} & t_2 + t_2^{-1} & 4 \\ t_2 + t_2^{-1} & t_1 t_2 + t_1^{-1} t_2^{-1} & 1 & 0 & t_1 + t_1^{-1} \\ t_1 + t_1^{-1} & 1 & t_1 t_2 + t_1^{-1} t_2^{-1} & 1 & t_2 + t_2^{-1} \\ t_2 + t_2^{-1} & 0 & 1 & t_1 t_2 + t_1^{-1} t_2^{-1} & t_1 + t_1^{-1} \\ 4 & t_1 + t_1^{-1} & t_2 + t_2^{-1} & t_1 + t_1^{-1} & 2(t_1 t_2^{-1} + t_1^{-1} t_2) - \frac{t_2 - t_2^{-1}}{t_1 - t_1^{-1}} \end{bmatrix}$$

and obtain

$$\det(\widetilde{\tau}_D(t^2)) = -\left(\frac{-4}{(t_1 - t_1^{-1})(t_2 - t_2^{-1})}\right)^5 (t_1 - t_1^{-1})(t_2 - t_2^{-1})(t_1 t_2 + t_1^{-1} t_2^{-1})^2.$$

Evaluating at $t_i = \omega_i^{1/2}$, and recalling that the signature can change value only when the determinant vanishes, we compute $\sigma_L(\omega_1, \omega_2) = -\text{sign}[\text{Re}((1 - \omega_1)(1 - \omega_2))]$, $\eta_L(\omega_1, \omega_2) = 1$ if $\omega_1\omega_2 = -1$ and $\eta_L(\omega_1, \omega_2) = 0$ otherwise. Finally, we also get $\pm\nabla_L(t_1, t_2) = t_1 t_2 + t_1^{-1} t_2^{-1}$.

The first point of this theorem provides a practical new way of computing multivariable signatures, but it also yields much simpler proofs of known properties of this invariant. For example, consider the following situation: let L' be the μ-colored link obtained from a $(\mu + 1)$-colored link L by identifying the colors μ and $\mu + 1$; then, for all $(\omega_1, \ldots, \omega_\mu) \in (S^1 \setminus \{1\})^\mu$, we have the equality

$$\sigma_{L'}(\omega_1, \ldots, \omega_\mu) = \sigma_L(\omega_1, \ldots, \omega_\mu, \omega_\mu) - \sum \text{lk}(K_\mu, K_{\mu+1}),$$

the sum being over all components $K_\mu \subset L$ of color μ and all components $K_{\mu+1} \subset L$ of color $\mu + 1$. The original proof of this fact is rather tedious, see [4, Proposition 2.5]. It is an amusing exercise to check that this fact immediately follows from Theorem 1.

The second point of this theorem implies a corollary that we now present. Given a connected colored diagram D, let K_D be the matrix whose rows are indexed by the crossings of D, whose columns are indexed by the regions of D, and whose coefficients are defined by the label of the
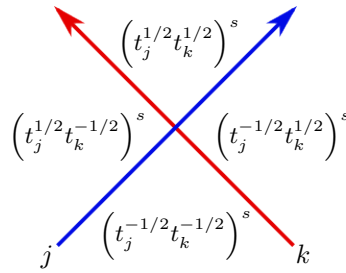
Figure 3. The labels in the definition of K_D around a vertex v with $s = \mathrm{sgn}(v)$.

corners in Figure 3. (If a region abuts a corner from two sides, then the corresponding labels should be added.) Finally, let $\widetilde{K}_D$ denote the square matrix obtained from K_D by removing two columns corresponding to two adjacent regions (separated by a strand of color 1).

Corollary 1. *If D is a connected diagram for a colored link L, then*

$$\det \widetilde{K}_D = \pm(t_1 - t_1^{-1})\nabla_L(t_1, \ldots, t_\mu).$$

Example 3. Consider one last time the 2-colored link illustrated in Figure 1. The matrix K_D is given by

$$\begin{bmatrix} t_1^{-1/2}t_2^{1/2} & t_1^{1/2}t_2^{1/2} & 0 & 0 & t_1^{1/2}t_2^{-1/2} & 0 & t_1^{-1/2}t_2^{-1/2} \\ t_1^{1/2}t_2^{-1/2} & t_1^{-1/2}t_2^{-1/2} & t_1^{1/2}t_2^{1/2} & 0 & t_1^{-1/2}t_2^{1/2} & 0 & 0 \\ t_1^{-1/2}t_2^{1/2} & 0 & t_1^{-1/2}t_2^{-1/2} & t_1^{1/2}t_2^{1/2} & t_1^{1/2}t_2^{-1/2} & 0 & 0 \\ t_1^{1/2}t_2^{-1/2} & 0 & 0 & t_1^{-1/2}t_2^{-1/2} & t_1^{-1/2}t_2^{1/2} & 0 & t_1^{1/2}t_2^{1/2} \\ 0 & 0 & 0 & 0 & 1 & 1 & t_1 + t_1^{-1} \end{bmatrix},$$

from which we compute once again $\pm\nabla_L(t_1, t_2) = t_1 t_2 + t_1^{-1}t_2^{-1}$.

Let us now put our results in the context of the preexisting literature.

In 2018, Kashaev [13] defined the matrix $\tau_D(x)$ in the case $\mu = 1$, and conjectured Theorem 1 in this special case. Recently, the first two named-authors [3] provided a proof of the second part of this conjecture by establishing a connection with Kauffman's determinantal model of the Alexander polynomial [15]; they also proved the first part of the Kashaev conjecture in a very restrictive case and indirect way. Immediately afterwards, the third named-author [16] gave a complete proof of the conjecture. Joining our efforts, we now extend the approach of [16] to the general multivariable case in Theorem 1 and in its proof.

As for Corollary 1, in the case $\mu = 1$ it is nothing but Kauffman's aforementioned model for the Alexander polynomial [15]. Interestingly, Kauffman did state a multivariable version of his model (a detailed proof was only given many years later by Sato [17]), but it is different from our model. However, Zibrowius [19] gave a state sum model for the multivariable Conway function which uses the same labels as the ones of Figure 3 up to a sign; using an extension of Kauffman's *Clock Theorem* [15], this state sum model can be turned into a determinantal model which coincides with the one of Corollary 1. This later fact can be found in the first version of Zibrowius's article on the ArXiv [18] (but not in the published version [19]). Therefore, and even though our proof is completely different, Corollary 1 is not a new result in the strict sense.

Let us finally mention that Friedl-Kausik-Quintanilha [11] recently provided an algorithm for the computation of generalized Seifert matrices (see Section 2.1) for colored links given as closures of colored braids. Since such matrices can be used to define σ_L, η_L and ∇_L, this method yields an algorithmic computation of these invariants. However, the remarkable feature of Theorem 1 remains: a new way of computing these invariants from a single symmetric matrix obtained directly from a colored diagram.

This paper is organised as follows. In Section 2, we recall the necessary background on generalized Seifert matrices (Section 2.1), multivariate signatures of colored links (Section 2.2), and the Conway function (Section 2.3). Section 3 contains the proof of our results, namely the first and
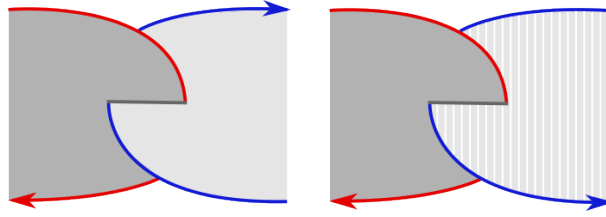
Figure 4. A positive clasp intersection (left), and a negative one (right).

second points of Theorem 1 in Section 3.1 and 3.2, respectively, and of Corollary 1 in Section 3.3. A slightly informal last Section 3.4 contains results on the Alexander module.

2. Background

The aim of this section is to briefly recall the necessary background for our work: we start in Section 2.1 with the definition of C-complexes and generalized Seifert forms, then move on to multivariate signatures in Section 2.2, before dealing with the Conway function in Section 2.3.

2.1. **Generalized Seifert surfaces and matrices.** Seifert surfaces and matrices are well known tools in the construction and study of (single-variable) abelian link invariants, such as the Levine-Tristram signature and the Alexander polynomial. Less well-known is the fact that multivariate invariants can be defined and studied via generalized Seifert surfaces, known as C-complexes. We now introduce these objects, following [8, 2].

To do so, we will use the notation $L = L_1 \cup \cdots \cup L_\mu$ for a μ-colored link, where L_i is the sublink of L consisting of all the components of color i.

Definition 2. A *C-complex* for a μ-colored link $L = L_1 \cup \cdots \cup L_\mu$ is a union $S = S_1 \cup \cdots \cup S_\mu$ of surfaces embedded in S^3 satisfying the following conditions:

  (i) for all i, the surface S_i is a (possibly disconnected) Seifert surface for L_i;
 (ii) for all $i \neq j$, the surfaces S_i and S_j are either disjoint or intersect in a finite number of *clasps*, see Figure 4;
(iii) for all i, j, k pairwise distinct, the intersection $S_i \cap S_j \cap S_k$ is empty.

Note that a C-complex for a 1-colored link L is nothing but a (possibly disconnected) Seifert surface for the oriented link L. The existence of a C-complex for any given colored link is easy to establish [2]. On the other hand, the corresponding notion of S-equivalence is more difficult to prove, see [10] for the recently corrected statement.

These C-complexes allow us to define *generalized Seifert forms* as follows. For any choice of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu) \in \{\pm 1\}^\mu$, let

$$\alpha^\varepsilon \colon H_1(S) \times H_1(S) \longrightarrow \mathbb{Z}$$

be the bilinear form given by $\alpha^\varepsilon(x, y) = \mathrm{lk}(x^\varepsilon, y)$, where x^ε denotes a well-chosen representative of the homology class $x \in H_1(S)$ pushed-off S_i in the ε_i-normal direction (see [4] for a more formal definition). We denote by A^ε the corresponding *generalized Seifert matrices*, defined with respect to a fixed basis of $H_1(S)$. One easily checks the equality

$$(1) \qquad\qquad A^{-\varepsilon} = (A^\varepsilon)^{\mathrm{T}}$$

for all $\varepsilon \in \{\pm 1\}^\mu$. Note that the two generalized Seifert matrices A^-, A^+ of a 1-colored link L coincide with the usual Seifert matrix A of the oriented link L and its transposed matrix A^{T}.

The general principle regarding these matrices is the following: what Seifert matrices can do in one variable for oriented links, generalized Seifert surfaces can do in μ-variables for μ-colored links. In Sections 2.2 and 2.3 we illustrate this principle with two examples of invariants.

2.2. **Signatures and nullities of colored links.** Fix a C-complex S for a μ-colored link L and a basis of $H_1(S)$. Consider an element $\omega = (\omega_1, \dots, \omega_\mu)$ of $\mathbb{T}_*^\mu := (S^1 \setminus \{1\})^\mu$, and set

$$(2) \qquad H(\omega) := \sum_{\varepsilon \in \{\pm 1\}^\mu} \prod_{i=1}^\mu (1 - \overline{\omega}_i^{\varepsilon_i}) A^\varepsilon \,.$$

Using (1), one easily checks that $H(\omega)$ is a Hermitian matrix and hence admits a well-defined signature $\mathrm{sign}(H(\omega)) \in \mathbb{Z}$ and nullity $\mathrm{null}(H(\omega)) \in \mathbb{Z}_{\geq 0}$.

Definition 3 ([4]). The *signature* and *nullity* of the μ-colored link L are functions

$$\sigma_L, \eta_L \colon \mathbb{T}_*^\mu \longrightarrow \mathbb{Z}$$

defined by $\sigma_L(\omega) := \mathrm{sign}(H(\omega))$ and $\eta_L(\omega) := \mathrm{null}(H(\omega))$, respectively.

The fact that these functions are well-defined invariants, i.e. do not depend on the choice of the C-complex S for L, relies on the aforementioned notion of S-equivalence [4, 10]. Note that in the case $\mu = 1$, the functions $\sigma_L, \eta_L \colon S^1 \setminus \{1\} \to \mathbb{Z}$ are the signature and nullity of the Hermitian matrix $(1 - \omega)A + (1 - \overline{\omega})A^\mathrm{T}$, i.e. they coincide with the Levine-Tristram signature and nullity of the oriented link L. We refer to the recent survey [5] for background on this classical invariant.

In a nutshell, all the remarkable properties of the Levine-Tristram signature extend to the multivariable setting. For example, the function σ_L is constant on the connected components of the complement in $\mathbb{T}_*^\mu$ of the zeros of the multivariable Alexander polynomial $\Delta_L(t_1, \dots, t_\mu)$ [4] (see Section 2.3 below). Also, if $(\omega_1, \dots, \omega_\mu) \in \mathbb{T}_*^\mu$ is not the root of any Laurent polynomial $p(t_1, \dots, t_\mu)$ with $p(1, \dots, 1) = \pm 1$, then $\sigma_L(\omega_1, \dots, \omega_\mu)$ and $\eta_L(\omega_1, \dots, \omega_\mu)$ are invariant under *topological concordance* of colored links [6].

2.3. **The Conway function of a colored link.** The one-variable Alexander polynomial $\Delta_L(t)$ of an oriented link L can be generalized to a μ-variable polynomial invariant $\Delta_L(t_1, \dots, t_\mu)$ of a μ-colored link L, a fact known to Alexander himself [1]. To do so, consider the exterior $X_L := S^3 \setminus \nu(L)$ of L and the surjective group homomorphism

$$\pi_1(X_L) \to \mathbb{Z}^\mu, \quad [\gamma] \mapsto (\mathrm{lk}(\gamma, L_1), \dots, \mathrm{lk}(\gamma, L_\mu)) \,.$$

This defines a regular $\mathbb{Z}^\mu$-cover $\widehat{X}_L$ of X_L whose homology groups are hence equipped with the structure of a module over the group ring $\mathbb{Z}[\mathbb{Z}^\mu] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$. In particular, the module $\mathscr{A}_L := H_1(\widehat{X}_L)$ is called the (multivariable) *Alexander module* of L (see Section 3.4), and a greatest common divisor of the elements of its order ideal is the *Alexander polynomial* of L.

Note however that this Laurent polynomial in μ-variables is only well-defined up to multiplication by units of the ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$, i.e. up to a sign and powers of the variables. This later indeterminacy can be easily overcome by harnessing the symmetry of Δ and requiring it to satisfy $\Delta_L(t_1^{-1}, \dots, t_\mu^{-1}) = \pm \Delta_L(t_1, \dots, t_\mu)$, but the sign issue is a non-trivial one.

The solution was suggested by Conway in his landmark paper [7]. He claimed the existence of a well-defined rational function ∇_L satisfying

$$(3) \qquad \nabla_L(t_1, \dots, t_\mu) \doteq \begin{cases} \frac{1}{t_1 - t_1^{-1}} \Delta_L(t_1^2) & \text{if } \mu = 1; \\ \Delta(t_1^2, \dots, t_\mu^2) & \text{if } \mu > 1, \end{cases}$$

where $\doteq$ stands for the equality up to multiplication by $\pm t_1^{\nu_1} \cdots t_\mu^{\nu_\mu}$ with $\nu_1, \dots, \nu_\mu \in \mathbb{Z}$. The first explicit construction of this *Conway function* was given by Hartley [12] using free differential calculus, but we will make use of the following geometric construction [2]. Given any *connected* C-complex $S = S_1 \cup \cdots \cup S_\mu$ for a L, consider the matrix

$$(4) \qquad A_S := \sum_{\varepsilon \in \{\pm 1\}^\mu} \prod_{i=1}^\mu \varepsilon_i t_i^{\varepsilon_i} A^\varepsilon \,.$$

Then, the Conway function of L is given by

$$(5) \qquad \nabla_L(t_1, \dots, t_\mu) = \mathrm{sgn}(S) \prod_{i=1}^\mu (t_i - t_i^{-1})^{\chi(S \setminus S_i) - 1} \det(-A_S) \,,$$
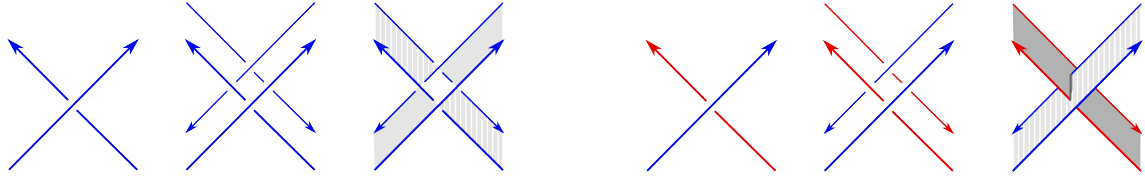
Figure 5. The construction of a C-complex S for $L\#_1 - L$ near a monochromatic crossing (left) and a bichromatic crossing (right).

where $\text{sgn}(S)$ denotes the product of the signs of the clasps of S (recall Figure 4). Note that in the case $\mu = 1$, equations (3) and (5) lead to the formula $\Delta_L(t) = \det(t^{-1/2}A - t^{1/2}A^{\text{T}})$, the classical definition of the Alexander-Conway polynomial of the oriented link L [14].

This geometric construction of the Conway function yields straightforward proofs of the various properties of this invariant. In particular, it yields a "geometric explanation" of the local relations that can be used to compute it from a link diagram, see [2] for more details.

3. Proofs of the main results

As evident from its title, this section contains the proofs of our results. More precisely, we start in Section 3.1 with the demonstration of the first part of Theorem 1 on signatures and nullities. Section 3.2 deals with the second part on the Conway function, while Section 3.3 contains the proof of Corollary 1 on the multivariable Kauffman model. Finally, Section 3.4 consists in a slightly informal discussion on the Alexander module.

3.1. **Signatures and nullities.** In this section we discuss how to compute the multivariable signature, proving part (i) of Theorem 1 which we now restate for convenience.

Proposition 1. *Let D be a diagram for a μ-colored link L. For any $\omega = (\omega_1, \ldots, \omega_\mu) \in (S^1 \setminus \{1\})^\mu$, the signature and nullity of L are given by*

$$\sigma_L(\omega) = \tfrac{1}{2}(\text{sign}(\widetilde{\tau}_D(\omega)) - w_{\text{m}}(D)) \quad and \quad \eta_L(\omega) = \tfrac{1}{2}\,\text{null}(\widetilde{\tau}_D(\omega)),$$

where $w_{\text{m}}(D)$ is the sum of the signs of all monochromatic crossings of D, and $\tau_D(\omega)$ stands for the evaluation of $\tau_D(x)$ at $x_j = \text{Re}(\omega_j^{1/2})$ and $x_{jk} = \text{Re}(\omega_j^{1/2}\omega_k^{1/2})$.

Proof. Fix an arbitrary μ-colored link L, and let $-L$ denote the μ-colored link L with reverse orientation but same coloring as L. Let $L\#_1 - L$ denote a connected sum of L and $-L$ along two components of color 1. Unlike for knots, the isotopy type of the connected sum of (colored) links is not well-defined. However, any two such connected sums have the same signature and nullity, as these invariants behave additively under this ill-defined operation [4, Proposition 2.12]. Since the signature and nullity are unchanged by reversing the orientation (see [4, Corollary 2.9]), the relations

$$(6) \qquad\qquad \sigma_{L\#_1 - L}(\omega) = 2\sigma_L(\omega) \qquad and \qquad \eta_{L\#_1 - L}(\omega) = 2\eta_L(\omega)$$

hold for any such connected sum.

The idea of the proof is to use a diagram D for L to construct a C-complex S for $L\#_1 - L$ whose first homology has a basis given by classes of loops corresponding to the regions and the monochromatric crossings of D – minus the two regions near the connected sum. By taking generalized Seifert matrices with respect to this basis, we show that the matrix $H(\omega)$ used in Definition 3 is congruent to a block-diagonal matrix of the form $\widetilde{\tau}_D(\omega) \oplus Z$ with $\sigma(Z) = -w_{\text{m}}(D)$ and $\text{null}(Z) = 0$. Combined with Equation (6), this completes the proof.

We now give the details. To construct a C-complex for $L\#_1 - L$ from D, we use the following procedure (see Figure 5 for the construction near crossings, and Figure 6 for an example).

(i) At each crossing of D, draw a copy of the corresponding crossing for $-L$ "a bit above and behind" the crossing of D.

(ii) Connect the remaining strands of $-L$ to each other following along the edges of D, possibly creating an additional crossing along each edge (with $-L$ passing under L). This yields a diagram for the disjoint sum of L and $-L$.
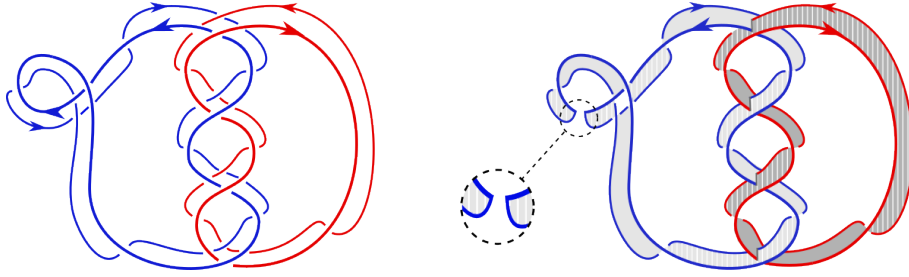
Figure 6. Left: the diagram for the disjoint sum of L and $-L$ obtained from D of Figure 1. Right: the corresponding C-complex S for $L\#_1 - L$.
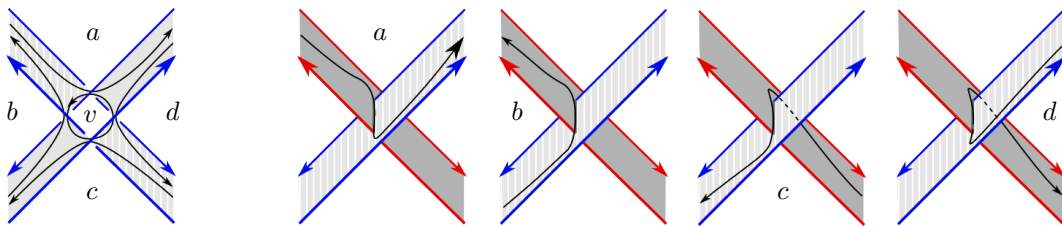

Figure 7. The five cycles of S near a monochromatic crossing (left, one image) and the four cycles of S near a bichromatic crossing (right, four images). The labels a, b, c, d, v for the regions are used for the local linking matrices in Figure 8.

(iii) Create a clasp intersection near each bichromatic crossing of D and apply the usual Seifert algorithm near each monochromatic crossing of D. This yields a C-complex for the disjoint sum of L and $-L$.

(iv) Finally, pick a point on a strand of color 1 in D and cut the corresponding surface at that place. The result is a C-complex S for (some version of) $L\#_1 - L$.

Note that S deformation retracts onto a graph defined as follows: take the 4-regular graph underlying the diagram D, add a loop at each vertex corresponding to a monochromatic vertex, and remove the edge along which the connected sum was performed. As a consequence, a natural basis for $H_1(S)$ is given by classes of cycles corresponding to the regions and monochromatic crossings of D – minus the two regions adjacent to where the connected sum happens. We use the convention that the cycles representing our basis of $H_1(S)$ are oriented counterclockwise in the plane of D where S is drawn, and denote by the same letter a region or monochromatic crossing and its corresponding cycle of $H_1(S)$. Using this explicit basis, we now study the local contribution to generalized Seifert matrices near crossings of D.

As illustrated in Figure 7, there are five homology classes in $H_1(S)$ coming into play near a monochromatic crossing of D, and four near a bichromatic crossing. The local contribution to the matrices A^ε near different types of crossings of D are given in Figure 8. By the symmetry (1) and the fact that the C-complex near a negative bichromatic crossing is the mirror image of the C-complex near a positive bichromatic crossing, the local linking of all possible cases can be computed from those in Figure 8.

For the remainder of the proof, we adopt the following convention: we say a crossing v has color $(j, k) = (j_v, k_v)$ if its incoming left strand is color j, and incoming right strand has color k, as in Figure 2. If $j = k$, we may simply say it has color j.

We now write $H(\omega)$ as a sum over crossings of D. For a crossing v, let A_v^ε denote the square matrix (of size equal to the first Betti number of S) given by the contribution to A^ε from the linking near v; in other words, A_v^ε is zero everywhere except in the 5×5 or 4×4 minor corresponding to the homology classes coming into play near v, where its values are given by the local contributions

lk	a	b	c	d	v
$a^{(1)}$	$-1/2$	$-1/2$	0	0	1
$b^{(1)}$	0	0	0	0	0
$c^{(1)}$	0	$-1/2$	$-1/2$	0	1
$d^{(1)}$	$1/2$	0	$1/2$	0	-1
$v^{(1)}$	0	1	0	0	-1

lk	a	b	c	d	v
$a^{(1)}$	$1/2$	$-1/2$	0	0	0
$b^{(1)}$	0	0	0	0	0
$c^{(1)}$	0	$-1/2$	$1/2$	0	0
$d^{(1)}$	$1/2$	0	$1/2$	0	-1
$v^{(1)}$	-1	1	-1	0	1

(a) Contributions near a positive (left) and negative (right) monochromatic crossing of color j, where $x^{(1)}$ stands for x^ε with $\varepsilon_j = 1$.

lk	a	b	c	d
$a^{(1,1)}$	$-1/2$	$-1/2$	0	0
$b^{(1,1)}$	0	0	0	0
$c^{(1,1)}$	0	$1/2$	$-1/2$	0
$d^{(1,1)}$	$1/2$	-1	$1/2$	0

lk	a	b	c	d
$a^{(1,-1)}$	0	0	0	0
$b^{(1,-1)}$	$-1/2$	$1/2$	0	0
$c^{(1,-1)}$	1	$-1/2$	0	$-1/2$
$d^{(1,-1)}$	$-1/2$	0	0	$1/2$

(b) Contributions near a positive bichromatic crossing of colors (j,k), with x^ε denoted by $x^{(\varepsilon_j,\varepsilon_k)}$.

Figure 8. The local contributions to linking numbers near crossings of D, where the curves are labeled as in Figure 7.

to linking numbers given by the matrices from Figure 8. We have:

$$(7) \qquad H(\omega) = \sum_{\varepsilon \in \{\pm 1\}^\mu} \prod_{i=1}^\mu (1 - \overline{\omega}_i^{\varepsilon_i}) A^\varepsilon = \sum_v \sum_{\varepsilon \in \{\pm 1\}^\mu} \prod_{i=1}^\mu (1 - \overline{\omega}_i^{\varepsilon_i}) A_v^\varepsilon =: \sum_v H_v \,,$$

where $H_v = \sum_\varepsilon \prod_i (1 - \overline{\omega}_i^{\varepsilon_i}) A_v^\varepsilon$, and the sums indexed by v always refer to the sum over all crossing of D. Note that A_v^ε is entirely specified by ε_{j_v} and ε_{k_v}. Thus we use $A_v^{(\alpha,\beta)}$ to denote A_v^ε for any ε with $\varepsilon_{j_v} = \alpha$ and $\varepsilon_{k_v} = \beta$. If v is monochromatic with $\varepsilon_{j_v} = \varepsilon_{k_v} = \alpha$, we simply write $A_v^{(\alpha)}$.

When v is monochromatic of color $j = j_v$, the matrix H_v can be rewritten as

$$H_v = \sum_{(\varepsilon_1,\cdots,\widehat{\varepsilon_j},\cdots,\varepsilon_\mu) \in \{\pm 1\}^{\mu-1}} \prod_{\substack{i=1 \\ i \neq j}}^\mu (1 - \overline{\omega}_i^{\varepsilon_i}) \left((1 - \overline{\omega}_j) A_v^{(1)} + (1 - \omega_j) A_v^{(-1)} \right)$$

$$= \left(\prod_{\substack{i=1 \\ i \neq j}}^\mu (1 - \overline{\omega}_i)(1 - \omega_i) \right) \left((1 - \overline{\omega}_j) A_v^{(1)} + (1 - \omega_j) A_v^{(-1)} \right) \,,$$

where the second equality uses the relation $(1 - \overline{\omega}_i) + (1 - \omega_i) = (1 - \overline{\omega}_i)(1 - \omega_i)$. Hence, using the notation $s_i := (1 - \omega_i)$, the matrix H_v for a monochromatic crossing v of color j is given by

$$(8) \qquad H_v = \prod_{\substack{i=1 \\ i \neq j}}^\mu |s_i|^2 \left(\overline{s}_j A_v^{(1)} + s_j A_v^{(-1)} \right) =: \prod_{\substack{i=1 \\ i \neq j}}^\mu |s_i|^2 A_v \,,$$

while for a bichromatic crossing of colors (j,k), it is given by

$$(9) \quad H_v = \prod_{\substack{i=1 \\ i \neq j,k}}^\mu |s_i|^2 \left(\overline{s}_j \overline{s}_k A_v^{(1,1)} + \overline{s}_j s_k A_v^{(1,-1)} + s_j \overline{s}_k A_v^{(-1,1)} + s_j s_k A_v^{(-1,-1)} \right) =: \prod_{\substack{i=1 \\ i \neq j,k}}^\mu |s_i|^2 A_v \,,$$

where we use the notation

$$A_v := \begin{cases} \overline{s}_j A_v^{(1)} + s_j A_v^{(-1)} & \text{if } v \text{ is monochromatic of color } j; \\ \overline{s}_j \overline{s}_k A_v^{(1,1)} + \overline{s}_j s_k A_v^{(1,-1)} + s_j \overline{s}_k A_v^{(-1,1)} + s_j s_k A_v^{(-1,-1)} & \text{if } v \text{ is bichromatic of colors } j,k \,. \end{cases}$$

Plugging Equations (8) and (9) into Equation (7), we get

$$(10) \qquad H(\omega) = \sum_v H_v = \sum_v \prod_{i \neq j_v, k_v}^\mu |s_i|^2 A_v \,.$$

$$\begin{bmatrix} & a & b & c & d \\ \hline a & \frac{\omega_j+\overline{\omega}_j}{2} & -\frac{1+\overline{\omega}_j}{2} & 1 & -\frac{1+\omega_j}{2} \\ b & -\frac{1+\omega_j}{2} & 1 & -\frac{1+\omega_j}{2} & \omega_j \\ c & 1 & -\frac{1+\omega_j}{2} & \frac{\omega_j+\overline{\omega}_j}{2} & -\frac{1+\omega_j}{2} \\ d & -\frac{1+\overline{\omega}_j}{2} & \overline{\omega}_j & -\frac{1+\overline{\omega}_j}{2} & 1 \end{bmatrix}$$

(a) v positive monochromatic crossing of color j

$$\begin{bmatrix} & a & b & c & d \\ \hline a & -\frac{\overline{s_j s_k}+s_j s_k}{2} & \frac{\overline{s_k}(\omega_j-\overline{\omega}_j)}{2} & \overline{s_k}s_j & \frac{s_j(\overline{\omega}_k-\omega_k)}{2} \\ b & \frac{(\overline{\omega}_j-\omega_j)s_k}{2} & \frac{s_j\overline{s_k}+\overline{s_j}s_k}{2} & \frac{s_j(\overline{\omega}_k-\omega_k)}{2} & -s_j s_k \\ c & \overline{s_j}s_k & \frac{(\omega_k-\overline{\omega}_k)\overline{s_j}}{2} & -\frac{\overline{s_j s_k}+s_j s_k}{2} & \frac{(\overline{\omega}_j-\omega_j)s_k}{2} \\ d & \frac{\overline{s_j}(\omega_k-\overline{\omega}_k)}{2} & -\overline{s_j s_k} & \frac{(\omega_j-\overline{\omega}_j)\overline{s_k}}{2} & \frac{s_j\overline{s_k}+\overline{s_j}s_k}{2} \end{bmatrix}$$

(b) v positive bichromatic crossing of colors (j,k)

Figure 9. The local contribution to $X - YZ^{-1}Y^*$ for a positive monochromatic and bichromatic crossing v. If v is a negative crossing, the matrix is the negative of the corresponding matrix for a positive crossing.

Writing the Hermitian matrix $H(\omega)$ as a block matrix of the form

$$H(\omega) = \begin{array}{c|c|c} & \text{regions} & \text{mono crossings} \\ \hline \text{regions} & X & Y \\ \hline \text{mono crossings} & Y^* & Z \end{array}$$

we see that Z is a diagonal matrix with coefficient corresponding to the monochromatic crossing v given by $-\prod_i |s_i|^2 \operatorname{sgn}(v)$. In particular, Z is invertible (and hence has nullity $\operatorname{null}(Z) = 0$), while its signature is equal to $\sigma(Z) = -w_{\mathrm{m}}(D)$. Furthermore, $H(\omega)$ is congruent to the block diagonal matrix

$$(11) \qquad\qquad MH(\omega)M^* = \begin{pmatrix} X - YZ^{-1}Y^* & 0 \\ 0 & Z^{-1} \end{pmatrix}$$

via $M = \begin{pmatrix} I & -YZ^{-1} \\ 0 & Z^{-1} \end{pmatrix}$. Since $\sigma(Z^{-1}) = \sigma(Z) = -w_{\mathrm{m}}(D)$ and $\operatorname{null}(Z^{-1}) = 0$, it remains to show that the matrix $X - YZ^{-1}Y^*$ coincides with $\widetilde{\tau}_D(\omega)$ up to transformations that do not affect the signature and nullity.

To determine $X - YZ^{-1}Y^*$, let us fix two regions a and b. Note that

$$(YZ^{-1}Y^*)_{a,b} = \sum_v Y_{a,v} Z_{v,v}^{-1} \overline{Y_{b,v}}\,,$$

and that it is only possible for both $Y_{a,v}$ and $Y_{b,v}$ to be nonzero if the regions a and b are both adjacent to the crossing v. Therefore, the matrix $YZ^{-1}Y^*$ is a sum over crossings, where the contribution at each crossing is a matrix that is zero everywhere except in the 4×4 minor corresponding to the four adjacent regions of the crossing. The same is then true for $X - YZ^{-1}Y^*$. For a bichromatic crossing v, there is no column of Y that corresponds to v, so this 4×4 minor is nothing but the nonzero 4×4 minor of A_v. For a monochromatic crossing v, the 4×4 minor of $X - YZ^{-1}Y^*$ is obtained by performing the corresponding matrix operations to the 5×5 minor of A_v. The computation is similar to the single variable case, see [16] for more details. The explicit values for these local contributions to $X - YZ^{-1}Y^*$ are given in Figure 9. Note that we only provide these minors in the case of positive crossings, as the minor for a negative crossing is obtained by the negative of the minor for a positive one.

We need to perform one last change of basis, which we now describe. If the sublink L_i winds around the region a a total of α_i times, then multiply the basis element corresponding to a with $\prod_{i=1}^{\mu}(-\omega_i^{-1/2})^{\alpha_i}$. This change of basis alters the matrices in Figure 9 in the following way. If v is a monochromatic crossing of color j, then L_j winds around b once more that it does around a and c,

and it winds around d once fewer. Thus the rows corresponding to b and d are multiplied by $-\omega_i^{-1/2}$ and $-\omega_i^{1/2}$ respectively, and the columns are multiplied by the conjugates $-\omega_j^{1/2}$ and $-\omega_j^{-1/2}$. If v is bichromatic crossing of colors (j,k), then L_j winds once more around a and b than it does around c and d, and L_k winds once more around b and c than it does around a and d. Thus we multiply the row for a by $-\omega_j^{-1/2}$, the row for c by $-\omega_k^{-1/2}$, and the row for b by $\omega_j^{-1/2}\omega_k^{-1/2}$, and we multiply the corresponding columns by the conjugates.

Remarkably, the sum of local contributions to $X - YZ^{-1}Y^*$ from Figure 9 can now be written in terms of the single matrix $\tau_v(\omega)$, the evaluation of $\tau_v(x)$ at $x_j = \mathrm{Re}(\omega_j^{1/2})$ and $x_{jk} = \mathrm{Re}(\omega_j^{1/2}\omega_k^{1/2})$, in both the monochromatic and bichromatic cases. Indeed, for a monochromatic crossing v, it coincides with $\mathrm{sgn}(v)\tau_v(\omega)$, while for a bichromatic crossing, it yields

$$4\,\mathrm{sgn}(v)\sqrt{1 - x_{j_v}^2}\,\sqrt{1 - x_{k_v}^2}\,\tau_v(\omega)\,.$$

The result of the matrix $X - YZ^{-1}Y^*$ after this change of basis thus gives

$$\sum_{v \text{ mono}}\prod_{i \neq j_v}|s_i|^2\,\mathrm{sgn}(v)\tau_v(\omega) + \sum_{v \text{ bi}}\prod_{i \neq j_v, k_v}|s_i|^2\,4\,\mathrm{sgn}(v)\sqrt{1 - x_{j_v}^2}\sqrt{1 - x_{k_v}^2}\,\tau_v(\omega)$$

$$= \frac{1}{4}\prod_{i=1}^{\mu}|s_i|^2\sum_v\frac{\mathrm{sgn}(v)}{\sqrt{1 - x_{j_v}^2}\sqrt{1 - x_{k_v}^2}}\tau_v(\omega) = \frac{1}{4}\prod_{i=1}^{\mu}|s_i|^2\,\widetilde{\tau}_D(\omega)\,.$$

The positive constant $\frac{1}{4}\prod_{i=1}^{\mu}|s_i|^2$ affects neither the signature nor the nullity, so we have

$$2\sigma_L(\omega) = \sigma(X - YZ^{-1}Y^*) - \sigma(Z^{-1}) = \sigma(\widetilde{\tau}_D(\omega)) - w_{\mathrm{m}}(D)\,,$$

$$2\eta_L(\omega) = \eta(X - YZ^{-1}Y^* - \eta(Z^{-1}) = \eta(\widetilde{\tau}_D(\omega))\,,$$

and the proof is complete.

$\square$

3.2. **The Conway function.** In this section we discuss how to compute the Conway function of a colored link from the matrix $\widetilde{\tau}_D$ and prove the second point of Theorem 1, which we now restate for convenience.

Proposition 2. *If D is a connected μ-colored diagram for a μ-colored link L, we have the equality*

$$\nabla_L^2(t_1, \ldots, t_\mu) = \frac{1}{(t_1 - t_1^{-1})^2}\prod_v(-\mathrm{sgn}(v))\frac{t_j - t_j^{-1}}{2}\frac{t_k - t_k^{-1}}{2} \cdot \det(\widetilde{\tau}_D(t^2))\,,$$

where the product is over all crossings of D, the indices j, k are the (possibly identical) colors of the two strands crossing at v, and $\tau_D(t^2)$ stands for the evaluation of $\tau_D(x)$ at

$$x_j = \frac{t_j + t_j^{-1}}{2}\,, \quad x_{jk} = \frac{t_jt_k + t_j^{-1}t_k^{-1}}{2}\,.$$

First recall that, for any $\omega = (\omega_1, \ldots, \omega_\mu) \in (S^1 \setminus \{1\})^\mu$, $\tau_D(\omega)$ stands for the evaluation of τ_D at

$$x_j = \mathrm{Re}(\omega_j^{1/2}) = \frac{\omega_j^{1/2} + \omega_j^{-1/2}}{2}\,, \quad x_{jk} = \mathrm{Re}(\omega_j^{1/2}\omega_k^{1/2}) = \frac{\omega_j^{1/2}\omega_k^{1/2} + \omega_j^{-1/2}\omega_k^{-1/2}}{2}\,.$$

Hence, $\tau_D(t^2)$ can be understood as $\tau_D(\omega)$ evaluated at $\omega_i = t_i^2$ (or equivalently, at $\omega_i = t_i^{-2}$), justifying the notation. We frequently make use of this change of variables in what follows.

Our starting point is Equation (5), which expresses the Conway function ∇_L in terms of the matrix A_S associated to a connected C-complex S for L, recall (4). First note that if $H(\omega)$ denotes the matrix defined in (2), which is used for computing the multivariable signature, we have

(12)
$$H(t^{-2}) := H(t_1^{-2}, \cdots, t_\mu^{-2}) = \sum_{\varepsilon \in \{\pm 1\}^\mu}\prod_{i=1}^{\mu}(1 - t_i^{2\varepsilon_i})A^\varepsilon = \sum_{\varepsilon \in \{\pm 1\}^\mu}\prod_{i=1}^{\mu}\varepsilon_i t_i^{\varepsilon_i}\varepsilon_i(t_i^{-\varepsilon_i} - t_i^{\varepsilon_i})A^\varepsilon$$

$$= \prod_{i=1}^{\mu}-(t_i - t_i^{-1})\sum_{\varepsilon \in \{\pm 1\}^\mu}\prod_{i=1}^{\mu}\varepsilon_i t_i^{\varepsilon_i}A^\varepsilon = (-1)^\mu\prod_{i=1}^{\mu}(t_i - t_i^{-1})A_S\,.$$

Hence, the Conway function can in fact be computed from the matrix H.

Now, in order to prove Proposition 2, we adopt the same strategy as in the computation of the signature: starting from a *connected* diagram D, we use the C-complex S for $L\#_1 - L$ constructed in the previous section to compute the Conway function of $L\#_1 - L$, and conclude by applying well-known formulas relating the Conway function of a connected sum to the Conway functions of the summands. Note that, by construction, requiring D to be connected precisely means that S is connected.

So, let $D = D_1 \cup \cdots \cup D_\mu$ be a connected, μ-colored diagram of a μ-colored link L and let $S = S_1 \cup \cdots \cup S_\mu$ be the C-complex for $L\#_1 - L$ defined in the proof of Proposition 1. As before, the notation $\#_1$ stands for the connected sum performed along a component of color 1. Let n_{m} and n_{b} denote the number of monochromatic and bichromatic crossings of D, respectively, and $n = n_{\mathrm{m}} + n_{\mathrm{b}}$ the total number of crossings. Similarly, let $n_{\mathrm{m},i}$ (resp. $n_{\mathrm{b},i}$) denote the number of monochromatic (resp. bichromatic) crossings of D *without* the color i.

Lemma 1. *With the notations above, the C-complex S satisfies:*

(i) $\operatorname{sgn} S = \prod\limits_{v \ bichr.} \operatorname{sgn} v$, *where the product is taken over all bichromatic crossings.*

(ii) *Its first Betti number is equal to $b_1(S) = n + n_{\mathrm{m}}$ and is even.*

(iii) $\chi(S \setminus S_1) = -n_{\mathrm{b},1} - 2n_{\mathrm{m},1}$ *and* $\chi(S \setminus S_i) = 1 - n_{\mathrm{b},i} - 2n_{\mathrm{m},i}$ *for all $i \neq 1$.*

Proof. The first equality is clear by construction, since S has one clasp for each bichromatic crossing of D, and the sign of the clasp is equal to the sign of the corresponding crossing. To check the second point let r denote the number of regions of D. By construction, we have $b_1(S) = (r-2)+n_{\mathrm{m}}$, while an Euler characteristic computations yields the equality $r - 2 = n$. Since n_{b} is always even, it follows that $b_1(S) = n + n_{\mathrm{m}} = n_{\mathrm{b}} + 2n_{\mathrm{m}}$ is also even. As for the third point, one just needs to notice that $S \setminus S_i$ deformation retracts onto a graph Γ_i constructed from (the planar projection of) the diagram $D \setminus D_i$ by adding one loop to each monochromatic crossing and, if $i \neq 1$, by removing one edge of color 1 (which corresponds to performing the connected sum). The number of vertices of Γ_i minus the number of its edges yields the result. $\square$

To shorten our formulas, we denote $s_{\mathrm{b}} := \prod\limits_{v \ bichr.} \operatorname{sgn} v$ and $s_{\mathrm{m}} := \prod\limits_{v \ mono.} \operatorname{sgn} v$, where the product is taken over all bichromatic (resp. monochromatic) crossings of D. We are now ready to prove Proposition 2.

Proof of Proposition 2. Recall that, by the proof of Proposition 1 (in particular (11)), there exists a matrix M such that
$$MH(\omega)M^* = \widetilde{\tau}'_D(\omega) \oplus Z^{-1},$$
where the matrix
$$Z = \prod_{i=1}^{\mu}(1 - \omega_i)(1 - \omega_i^{-1})\operatorname{diag}(-\operatorname{sgn}(v))$$
is indexed by the monochromatic crossings of D, and $\det(M) = \det(M^*) = \det(Z^{-1})$. Therefore,
$$\det(H(\omega)) = \det(Z)\det(\widetilde{\tau}'_D(\omega)) = (-1)^{n_{\mathrm{m}}} s_{\mathrm{m}}\Big(\prod_{i=1}^{\mu}(1 - \omega_i)(1 - \omega_i^{-1})\Big)^{n_{\mathrm{m}}}\det(\widetilde{\tau}'_D(\omega)).$$

Evaluating H at $\omega_i = t_i^{-2}$, and using the equality $(1 - t_i^2)(1 - t_i^{-2}) = -(t_i - t_i^{-1})^2$, we obtain

(13) $$\det(H(t^{-2})) = (-1)^{n_{\mathrm{m}}+\mu n_{\mathrm{m}}} s_{\mathrm{m}} \prod_{i=1}^{\mu}(t_i - t_i^{-1})^{2n_{\mathrm{m}}}\det(\widetilde{\tau}'_D(t^2)).$$

Furthermore, since $\widetilde{\tau}'_D$ is a matrix of size n and, by definition,
$$\widetilde{\tau}'_D(\omega) = \frac{1}{4}\prod_{i=1}^{\mu}(1 - \omega_i)(1 - \omega_i^{-1})\widetilde{\tau}_D(\omega),$$
we have

(14) $$\det(\widetilde{\tau}'_D(t^2)) = (-1)^{\mu n}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{2n}\det(\tfrac{1}{4}\widetilde{\tau}_D(t^2)).$$

Putting everything together, and writing t for $(t_1, \ldots, t_\mu)$, we obtain

$$
\nabla_L^2(t) = \nabla_L(t)\nabla_{-L}(t) = \frac{1}{t_1 - t_1^{-1}}\nabla_{L\#_1 - L}(t)
$$

$$
\overset{(5)}{=} \frac{1}{t_1 - t_1^{-1}}(-1)^{b_1(S)}\operatorname{sgn}(S)\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{\chi(S\setminus S_i)-1}\det(A_S)
$$

$$
\overset{(12)}{=} \frac{s_{\mathrm b}}{t_1 - t_1^{-1}}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{\chi(S\setminus S_i)-1}(-1)^{\mu b_1(S)}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{-b_1(S)}\det(H(t^{-2}))
$$

$$
\overset{(13)}{=} \frac{s_{\mathrm b}}{t_1 - t_1^{-1}}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{\chi(S\setminus S_i)-1-b_1(S)}(-1)^{n_{\mathrm m}+\mu n_{\mathrm m}}s_{\mathrm m}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{2n_{\mathrm m}}\det(\widetilde{\tau}_D'(t^2))
$$

$$
\overset{(14)}{=} \frac{(-1)^{n_{\mathrm m}+\mu n_{\mathrm m}}s_{\mathrm b}s_{\mathrm m}}{t_1 - t_1^{-1}}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{\chi(S\setminus S_i)-1-b_1(S)+2n_{\mathrm m}}(-1)^{\mu n}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{2n}\det(\tfrac{1}{4}\widetilde{\tau}_D(t^2))
$$

$$
= \frac{(-1)^{n_{\mathrm m}}}{(t_1 - t_1^{-1})^2}\frac{s_{\mathrm b}s_{\mathrm m}}{4^n}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{-n_{\mathrm{b},i}-2n_{\mathrm m,i}+n+n_{\mathrm m}}\det(\widetilde{\tau}_D(t^2))\,,
$$

where in the first line we used Corollary 2 and Proposition 5 of [2]. The following equalities derive, as indicated, from Equations (5), (12), (13) and (14), together with the first point of Lemma 1 in the third line (with (12)), the second point in the fourth line (with (13)), and the second and third points in the last line.

To conclude, we note that the exponent $-n_{\mathrm{b},i} - 2n_{\mathrm m,i} + n + n_{\mathrm m}$ appearing in the last line is equal to the number of bichromatic crossings involving a strand of color i plus twice the number of monochromatic crossings of color i. Therefore,

$$
\frac{s_{\mathrm b}s_{\mathrm m}}{4^n}\prod_{i=1}^{\mu}(t_i - t_i^{-1})^{-n_{\mathrm{b},i}-2n_{\mathrm m,i}+n+n_{\mathrm m}} = \prod_v \operatorname{sgn}(v)\frac{t_j - t_j^{-1}}{2}\frac{t_k - t_k^{-1}}{2}\,,
$$

where the product on the right-hand side is over all crossings of D and the indices j, k are the two (possibly identical) colors of strands crossing at v. The proposition now follows from observing that $(-1)^{n_{\mathrm m}} = (-1)^n$ since $n = n_{\mathrm m} + n_{\mathrm b}$ and $n_{\mathrm b}$ is even. $\qquad\square$

3.3. **The multivariate Kauffman model.** Having finished the proof of Theorem 1, we now turn our attention to Corollary 1.

Starting from a *connected* diagram D of a μ-colored link, let K_D (or simply K) be the matrix defined by the labels in Figure 3, and $\widetilde{K}$ the square matrix obtained from K by removing two columns corresponding to two adjacent regions of D separated by a strand of color 1. Corollary 1 is a direct consequence of Proposition 2 together with the following lemma.

Lemma 2. *Let $S = (S_{v,v})$ be the diagonal matrix indexed by the crossings of D with coefficients*

$$
S_{v,v} = \frac{-4\operatorname{sgn}(v)}{(t_j - t_j^{-1})(t_k - t_k^{-1})}\,,
$$

where j and k are the colors of the two strands meeting at v. Then, we have

$$
\tau_D(t^2) = K^{\mathrm T}SK\,.
$$

We start by proving Corollary 1, before addressing the proof of Lemma 2.

Proof of Corollary 1. By Lemma 2, we have $\widetilde{\tau}_D(t^2) = \widetilde{K}^{\mathrm T}S\widetilde{K}$, yielding the equality

$$
(\det \widetilde{K})^2 = \det(S)^{-1}\det(\widetilde{\tau}_D(t^2))\,.
$$

Since moreover

$$
\det(S)^{-1} = \prod_v(-\operatorname{sgn}(v))\frac{t_j - t_j^{-1}}{2}\frac{t_k - t_k^{-1}}{2}\,,
$$

Proposition 2 implies $(\det \widetilde{K})^2 = (t_1 - t_1^{-1})^2\nabla_L^2(t_1, \ldots, t_\mu)$, and Corollary 1 follows. $\qquad\square$

Proof of Lemma 2. Recall that $K = (K_{v,a})$ is a matrix with rows indexed by the crossings of D and columns indexed by the regions of D. Let us fix two regions a and b of D. By definition, the corresponding coefficient of $K^{\mathrm{T}}SK$ is

$$(15) \qquad (K^{\mathrm{T}}SK)_{a,b} = \sum_v \frac{-4\,\mathrm{sgn}(v)}{(t_j - t_j^{-1})(t_k - t_k^{-1})} K_{v,a} K_{v,b}\,,$$

while the corresponding coefficient of $\tau_D(t^2)$ is

$$(16) \qquad (\tau_D(t^2))_{a,b} = \sum_v \frac{-4\,\mathrm{sgn}(v)}{(t_j - t_j^{-1})(t_k - t_k^{-1})} (\tau_v(t^2))_{a,b}\,.$$

In both cases, the sum is over all crossings of D (and j, k denote the colors of the strands crossing at v), but the only non-zero contributions come from the crossings adjacent to both a and b.

Comparing the labels of Figure 2 evaluated at $x_j = \frac{t_j + t_j^{-1}}{2}$ and $x_{jk} = \frac{t_j t_k + t_j^{-1} t_k^{-1}}{2}$ with the labels of Figure 3, one notices an interesting relation. To state it precisely, let $\mathbb{Q}(t)$ denote the field of fractions of $\mathbb{Z}[t_1^{\pm 1}, \cdots, t_\mu^{\pm 1}]$, and let $\varphi : \mathbb{Q}(t) \to \mathbb{Q}(t)$ be the involution induced by $t_i \mapsto t_i^{-1}$ for all i. We claim that the following equality holds:

$$(17) \qquad (\tau_v(t^2))_{a,b} = \frac{K_{v,a} K_{v,b} + \varphi(K_{v,a} K_{v,b})}{2}\,.$$

The proof of this claim is divided into three cases, depending on the relative positions of the regions a and b. Let us first assume that a and b are two different regions of the same checkerboard color. In such a case, we have $K_{v,a} K_{v,b} = 1 = (\tau_v(t^2))_{a,b}$ for each crossing v incident to both a and b, so (17) holds. Let us now assume that a and b are regions with different checkerboard colors, meeting at a crossing v with strands of colors j and k. If a and b are adjacent to the strand of color j (resp. k), we get $K_{v,a} K_{v,b} = t_k^{\pm 1}$ (resp. $t_j^{\pm 1}$). Since the coefficient of $\tau_v(t^2)$ is $x_k = \frac{t_k + t_k^{-1}}{2}$ (resp. $x_j = \frac{t_j + t_j^{-1}}{2}$), Equation (17) holds in this case as well. Finally, let us assume that $a = b$. For a crossing v incident to a, we get either $K_{v,a}^2 = (t_j t_k)^{\pm 1}$ or $K_{v,a}^2 = (t_j^{-1} t_k)^{\pm 1}$, depending to the position of a around v. Similarly, the corresponding coefficient of $\tau_v(t^2)$ is either $x_{jk} = \frac{t_j t_k + t_j^{-1} t_k^{-1}}{2}$ or $2x_j x_k - x_{jk} = \frac{t_j t_k^{-1} + t_j^{-1} t_k}{2}$, respectively. This concludes the proof of (17).

Equations (15), (16) and (17) immediately imply the equality

$$\tau_D(t^2) = \frac{K^{\mathrm{T}}SK + \varphi(K^{\mathrm{T}}SK)}{2}\,,$$

where φ is applied to matrices coefficientwise. To conclude the proof of Lemma 2, it remains to check that $\varphi(K^{\mathrm{T}}SK) = K^{\mathrm{T}}SK$. This fact being suprisingly technical, we make it the object of a final separate lemma. $\qquad\square$

Lemma 3. *Let $\mathbb{Q}(t)$ denote the field of fractions of $\mathbb{Z}[t_1^{\pm 1}, \cdots, t_\mu^{\pm 1}]$, and let $\varphi : \mathbb{Q}(t) \to \mathbb{Q}(t)$ be the involution induced by $t_i \mapsto t_i^{-1}$ for all i. Then, we have the equality $\varphi(K^{\mathrm{T}}SK) = K^{\mathrm{T}}SK$.*

Proof. We have to show that, for any two regions a and b of D, the coefficient

$$(K^{\mathrm{T}}SK)_{a,b} = \sum_v \frac{-4\,\mathrm{sgn}(v)}{(t_j - t_j^{-1})(t_k - t_k^{-1})} K_{v,a} K_{v,b}$$

is invariant under φ, where the sum is taken over all crossings adjacent to both a and b. We will consider several cases, according to the relative positions of a and b with respect to the crossings.

First of all, if a and b are two regions of the same checkerboard color, each common crossing v contributes a term $\frac{-4\,\mathrm{sgn}(v)}{(t_j - t_j^{-1})(t_k - t_k^{-1})}$ to the coefficient $(K^{\mathrm{T}}SK)_{a,b}$. Since all these terms are invariant under φ, this case is checked.

Next, suppose that a and b have different checkerboard colors. Each edge of D adjacent to both regions gives two contributions to the sum, one for each crossing adjacent to the edge. So, let us consider a common edge with endpoints v and v', and suppose without loss of generality that the colors and orientations of the strands are as in Figure 10, left. The two contributions sum up to

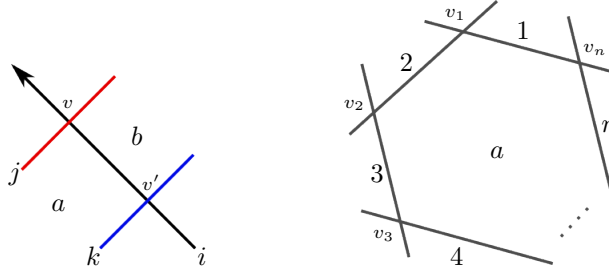$$\frac{-4s\,t_j^{-s}}{(t_i - t_i^{-1})(t_j - t_j^{-1})} + \frac{-4s'\,t_k^{s'}}{(t_i - t_i^{-1})(t_k - t_k^{-1})}\,,$$

Figure 10. The conventions in the proof of Lemma 3.

where $s = \mathrm{sgn}(v)$ and $s' = \mathrm{sgn}(v')$. Proving that the term displayed above is invariant under φ is clearly equivalent to showing that

$$G := s t_j^{-s}(t_k - t_k^{-1}) + s' t_k^{s'}(t_j - t_j^{-1})$$

satisfies $\varphi(G) = -G$. Expanding the products and denoting by χ the characteristic function, one checks that G is equal to

$$(t_j t_k - t_j^{-1} t_k^{-1})(\chi_{s'=1} - \chi_{s=-1}) + (t_j t_k^{-1} - t_j^{-1} t_k)(\chi_{s=-1} - \chi_{s'=-1}),$$

which is clearly antisymmetric, thus finishing this case.

Finally, let us consider the diagonal coefficient corresponding to a region a. Suppose that, when moving around the boundary of a counterclockwise, one encounters n crossings $v_1, \ldots, v_n$ of respective signs $s_1, \ldots, s_n$. (It can happen that a abuts the same crossing from two sides, but since the corresponding labels are added, our computations remain valid in this case.) Let us also number the edges of the boundary from 1 to n as in Figure 10, right. To each edge, we assign a sign $\varepsilon_i \in \{\pm 1\}$, where $\varepsilon_i = 1$ if the edge i is oriented coherently with the counterclockwise orientation of the boundary of a, and $\varepsilon_i = -1$ otherwise. Without loss of generality, we can assume that all the edges have different colors, that we also denote by $1, \ldots, n$; in the general case, if two colors coincide, one simply needs to identify the corresponding variables in the following computations, a transformation which does not affect the symmetry.

With these notations and the help of Figure 3, one computes

$$(K^{\mathrm{T}} S K)_{a,a} = \sum_{i=1}^{n} \frac{-4 s_i (t_i^{\varepsilon_{i+1}} t_{i+1}^{-\varepsilon_i})^{s_i}}{(t_i - t_i^{-1})(t_{i+1} - t_{i+1}^{-1})} .$$

As before, proving that this coefficient is invariant under φ is equivalent to showing that

$$G := \prod_{j=1}^{n} (t_j - t_j^{-1}) \sum_{i=1}^{n} \frac{s_i (t_i^{\varepsilon_{i+1}} t_{i+1}^{-\varepsilon_i})^{s_i}}{(t_i - t_i^{-1})(t_{i+1} - t_{i+1}^{-1})}$$

satisfies $\varphi(G) = (-1)^n G$. Expanding as a sum of monomials, we obtain

$$G = \sum_{i=1}^{n} \prod_{j \neq i, i+1} (t_j - t_j^{-1}) s_i (t_i^{\varepsilon_{i+1}} t_{i+1}^{-\varepsilon_i})^{s_i} = \sum_{i=1}^{n} \sum_{\alpha \in \{\pm 1\}^{n-2}} \prod_{j \neq i, i+1} \alpha_j t_j^{\alpha_j} s_i (t_i^{\varepsilon_{i+1}} t_{i+1}^{-\varepsilon_i})^{s_i}$$

$$= \sum_{\beta \in \{\pm 1\}^n} c_\beta t_1^{\beta_1} \cdots t_n^{\beta_n}$$

for some coefficient c_β. To compute these coefficients explicitly, let us define for each $\beta \in \{\pm 1\}^n$ the (possibly empty) set of indices $I_\beta = \{i \in \{1, \ldots, n\} \mid \beta_i = \varepsilon_{i+1} s_i, \beta_{i+1} = -\varepsilon_i s_i\}$. We then have

$$c_\beta = \sum_{i \in I_\beta} \beta_1 \ldots \beta_{i-1} s_i \beta_{i+2} \ldots \beta_n = \sum_{i \in I_\beta} s_i \beta_i \beta_{i+1} (\beta_1 \ldots \beta_n) = \beta_1 \ldots \beta_n \sum_{i \in I_\beta} s_i \beta_i \beta_{i+1} = \beta_1 \ldots \beta_n d_\beta ,$$

with $d_\beta = \sum_{i \in I_\beta} s_i \beta_i \beta_{i+1}$. The desired equality $\varphi(G) = (-1)^n G$ is equivalent to $c_{-\beta} = (-1)^n c_\beta$ for all $\beta \in \{\pm 1\}^n$, which in turns is equivalent to $d_{-\beta} = d_\beta$.

Therefore, we are left with the proof of the equality $d_{-\beta} = d_\beta$ for all $\beta \in \{\pm 1\}^n$. Given any such β and any index $i \in \{1, \ldots, n\}$, define $\tilde\beta$ by $\tilde\beta_i = -\beta_i$ and $\tilde\beta_j = \beta_j$ for $j \neq i$. A straightforward

but slightly cumbersome computation yields

$$
d_\beta - d_{\tilde\beta} = \begin{cases} -\varepsilon_i \beta_i, & \text{if } (\beta_{i-1}, \beta_{i+1}) = (-\varepsilon_i s_{i-1}, -\varepsilon_i s_i) \\ \varepsilon_i \beta_i, & \text{if } (\beta_{i-1}, \beta_{i+1}) = (\varepsilon_i s_{i-1}, \varepsilon_i s_i) \\ 0, & \text{otherwise.} \end{cases}
$$

This expression is invariant if we replace β by $-\beta$. It thus follows that, for any two $\beta, \beta' \in \{\pm 1\}^n$, we have $d_\beta - d_{\beta'} = d_{-\beta} - d_{-\beta'}$. To prove the equality $d_{-\beta} = d_\beta$ for all β, we therefore only need to check it for a single β. Taking $\beta = (1, \ldots, 1)$, we get

$$
d_\beta - d_{-\beta} = \sum_{i:(\varepsilon_i, \varepsilon_{i+1})=(-s_i, s_i)} s_i - \sum_{i:(\varepsilon_i, \varepsilon_{i+1})=(s_i, -s_i)} s_i = \sum_{i:\varepsilon_i \neq \varepsilon_{i+1}} \varepsilon_{i+1} = 0 \,,
$$

since there is an even number of crossings at which ε_i changes sign, going from 1 to -1 in exactly half of the cases and form -1 to 1 in the others. $\qquad\square$

3.4. **The Alexander module.** We conclude this article with a slighlty informal discussion on yet another abelian link invariant, namely the Alexander module (recall its definition from Section 2.3).

The question we address is whether it is possible to obtain a presentation of (the square of) the Alexander module $\mathscr{A}_L$ of a μ-colored link L from the matrix $\widetilde{\tau}_D(x)$ associated to a colored diagram D for L. As we will see, the answer is positive in the case $\mu = 1$, but not in general.

First, recall that $\mathscr{A}_L$ does not admit a square presentation matrix over $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ if $\Delta_L \neq 0$ and $\mu \geq 4$ [9]. For this reason alone, there is no hope of answering the above question positively in general. However, the module $\mathscr{A}_L$ does admit a square presentation matrix over the localised ring

$$
\Lambda_S := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}, (t_1 - 1)^{-1}, \ldots, (t_\mu - 1)^{-1}] \,.
$$

Moreover, by Corollary 3.6 of [4], the generalized Seifert matrices can be used to compute such a square presentation matrix. Therefore, it is natural to hope that the strategy developed in this work could be applied to this invariant as well. However, since the change of variables $2x_j = t_j^{1/2} + t_j^{-1/2}$ makes use of fractional powers of the variables, we will need to work over the slightly larger ring

$$
\Lambda_S' := \mathbb{Z}[t_1^{\pm 1/2}, \ldots, t_\mu^{\pm 1/2}, (t_1 - 1)^{-1}, \ldots, (t_\mu - 1)^{-1}] \,.
$$

In the case $\mu = 1$, this program can be carried out, yielding the following result. Let D be a connected diagram for an oriented link L. As one easily checks, the coefficients of the matrix $\tau_D(x)$ are polynomials in $2x =: y$. Let $\mathscr{M}_D$ denote the $\mathbb{Z}[y]$-module presented by the matrix $\widetilde{\tau}_D(y)$. Then, we have an isomorphism of Λ_S'-modules

$$
\mathscr{M}_D \otimes_{\mathbb{Z}[y]} \Lambda_S' \simeq \mathscr{A}_L^{\oplus 2} \otimes_\Lambda \Lambda_S' \,,
$$

where $\Lambda_S' = \mathbb{Z}[t^{\pm 1/2}, (t-1)^{-1}]$ is a $\mathbb{Z}[y]$-module via the ring homomorphism $\mathbb{Z}[y] \to \Lambda_S'$ mapping y to $t^{1/2} + t^{-1/2}$, and a Λ-module via the natural inclusion $\Lambda \to \Lambda_S'$. Less formally, one can say that the matrix $\widetilde{\tau}_D(y)$ is a presentation matrix of $\mathscr{A}_L^{\oplus 2}$ over Λ_S' via the substitution $y = t^{1/2} + t^{-1/2}$. Moreover, if $L = K$ is a knot, then the multiplication by $(t-1)$ is invertible in $\mathscr{A}_K$. As a consequence, the matrix $\widetilde{\tau}_D(y)$ presents $\mathscr{A}_K^{\oplus 2}$ over the ring $\mathbb{Z}[t^{\pm 1/2}]$.

Unfortunately, these results do not carry over to the case $\mu > 1$. Indeed, let D be a μ-colored diagram for a μ-colored link L, and assume that each pair of colors meet in D. Then, using Corollary 3.6 of [4], it is possible to prove that the matrix $\widetilde{\tau}_D(x)$ presents the Alexander module of $L \#_1 - L$ over the ring

$$
\mathbb{Z}[\tfrac{1}{2}, t_1^{\pm 1/2}, \ldots, t_\mu^{\pm 1/2}, (t_1 - 1)^{-1}, \ldots, (t_\mu - 1)^{-1}]
$$

under the substitutions $x_j = \frac{t_j + t_j^{-1}}{2}$ and $x_{jk} = \frac{t_j t_k + t_j^{-1} t_k^{-1}}{2}$. However, the isomorphism

$$
\mathscr{A}_{L \#_1 L'} \simeq \mathscr{A}_L \oplus \mathscr{A}_{L'}
$$

that we used in the case $\mu = 1$ is no longer valid in general for $\mu > 1$. In other words, the addivity under connected sum enjoyed by the other abelian invariants considered in this work does not extend to the Alexander module in general.

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