Notes on relative Kazhdan's property (T)

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Abstract

This note is in major part a summary (which is probably non-exhaustive) of results and examples known concerning the relative property (T) for pairs of topological groups[1].

Contents

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0 Introduction

Kazhdan's property (T) for pairs is often an important step when one wants to prove that a given group is a Kazhdan one. The property (T) of the pair $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ is implicit and nevertheless crucial in the original paper of D. Kazhdan [Kaz67] (see also [BHV03]). But later this property was exploited very successfully in other area of mathematics. One of the first application of this property was the resolution of the Ruziewicz problem for $\mathbb{R}^n$ when $n \geq 3$ which is due to G.A. Margulis (see [M82]). On the other hand Margulis used the property (T) of the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ to give the first explicit construction of expanding graphs (see [HV89] and [L94]). The same pair plays also an important role in the paper of S. Popa [P03] where he gives an explicit example of type II_1 factor with trivial fundamental group (see also [NPS03] and [PP04]).

For the background about property (T) and unitary representations see [BHV04].

All groups considered here are supposed to be Hausdorff, locally compact and σ-compact.

0.1 Some notations

- H will be a closed subgroup of a topological group G.
- If π is an orthogonal or a unitary representation of G, $\mathcal{H}_\pi$ will denote the Hilbert space of the representation π, $\mathcal{H}_\pi^H$ will denote the subspace of $\pi(H)$-invariant vectors and $\mathcal{H}_\pi^1$ the unit sphere of $\mathcal{H}_\pi$.
- $\overline{\pi}$ will denote the conjugate representation of π in $\overline{\mathcal{H}_\pi}$ (the conjugate Hilbert space of $\mathcal{H}_\pi$).
- If $\mathcal{H}$ is a complex (resp: real) Hilbert space we will denote by $\mathcal{U}(\mathcal{H})$ (resp: $\mathcal{O}(\mathcal{H})$) the unitary (resp: orthogonal) group of $\mathcal{H}$.
- If $\mathcal{K} \subset \mathcal{H}_\pi$ is a closed invariant subspace then $\pi^{\mathcal{K}}$ will denote the subrepresentation of π corresponding to $\mathcal{K}$.
- $\mathrm{Ind}_H^G(\pi)$ will denote the induced representation of π by H on G.
- 1_G will denote the unit representation of G.
- λ_G will denote the left regular representation of G in $L^2(G)$.
- $\lambda_{G/H}$ will denote the quasi-regular representation of G in $L^2(G/H)$.
- $\widehat{G}$ will denote the set of all classes of irreducible unitary representations of G and $\widetilde{G}$ will denote the set of all classes of unitary representations of G in separable Hilbert spaces (as G is second countable, this is not restrictive to consider only unitary representations in separable Hilbert spaces).

0.2 Some definitions

- Given $\varepsilon > 0$ and $Q \subset G$ compact, a unit vector $\xi \in \mathcal{H}_\pi^1$ is said to be (Q, ε)-invariant if
$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| \leq \varepsilon$$

- We write $1_G \prec \pi$ if π is a unitary representation of G which has (Q, ε)-invariant vectors for every $\varepsilon > 0$ and every compact $Q \subset G$.

- We write $1_G \leq \pi$ if $\mathcal{H}_\pi$ contains a unit vector ξ such that $\pi(g)\xi = \xi$ $\forall g \in G$.

- We say that the pair (G, H) has Kazhdan's property (T) if whenever $1_G \prec \pi$ one has $1_H \leq \pi_{|_H}$.

0.2.1. Remarks. G is a Kazhdan group if and only if the pair (G, G) has property (T). An immediate consequence of the above definition is that for any $H_2 \subset H_1 \subset G_1 \subset G_2$ if (G_1, H_1) has Kazhdan's property (T) then (G_2, H_2) has Kazhdan's property (T) too. In particular if G or H is a Kazhdan group then the pair (G, H) has Kazhdan's property (T).

- We will say that G has property (TR) if G contains a non-compact subgroup H such that the pair (G, H) has Kazhdan's property (T).

- If K is another subgroup of G then we will say that the triple (G, H, K) has property (T) if for every unitary representation π of G such that $1_H \prec \pi_{|_H}$ one has $1_K \leq \pi_{|_K}$.

0.2.2. Remarks. If (H, K) has Kazhdan's property (T) then the triple (G, H, K) has Kazhdan's property (T). On the other hand if (G, H, K) has Kazhdan's property (T) then (G, K) has Kazhdan's property (T). When $G = H \ltimes K$ this notion of property (T) for triplets is sometimes called "strong property (T) (see for instance [BFGM05]).

1 Some equivalent characterizations

It is well known that Kazhdan's property (T) has numerous different characterizations and according to the situation one characterization is more adapted than others. That's why it is interesting and natural to ask if similar formulations holds for the relative property (T). This is actually the case for the most classical ones (with slight modifications).

1.1 Kazhdan's constants and Fell's topology

The first definition of the property (T) given by Kazhdan was in terms of Fell's topology (see [BHV03] appendix F) and its analogue for the relative property (T) appeared already implicitly in [Kaz67]. The following appears explicitly in [M82] (the proof is exactly the same that for usual property (T), see for instance proposition 1.11 in [V94], theorem 1.2.5 in [BHV03]).

1.1.1. Theorem. *The following conditions are equivalent:*

(*i*) *The pair* $(G,\ H)$ *has Kazhdan's property* (T)*;*

(*ii*) *There exists a neighborhood* V *of* 1_G *in* $\widetilde{G}$ *such that for any* $\pi \in V$ *one has* $1_H \leq \pi_{|_H}$*;*

(*iii*) *There exists a neighborhood* V *of* 1_G *in* $\widehat{G}$ *such that for any* $\pi \in V$ *one has* $1_H \leq \pi_{|_H}$.

1.1.2. Remark. If $\widehat{\mathcal{R}}_H := \{\pi \in \widehat{G}\ s.t.\ 1_H \not\leq \pi_{|_H}\}$, then the Kazhdan's property (T) of the pair (G, H) say that 1_G is isolated in $\{1_G\} \cup \widehat{\mathcal{R}}_H$. This correspond exactly to the property $(T; \mathcal{R})$ for $\mathcal{R} = \widehat{\mathcal{R}}_H$ (see [L94] definition 3.1.10).

It is sometime very useful to quantify the existence of such a neighborhood in the following way:

1.1.3. Proposition. *The following statements are equivalent:*

(*i*) *The pair* (G,H) *has Kazhdan's property* (T)*;*

(*ii*) *There exist a compact* $Q\subset G$ *and an* $\varepsilon>0$ *such that every unitary representation* π *of* G *which has* (Q,ε)*-invariant vectors has actually a non-zero invariant vector (that is* $1_H\le\pi_{|_H}$*);*

(*iii*) *There exist a compact* $Q\subset G$ *and an* $\varepsilon>0$ *such that every* $\pi\in\widehat{G}$ *which has* (Q,ε)*-invariant vectors actually satisfies* $1_H\le\pi_{|_H}$.

1.1.4. Remark. If in the previous two results we suppose H normal in G then in (iii), "$1_H\le\pi_{|_H}$" may be replaced by "$\pi(h)=1,\ \forall h\in H$". It is false in general but this mistake actually appears in the definition 1 of [M82] and in the definition 18 (p. 16) of [HV89]. A counter-example which was pointed to me by P. de la Harpe is the case of the pair $(SL_2(\mathbb{R}),SO(2))$. This pair has Kazhdan's property (T) while $SO(2)$ is compact but each neighborhood of $1_{SL_2(\mathbb{R})}$ in $\widehat{SL_2(\mathbb{R})}$ contains an element of the complementary series $\{\kappa_s:0<s<1\}$ of $SL_2(\mathbb{R})$ (κ_s "converges" to $1_{SL_2(\mathbb{R})}$ as s goes to 1). But for $s\in]0,1[$, $\kappa_{s|_{SO(2)}}$ is not equal to $1_{SO(2)}$ (see [Fo95] p. 244-248).

1.1.5. Definition. A pair (Q,ε) as in the previous proposition is sometime called a "Kazhdan pair for (G,H)".

As opposed to what occurs for the usual property (T), the property (T) of a pair (G,H) does not impose that the group G or the group H is compactly generated (see paragraph 2.2 below) but if we restrict ourselves to the compactly generated case proposition 1.1.2 is written as follows (the proof is an obvious adaptation of proposition 1.9 in [V94], see also proposition 15 in [HV89]):

1.1.6. Proposition. *If* G *is compactly generated by a compact subset* S_G *then* (G,H) *has Kazhdan's property* (T) *if and only if there exists an* $\varepsilon>0$ *such that* (S_G,ε) *is a kazhdan pair for* (G,H).

1.1.7. Definition. Let us note $\widetilde{\mathcal{R}}_H:=\{\pi\in\widetilde{G}\ s.t.\ 1_H\not\leq\pi_{|_H}\}$.
Then we define the Kazhdan constants associated to (G,H) and S_G:

$$\mathcal{K}((G,H),S_G):=\inf_{\pi\in\widetilde{\mathcal{R}}_H}\ \inf_{\xi\in\mathcal{H}^1_\pi}\ \max_{g\in S_G}\|\pi(g)\xi-\xi\|$$

and

$$\widehat{\mathcal{K}}((G,H),S_G):=\inf_{\pi\in\widehat{\mathcal{R}}_H}\ \inf_{\xi\in\mathcal{H}^1_\pi}\ \max_{g\in S_G}\|\pi(g)\xi-\xi\|$$

1.1.8. Remarks. $\mathcal{K}((G,H),S_G)$ is actually a minimum on $\widetilde{\mathcal{R}}_H$.
$\mathcal{K}((G,H),S_G)$ and $\widehat{\mathcal{K}}((G,H),S_G)$ are respectively the largest ϵ in (ii) and (iii) of proposition 1.1.3 when $Q=S_G$.
Explicit calculations or estimations of these constants give estimations on isoperimetric constants of expanding graphs but are difficult to obtain in general (see [L94], see also [Bu91] and [V94]).

We can then summarize what precedes by:

1.1.9. Theorem. *If G is compactly generated by a compact subset S_G then the following are equivalent:*

(i) *(G,H) has Kazhdan's property (T);*

(ii) $\mathcal{K}((G,H),S_G) > 0$;

(iii) $\widehat{\mathcal{K}}((G,H),S_G) > 0$.

1.1.10. Examples. *(i)* For any given finite (non empty) set $F \subset SL_2(\mathbb{Z})$ we define:

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_1} \sup_{B \in \mathcal{B}(\mathbb{P}_1(\mathbb{R}))} \max_{\gamma \in F} |\mu(\gamma B) - \mu(B)|$$

where $\mathcal{M}_1$ denotes the set of probability measures on $\mathbb{P}_1(\mathbb{R})$ and $\mathcal{B}(\mathbb{P}_1(\mathbb{R}))$ the σ-algebra of Borel subsets of $\mathbb{P}_1(\mathbb{R})$. Let D denote a fundamental domain for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. In [Bu91] M. Burger proves that if $\alpha(F) > 0$ and if Γ denotes the subgroup of $SL_2(\mathbb{Z})$ generated by F then the pair $(\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2)$ has Kazhdan's property (T) and moreover:

$$\mathcal{K}(\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2, P_F) \geq \sqrt{2 - 2\sqrt{1 - \alpha(F)^2}}$$

with $P_F := \{(z,\gamma) \in \mathbb{Z}^2 \times F \ / : \ m[(z + \gamma(D)) \cap D] > 0\}$, m denoting the Lebesgue measure on $\mathbb{R}^2$.

(ii) Let $n \geq 2$, let E_n be the set of all elementary matrices with ± 1 off the diagonal and let B_n the standard basis of $\mathbb{Z}^n$. Then $S_n := E_n \cup B_n$ generates $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ and it is shown in [Kas03] theorem 4.1 that:

$$\mathcal{K}((SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n, \mathbb{Z}^n), S_n) \geq \frac{1}{\sqrt{n+25}+3}$$

(iii) If $p,q \geq 2$ and if $T_{p,q} := (E_p \times \{I_p\}) \cup (\{I_p\} \times E_q) \cup B_{pq}$, then we obtain a system of generators for $(SL_p(\mathbb{Z}) \times SL_q(\mathbb{Z})) \ltimes \mathbb{Z}^{pq}$ and the same method gives (see [Kas03] theorem 5.1):

$$\mathcal{K}(((SL_p(\mathbb{Z}) \times SL_q(\mathbb{Z})) \ltimes \mathbb{Z}^{pq}, \mathbb{Z}^{pq}), T_{p,q}) \geq \frac{1}{\sqrt{5(p+q)/2+60}+6}$$

In the discrete case we can be more precise about the equivalence between *(ii)* and *(iii)* of theorem 1.1.9 (it is just a simple generalization of proposition 2 in [BH94]):

1.1.11. Proposition. *If G is finitely generated by a finite subset S_G then:*

$$\widehat{\mathcal{K}}((G,H),S_G) \geq \mathcal{K}((G,H),S_G) \geq \frac{1}{\sqrt{|\ S_G\ |}}\ \widehat{\mathcal{K}}((G,H),S_G).$$

1.2 Functions of positive type and conditionally of negative type

As in the framework of classical property (T), the Kazhdan's property (T) for pairs of groups admits a traduction in terms of functions on groups. For convenience we recall the definitions of positive type and conditionally of negative type functions:

1.2.1. Definition. Let $\varphi : G \to \mathbb{C}$ be a continuous function.
We say that φ is a function of positive type on G (or positive definite function) if $\forall n \geq 1$, $\forall g_1$, $\ldots$, $g_n \in G$, $\forall \alpha_1, \ldots , \alpha_n \in \mathbb{C}$:

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \varphi(g_i^{-1} g_j) \geq 0$$

We say that φ is normalized if $\varphi(1) = 1$

1.2.2. Definition. Let $\psi : G \to \mathbb{C}$ be a continuous function.
We say that ψ is conditionally of negative type function on G if:

(*i*) $\psi(g^{-1}) = \overline{\psi(g)}$, $\forall g \in G$, and

(*ii*) $\forall n \geq 1$, $\forall g_1, \ldots , g_n \in G$, $\forall \alpha_1, \ldots , \alpha_n \in \mathbb{C}$ / $\sum_{i=1}^{n} \alpha_i = 0$:

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \psi(g_i^{-1} g_j) \leq 0$$

Using the so-called Schoenberg's theorem and GNS construction which unify functions of positive type, functions conditionally of negative type and unitary representations (see [BHV03]), one can show (see [J04] theorem 1.2, see also the appendix of [PP04]):

1.2.3. Theorem. *The following conditions are equivalent:*

(*i*) *(G, H) has Kazhdan's property (T);*

(*ii*) *If $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of normalized positive definite functions on G which converges to 1 uniformly on compact sets of G, then $(\varphi_{n|_H})_{n\in\mathbb{N}}$ converges to 1 uniformly on H;*

(*iii*) *If ψ is a conditionally of negative type function on G then $\psi_{|_H}$ is bounded.*

Actually the proofs in [J04] and in [PP04] of the above theorem highlight the very useful (even fundamental) fact that the invariant vectors are arbitrary closed to almost invariant vectors. More precisely:

1.2.4. Proposition. *The following are equivalent:*

(*i*) *(G, H) has Kazhdan's property (T);*

(*ii*) *For every $\delta > 0$ there exists a Kazhdan pair $(Q(\delta), \epsilon(\delta))$ such that for every unitary representation π of G which has a $(Q(\delta), \epsilon(\delta))$- invariant $\xi \in \mathcal{H}_\pi^1$ one has $\|\xi - P_H(\xi)\| \leq \delta$, where P_H denotes the orthogonal projection on $\mathcal{H}_\pi^H$.*

Moreover, if H is normal in G, one can find a Kazhdan pair which is "uniform", that is (i) and (ii) are both equivalent to:

(*iii*) *There exists a Kazhdan pair (Q_0, ϵ_0) such that for every $\delta > 0$ and for every unitary representation π of G which has a $(Q_0, \delta\epsilon_0)$- invariant $\xi \in \mathcal{H}_\pi^1$ one has $\|\xi - P_H(\xi)\| \leq \delta$.*

1.3 Cohomology and Serre's property (FH)

This section is devoted to the relative property (T) from a cohomological viewpoint.

1.3.1. Definition. If $\mathcal{H}$ is a complex Hilbert space we denote by $\mathcal{PU}(\mathcal{H})$ the quotient group $\mathcal{U}(\mathcal{H})/\mathbb{T}$ (where $\mathbb{T}$ denotes the group of complex numbers of module 1) and we denote by $p : \mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ the projection map ($\mathcal{PU}(\mathcal{H})$ is endowed with the strong operator topology).

1.3.2. Definition. A projective (unitary) representation of G is a continuous group homomorphism $\pi : G \to \mathcal{PU}(\mathcal{H})$.
We say that such a π admits a lifting if there is a continuous unitary representation $\widetilde{\pi} : G \to \mathcal{U}(\mathcal{H})$ such that $\pi = p \circ \widetilde{\pi}$.

For more about projective representations see [Kir76] paragraph 14.

1.3.3. Definition.

(i) A $\mathbb{T}$-valued 2-cocycle on G is a map $\gamma : G \times G \to \mathbb{T}$ satisfying the following 2-cocycle relation:
$$\gamma(g,h)\gamma(gh,k) = \gamma(g,hk)\gamma(h,k), \forall g,h,k \in G$$
We denote by $Z^2(G,\mathbb{T})$ the set of all $\mathbb{T}$-valued 2-cocycles.

(ii) A $\mathbb{T}$-valued 2-cocycle is a 2-coboundary if there exists a map $\mu : G \to \mathbb{T}$ such that:
$$\mu(g)\mu(h) = \gamma(g,h)\mu(gh), \forall g,h \in G$$
We denote by $B^2(G,\mathbb{T})$ the set of all 2-coboundaries.

(iii) The second cohomology group of G in $\mathbb{T}$ is $H^2(G,\mathbb{T}) := Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$.

1.3.4. Definition. Given a projective representation $\pi : G \to \mathcal{PU}(\mathcal{H})$, for each $g \in G$ one can choose $u_g \in \mathcal{U}(\mathcal{H})$ such that $p(u_g) = \pi(g)$ and then for each $(g,h) \in G \times G$ one can find $\gamma(g,h) \in \mathbb{T}$ such that:
$$u_g u_h = \gamma(g,h) u_{gh}$$
It is easy, using associativity in $\mathcal{U}(\mathcal{H})$, to show that γ is a 2-cocycle. Moreover the 2-cohomology class $[\gamma] \in H^2(G,\mathbb{T})$ depends only on π. We call $[\gamma]$ the 2-cohomology class associated to π.

1.3.5. Proposition. *Let π be a projective representation and let $[\gamma]$ be the 2-cohomology class associated to π, then: π admits a lifting if and only if $[\gamma] = 1$.*

The following theorem is a recent characterization of property (T) for pairs in terms of projective representations (see [NPS03] theorem 3.1):

1.3.6. Theorem. *If G is an infinite discrete group then the following are equivalent:*

(i) *(G,H) has property (T);*

(ii) *$\forall \epsilon > 0$, $\exists F(\epsilon) \subset G$ finite, $\exists \delta(\epsilon) > 0$ such that if π is a projective representation of G with associated 2-cocycle γ, and if $\xi \in \mathcal{H}$ is a unit vector satisfying: $d(\pi(g)\xi, \mathbb{C}\xi) \leq \delta(\epsilon)$, $\forall g \in F(\epsilon)$, then: $\exists \xi_0 \in \mathcal{H}$, $\exists \mu : G \to \mathbb{T}$ such that: $\|\xi - \xi_0\| \leq \epsilon$, $\pi(h)\xi_0 = \mu(h)\xi_0$ and $\mu(h)\mu(h') = \gamma(h,h')\mu(hh')$, $\forall h,h' \in H$ (in particular $[\gamma_{|H\times H}] = 1$ in $H^2(H,\mathbb{T})$).*

We will give now a characterization of property (T) for pairs which is more geometric and which can be formulated in 1-cohomological terms. First let us recall some definitions:

1.3.7. Definition. Let $\mathcal{H}$ be an (real or complex) affine Hilbert space. The Hilbert space acting simply transitively on $\mathcal{H}$ will be denote by $\mathcal{H}$ too.
Let $\mathcal{I}(\mathcal{H})$ denote $\mathcal{O}(\mathcal{H}) \ltimes \mathcal{H}$ if $\mathcal{H}$ is real and $\mathcal{U}(\mathcal{H}) \ltimes \mathcal{H}$ if $\mathcal{H}$ is complex.
An affine isometric action of G on $\mathcal{H}$ is a group homomorphism
$\alpha : G \to \mathcal{I}(\mathcal{H})$ such that the mapping

$$G \to \mathcal{H}, g \mapsto \alpha(g)\xi$$

is continuous for every $\xi \in \mathcal{H}$. For every $g \in G$ we have:

$$\alpha(g)\xi = \pi(g)\xi + b(g), \forall \xi \in \mathcal{H}$$

and we say that $\pi : G \to \mathcal{O}(\mathcal{H})$ (resp: $\pi : G \to \mathcal{U}(\mathcal{H})$) is the linear part of α.

1.3.8. Definition. We will say that a pair (G, H) has Serre's property (FH) if every affine isometric action of G on a (real or complex) Hilbert space admits an H-fixed point.

1.3.9. Definition. Let π be an orthogonal or a unitary representation of G on a (real or complex) Hilbert space $\mathcal{H}$.

(*i*) A continuous map $b : G \to \mathcal{H}$ is called a 1-cocycle with respect to π if:

$$b(gh) = b(g) + \pi(g)b(h), \forall g, h \in G$$

(*ii*) A 1-cocycle with respect to π, $b : G \to \mathcal{H}$, is called a 1-coboundary if there exists a $\xi \in \mathcal{H}$ such that:

$$b(g) = \pi(g)\xi - \xi, \forall g \in G$$

(*iii*) We denote by $Z^1(G, \pi)$ the vector space of all 1-cocycles with respect to π. We denote by $B^1(G, \pi)$ the vector space of all 1-coboundaries with respect to π. The quotient vector space $H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$ is called the first cohomology group of G with coefficients in π.

1.3.10. Remarks. The 1-cohomology comes from the correspondence between the affine isometric actions and their linear parts, more precisely if $\alpha : G \to \mathcal{I}(\mathcal{H})$ is an affine isometric action of G then an orthogonal representation $\pi : G \to \mathcal{O}(\mathcal{H})$ (resp: a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$) is the linear part of α if and only if there exists a 1-cocycle $b : G \to \mathcal{H}$ such that: $\alpha(g)\xi = \pi(g)\xi + b(g), \forall \xi \in \mathcal{H}$. On the other hand, it is clear that α has a fixed point in $\mathcal{H}$ if and only if $b_{|_H}$ belongs to $B^1(H, \pi_{|_H})$. Then the property (FH) can be reformulated in the following way:

1.3.11. Proposition. *The pair (G, H) has property (FH) if and only if for any orthogonal or unitary representation π of G,*

$$Res_G^H : H^1(G, \pi) \to H^1(H, \pi_{|_H}), [b] \mapsto [b_{|_H}]$$

is identically zero.

The following theorem is due (in its original formulation for usual property (T)) to A. Guichardet ([G72] theorem 1 for $(FH) \Rightarrow (T)$) and P. Delorme ([D77] theorem V.1 for $(T) \Rightarrow (FH)$). The proof of the following statement is essentially based on the previous proposition and on the theorem 1.2.3:

1.3.12. Theorem. *The following are equivalent:*

(*i*) *The pair* (G, H) *has Kazhdan's property* (T)*;*

(*ii*) *For any orthogonal or unitary representation* π *of* G*,* $Res_G^H : H^1(G, \pi) \to H^1(H, \pi_{|_H})$ *is identically zero;*

(*iii*) *For any orthogonal or unitary representation* π *of* G*, the range of* Res_G^H *is separated;*

(*iv*) *The pair* (G, H) *has property* (FH)*.*

1.3.13. Remarks. It is unreasonable to think that the previous theorem holds replacing in (*ii*) "for any orthogonal or unitary representation" by "for any *irreducible* orthogonal or unitary representation" and this is false in general, indeed, if $H = G$ is an infinite abelian torsion group (such a group is not finitely generated) then one has $H^1(G, \chi) = 0$ for every unitary character $\chi \in \widehat{G}$, but the pair (G, G) (or equivalently G) doesn't have property (T). On the other hand, by a result of Y. Shalom (see [Sh00]), when $H = G$ one can test the property (T) on irreducible representations, this is allowed by the characterization of property (T) in terms of reduced cohomology, and unfortunately there is no such characterization for the relative case (nevertheless, see [C05]).

1.4 Amenable representations

In this section we try to give a characterization of the relative property (T) in terms of "amenable" (in the Bekka terminology, see [B90]) representations which generalizes a result of M.B. Bekka and A. Valette (see [BV93]). This characterization is very useful and in particular it is fundamental to obtain the characterizations of the relative property (T) in sections 1.5 and some other dynamical ones in section 3.

1.4.1. Definition. Let $p, q \geq 0$ denote integers. We will say that a unitary representation π of G is (p, q)-amenable if $1_G \prec \pi^{p,q} := \pi^p \otimes \overline{\pi}^q$ (where π^p denotes $\pi \otimes \ldots \otimes \pi$ p times and $\overline{\pi}^q$ denotes $\overline{\pi} \otimes \ldots \otimes \overline{\pi}$ q times, $\pi^0 := 1_G$). A unitary representation π is just said to be amenable (in the Bekka sense) if it is $(1, 1)$-amenable.

1.4.2. Remark. The terminology "amenable" comes from a result of Bekka which asserts that a group is amenable if and only if all its unitary representations are amenable (see theorem 2.2 in [B90]).

1.4.3. Definition. Let $p, q, r \geq 0$. A pair (G, H) has property $(T^{p,q})$ if whenever π is a (p, q)-amenable unitary representation of G then $1_H \leq (\pi^{p,q})_{|_H} \simeq (\pi_{|_H})^{p,q}$.
A pair (G, H) has property $(T^r_{\mathbb{R}})$ if whenever π is a $(r, 0)$-amenable orthogonal representation of G then $1_H \leq (\pi^r)_{|_H} \simeq (\pi_{|_H})^r$ (where $\pi^r := \pi^{r,0}$).

1.4.4. Remarks. Property $(T^{1,0})$ for a pair (G, H) is exactly the relative property (T). If $p, q, r - 1 \geq 1$ then the conclusions in the definitions of property $(T^{p,q})$ and property $(T^r_{\mathbb{R}})$ are both equivalent to: "$\pi_{|_H}$ contains a finite dimensional subrepresentation". Actually, we have the following general lemma (see proposition A.1.10 in [BHV03]):

1.4.5. Lemma. *Let* π *be a unitary (resp: orthogonal) representation of* G*. The following are equivalent:*

(*i*) $\pi_{|_H}$ *contains a finite dimensional subrepresentation;*

(ii) $\pi^{1,1}_{|_H} \geq 1_H$ *(resp:* $\pi^{2,0}_{|_H} \geq 1_H$*);*

(iii) *There exists a unitary (resp: orthogonal) representation* σ *of* G *such that:* $(\pi \otimes \sigma)_{|_H} \geq 1_H$.

Clearly the property (T) of a pair (G, H) implies properties $(T^{p,q})$ and $(T^r_{\mathbb{R}})$ for every $r, p, q \geq 0$, and a priori property $(T^{p,q})$ and property $(T^r_{\mathbb{R}})$ seem to be strictly weaker that property (T) when $r - 1, p, q \geq 1$. Indeed the converse is also true and the point is the following:

Let α be an affine isometric action of G on an affine Hilbert space $\mathcal{H}$, then $\forall g \in G$, $\forall \xi \in \mathcal{H}$,

$$\alpha(g)\xi = \pi(g)\xi + b(g)$$

where π denotes the linear part of α and $b \in Z^1(G, \pi)$ denotes the associated 1-cocycle. Then $\psi : G \to \mathbb{R}_+, g \mapsto \|b(g)\|^2$ is conditionally of negative type on G and, by the Schoenberg's theorem, for every $t > 0$ the function $exp(-t\psi)$ is of positive type. For every $t > 0$, let us denote by $(\pi_t, \mathcal{H}_t, \xi_t)$ the GNS triple associated to the function $exp(-t\psi)$ (see [BHV03] theorem C.4.10). So, we have the following statement where "$(ii) \Leftrightarrow (iii)$" is exactly lemma 2.1 in [J04]:

1.4.6. Proposition. *With the same notations, the following are equivalent:*

(i) α *has* H*-fixed points;*

(ii) $b_{|_H}$ *is bounded;*

(iii) *There exists a* $t > 0$ *such that* $\pi_{t|_H} \geq 1_H$*;*

(iv) *There exists a* $t > 0$ *such that* $\pi_{t|_H}$ *contain a finite dimensional subrepresentation.*

Combining proposition 3.1.5 with lemma 3.1.4, we can show the expected characterization of the relative property (T):

1.4.7. Theorem. *The following conditions are equivalent:*

(i) (G, H) *has property* (T)*;*

(ii) *For every* $p, q \geq 0$ *the pair* (G, H) *has property* $(T^{p,q})$*;*

(iii) *There exist* $p, q \geq 0$ *such that* (G, H) *has property* $(T^{p,q})$*;*

(iv) *There exists* $r \geq 0$ *such that* (G, H) *has property* $(T^r_{\mathbb{R}})$*.*

1.4.8. Remarks. In particular this theorem says that the property (T) of a pair (G, H) is equivalent to: " for every unitary representation π of G, one has $1_G \prec \pi \Rightarrow \pi_{|_H}$ contains a finite dimensional representation".
It is important to note that this characterization is false if we consider only irreducible representations (see theorem 3 of [BV93]).

1.4.9. Corollary. *If* (G, H) *does not have property* (T) *then there exists a unitary representation* $\pi := \pi_H$ *of* G *with the following properties:*

(i) $1_G \prec \pi$,

(ii) $1_H \not\leq (\pi \otimes \overline{\pi})_{|_H}$ *(that is: no finite dimensional representation can be contained in* $\pi_{|_H}$*;* π *is* H*-weakly mixing).*

1.5 Relative property (T) for C^*-algebras

The notion of property (T) in the framework of C^*-algebras was first introduced by A. Connes for type II_1-factors (see [C80]). It was related to the classical notion of Kazhdan's property for groups by a result due to A. Connes and V. Jones (see [CJ85]) asserting that for countable discrete ICC groups (groups in which all the non-trivial conjugacy classes are infinite, or equivalently groups for which the associated von Neumann algebra is a factor) the Kazhdan's property (T) of the group is equivalent to the Connes's property (T) of his von Neunmann algebra. P. Jolissaint has extended this result showing actually that a countable discrete group has property (T) and has finitely many finite conjugacy classes if and only if his von Neumann algebra has property (T) (see [J93]). Naturally there is a corresponding notion of relative property (T) for pairs of C^*-algebras and this last (actually a stronger notion called "rigid embedding") was extensively studied by S. Popa (see for instance [P03] and [PP04]).
More recently B. Bekka has defined a notion of relative property (T) for arbitrary unital C^*-algebras and he has deduced a characterization of property (T) for countable discrete groups in terms of the classical operator-algebraic objects associated to their left regular representations (see [B05]).

1.5.1. Definition. Let A be a C^*-algebra. A Hilbert bimodule over A (or A-bimodule) is a Hilbert space $\mathcal{H}$ carrying two commuting actions of A usually denoted $\xi \mapsto a\xi b$ for every $a, b \in A$.

1.5.2. Definition. Let A be a unital C^*-algebra endowed with a tracial state (ie: a positive linear functional $\tau : A \mapsto \mathbb{C}$ satisfying $\tau(1) = 1$ and $\tau(ab) = \tau(ba)$ for every $a, b \in A$). Then A is said to have property (T) if there exists a finite subset F of A and a $\epsilon > 0$ such that the following holds: if a A-bimodule $\mathcal{H}$ contains a unit vector ξ being (F, ϵ)-central (ie: $\|a\xi - \xi a\| < \epsilon$ for every $a \in F$) then $\mathcal{H}$ contains a non-zero central vector (ie: a vector η such that $a\eta = \eta a$ for every $a \in A$).

1.5.3. Remark. This definition makes sense for any C^*-algebra but it is easy to show that with this definition any unital C^*-algebra without tracial state has property (T).

1.5.4. Definition. Let Γ be a countable discrete group.

- (i) The reduced C^*-algebra of Γ, denoted by $C^*_{\text{red}}(\Gamma)$, is the norm closure in $\mathcal{L}(l^2(\Gamma))$ of the subspace $< \lambda_\Gamma(\Gamma) >$ spanned by $\lambda_\Gamma(\Gamma)$.
- (ii) The closure for the weak operator topology of $< \lambda_\Gamma(\Gamma) >$ (or equivalently $\lambda_\Gamma(\Gamma)''$) is called the von Neumann algebra $\text{VN}(\Gamma)$ of Γ.
- (iii) The maximal (or full) C^*-algebra of Γ, denoted by $C^*_{\max}(\Gamma)$, is the completion of the group algebra $\mathbb{C}[\Gamma]$ with respect to the norm defined by:

$$\|\sum_{\gamma\in\Gamma} a_\gamma \gamma\| := \sup_\pi \|\sum_{\gamma\in\Gamma} a_\gamma \pi(\gamma)\|$$

  π running over all the (equivalence classes of) cyclic representations of Γ.

The main result of [B05] is the following:

1.5.5. Theorem. *Let Γ be a countable discrete group and let Λ be a subgroup of Γ. Then the following assertions are equivalent:*

- (i) *The pair (Γ, Λ) has Kazhdan's property (T);*

(ii) *The pair* $(C^*_{max}(\Gamma), C^*_{max}(\Lambda))$ *has property (T);*

(iii) *The pair* $(C^*_{red}(\Gamma), C^*_{red}(\Lambda))$ *has property (T);*

(iv) *The pair* $(VN(\Gamma), VN(\Lambda))$ *has property (T).*

1.5.6. Remarks. The notions of relative property (T) for groups and for C^*-algebras associated to it are related via the correspondence between unitary representations and Hilbert bimodules. More precisely, given a Hilbert $C^*_{\max}(\Gamma)$-bimodule $\mathcal{H}$ and viewing Γ as a subset of $C^*_{\max}(\Gamma)$, one can define two commuting unitary representations π_1 and π_2 of Γ in $\mathcal{H}$ by:

$$\pi_1(\gamma)\xi := \gamma\xi \qquad \pi_2(\gamma)\xi := \xi\gamma^{-1}$$

Hence a non-zero invariant vector for the unitary representation $\lambda \mapsto \pi_1(\lambda)\pi_2(\lambda)$ is a non-zero $C^*_{\max}(\Lambda)$-central vector.

Conversely, given a unitary representation π of Γ in a Hilbert space $\mathcal{H}$, one can define two commuting unitary representations π_1 and π_2 of Γ in $l^2(\Gamma) \otimes \mathcal{H} \simeq l^2(\Gamma, \mathcal{H})$ by:

$$\pi_1(\gamma)\xi(x) := \xi(\gamma^{-1}x) \qquad \pi_2(\gamma)\xi(x) := \pi(\gamma)\xi(x\gamma)$$

Then, π_1 and $\pi_2 \simeq \lambda_\Gamma \otimes \pi$ being multiples of λ_Γ, they extend to commuting representations of $\mathrm{VN}(\Gamma)$ which give a natural structure of $\mathrm{VN}(\Gamma)$-bimodule to $l^2(\Gamma, \mathcal{H})$. The deep part of the preceding theorem is the implication $(iv) \Rightarrow (i)$ and the key point is the characterization of relative property (T) by amenable representations, because a non-zero central vector for the above $\mathrm{VN}(\Gamma)$-bimodule $l^2(\Gamma, \mathcal{H})$ only provides a finite dimensional sub-representation of π_Λ but this is sufficient to conclude by corollary 1.4.9.

2 General results about relative property (T)

In this section we gather some results known about relative Kazhdan's property (T) which come from results about the usual property (T) with appropriate adaptations. We have collected also most of the methods and criterions known to construct examples.

2.1 Some criterions and non-obvious examples

The following criterion is very useful and is one of the most classical methods to prove the Property (T) of a pair (see [BHV03] theorem 1.4.5 and [M89] corollary 4.5):

2.1.1. Theorem. *Let H be an abelian normal subgroup of G. We consider the natural dual action of G on $\widehat{H}$ induced by the conjugation. If the Dirac measure at the unit character 1_H is the unique invariant mean on the Borel subsets of $\widehat{H}$, then the pair (G, H) has Kazhdan's property (T).*

2.1.2. Remark. The preceding theorem never holds when H is an infinite discrete group. More precisely, when H is discrete and infinite, $\widehat{H}$ is a compact infinite group and its Haar measure may be viewed as an invariant mean on $\widehat{H} \setminus \{0\}$ ($\{0\}$ being of null measure). In particular the previous theorem can't be used to prove the property (T) of the so-called pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ (see examples 1.1.10) but it can be used to prove the property (T) of the pair $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$. The property (T) of the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ was originally deduced from the property (T) of this last pair using the fact that $SL_2(\mathbb{Z})$ is a lattice in $SL_2(\mathbb{R})$ (an alternative proof not using this fact can be found in [Sh99a], see also [Kas03] and [Bu91]).

In the case of semi-direct products one has the following similar criterion which actually was the first for the relative property (T) :

2.1.3. Theorem. *Let $G := K \ltimes H$ a semi-direct product, the group H being abelian. We consider the natural dual action of G on $\widehat{H}$ induced by the conjugation. Consider the following conditions:*

- (i) *Every orbit is locally closed (intersection of an open and a closed set) in $\widehat{H}$,*
- (ii) *For each $\chi \in \widehat{H} \smallsetminus \{0\}$, the stabilizer $\{g \in K/g\chi = \chi\}$ is amenable,*
- (iii) *K is not amenable.*

If (i), (ii) and (iii) are satisfied then (G, H) has Kazhdan's property (T)

2.1.4. Remarks. In the previous criterion, the fact that H is an abelian group and the hypothesis (i) allow us to use the "Mackey machine". Then we have a complete description of the dual group $\widehat{G}$ (see paragraph 6.6 of [Fo95]).
Margulis used this criterion in [M82] (theorem 2) to prove the property of the pair $(O_3(\mathbb{Q}_5) \ltimes \mathbb{Q}_5^3, \mathbb{Q}_5^3)$.

A very important particular case of theorem 2.1.1 is the following: (see [HV89] proposition p.22):

2.1.5. Proposition. *If $\mathbb{K}$ is a local field, let V be a finite dimensional vector space on $\mathbb{K}$ and let H be a subgroup of $GL(V)$ such that there is no invariant probability measure on $\mathbb{P}\widehat{V}$ then $(H \ltimes V, V)$ has Kazhdan's property (T). Actually the triple $(H \ltimes V, H, V)$ has Kazhdan's property (T).*

Using the previous proposition, in the case $\mathbb{K} = \mathbb{R}$ we have the more general and more precise statement (see [V94] proposition 2.3):

2.1.6. Proposition. *Let K be a non-compact semi-simple Lie group, let V be a finite dimensional vector space and let $\rho : K \to GL(V)$ be a continuous homomorphism. Then the following are equivalent:*

- (i) *The pair $(K \ltimes_\rho V, V)$ has Kazhdan's property (T);*
- (ii) *There is no K-invariant measure on $\mathbb{P}\widehat{V}$.*

There exists a little bit more stronger result than proposition 2.1.1 which is due to Y. Shalom (see [Sh99b] theorem 5.5):

2.1.7. Theorem. *Let $G := K \ltimes H$ a semi-direct product, the group H being abelian. If there is no invariant mean for the dual action of G on $\widehat{H}$ on*
$\widehat{H} \smallsetminus \{0\}$ then the triple (G, K, H) has Kazhdan's property (T). Actually, a stronger property is true:
In the Shalom's terminology, the pair (K, H) has "strong relative property (T)".

2.1.8. Remark. The triple (G, K, H) never has property (T) when H is infinite and discrete, indeed one can take the natural representation of the semi-direct product $G := K \ltimes H$ on $L^2(H)$ coming from the "affine" action on H. By discreteness, there is actually a K-invariant vector, the dirac function at the identity element, but there is no non-zero H-invariant vector because such a vector would be a non-zero constant function in $L^2(H)$ by transitivity of the action of

H on himself and this is impossible if H is infinite. In particular this show that the implication "(G,H,K) has property $(T) \Rightarrow (G,K)$ has property (T)" is strict in general, indeed the case $(G,H,K) = (SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, SL_2(\mathbb{Z}), \mathbb{Z}^2)$ gives a counter-example to the converse.

2.1.9. Examples. All the following pairs non obviously have Kazhdan's property (T):

- *(i)* $(SL_2(\mathbb{K}) \ltimes \mathbb{K}^2, \mathbb{K}^2)$ for any local field $\mathbb{K}$ (see [BHV03] corollary 1.4.13 and [M89] corollary 4.6);
- *(ii)* $(SL_2(\mathbb{K}) \ltimes S_2^*(\mathbb{K}^2), S_2^*(\mathbb{K}^2))$ where $S_2^*(\mathbb{K}^2)$ denote the space of all symmetric bilinear forms on $\mathbb{K}^2$ (see [BHV03] corollary 1.4.18);
- *(iii)* $(SL_2(\mathbb{K}) \ltimes V_n(\mathbb{K}), V_n(\mathbb{K}))$ for all $n \geq 2$, where $\mathbb{K}$ is a local field of characteristic zero ($\mathbb{R}$, $\mathbb{C}$ or every finite extension of $\mathbb{Q}_p$), and where $V_n(\mathbb{K})$ is the unique $(n+1)$-dimensional irreducible linear representation of $SL_2(\mathbb{K})$ (see for instance [C04]).
- *(iv)* $(SL_2(\mathbb{Z}[t]/(t^2)), H)$, where H denotes the kernel of the natural homomorphism $SL_2(\mathbb{Z}[t]/(t^2)) \mapsto SL_2(\mathbb{Z})$. This can be deduced from the preceding example by embedding $(SL_2(\mathbb{Z}[t]/(t^2))$ as a lattice in the Lie group $(SL_2(\mathbb{R}[t]/(t^2))$ and by noticing that $(SL_2(\mathbb{R}[t]/(t^2))$ is actually isomorphic to $SL_2(\mathbb{R}) \ltimes \mathfrak{sl}_2(\mathbb{R})$, the action being the adjoint one (ie: $\mathfrak{sl}_2(\mathbb{R})$ can be viewed as $V_2(\mathbb{R})$).

2.1.10. Remarks. Moreover the property (T) of the pairs in (i) and (ii) give respectively the property (T) of the groups $SL_n(\mathbb{K})$ for all $n \geq 3$ and the property (T) of the groups $Sp_{2n}(\mathbb{K})$ for all $n \geq 2$ (see [BHV03] p.23-33).

In the same way we can mention the following recent result of A. Valette (see [V04] theorem 1):

2.1.11. Theorem. *Let G be an absolutely non-compact semi-simple Lie group with trivial center (absolutely simple means that G is simple and not locally isomorphic , as a real Lie group, to a complex Lie group) and let Γ be an arithmetic lattice in G. Then there exist an integer $N \geq 2$ and a subgroup Λ of finite index in Γ such that the pair $(\Lambda \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has Kazhdan's property (T).*

2.1.12. Remark. As S. Popa in [P03], A.Valette use the previous theorem in particular to construct a sequence of non-isomorphic type II_1 group's factors all with trivial fundamental group (see [V04] theorem 3).

T. Fernós have very recently improved the theorem 2.1.11 above (see theorem 1 in [Fe04]):

2.1.13. Theorem. *Let Γ be a finitely generated group. The following condtions are equivalent:*

- *(i)* *There exists $\phi : \Gamma \to SL_n(\mathbb{R})$ such that the Zariski closure $\overline{\phi(\Gamma)}^Z(\mathbb{R})$ is non-amenable;*
- *(ii)* *There exists an abelian group A of non-zero finite $\mathbb{Q}$-rank and an action of Γ on A such that the pair $(\Gamma \ltimes A, A)$ has Kazhdan's property (T)*

Now we give a generalization of theorem 2.1.1 in the case of semi-direct products which is due to M. Burger (see [Bu91] paragraph 5):

2.1.14. Theorem. *Let $G := K \ltimes H$ a semi-direct product, the group H being abelian, and let $\widehat{H}_\infty$ denote the Alexandroff compactification of $\widehat{H}$ (we denote by "∞" the point at infinity).*

We suppose that K acts by homeomorphisms on a compact space X. Moreover we suppose that there is a continuous K-equivariant map $p : X \to \widehat{H}_\infty$ such that:

(i) *p realizes an homeomorphism between $X \smallsetminus (p^{-1}(1_H) \cup p^{-1}(\infty))$ and $\widehat{H}_\infty \smallsetminus \{1_H, \infty\}$;*

(ii) *There is no K-invariant probability measure on $p^{-1}(1_H) \cup p^{-1}(\infty)$.*

Then the pair (G, H) has Kazhdan's property (T).

2.1.15. Remarks. In particular, when H is discrete, we have $\widehat{H}_\infty = \widehat{H}$, then the point at infinity disappears in the assumptions and, as opposed to theorem 2.1.7, this result may be used in this case. Actually if Γ is a subgroup of $SL_2(\mathbb{Z})$ such that there is no Γ-invariant probability measure on $\mathbb{P}_{n-1}(\mathbb{R})$ for $n \geq 2$ fixed (by corollary 3.2.2 of [Z84a] this is equivalent to say that Γ is not virtually cyclic) then the previous theorem gives a direct proof of the property (T) of the pair $(\Gamma \ltimes \mathbb{Z}^n, \mathbb{Z}^n)$ for every $n \geq 2$ (the space X is the blowing up of the point $1_{\mathbb{Z}^n} \in \widehat{\mathbb{Z}^n} \cong \mathbb{T}^n$ and the map p is the natural projection map, then we have $p^{-1}(1_{\mathbb{Z}^n}) \cong \mathbb{P}_{n-1}(\mathbb{R})$ and the conditions (i) and (ii) are satisfied). Moreover quantifications of the "non-invariance" of probability measures provide estimations of relative Kazhdan constants (see 1.1.10).

We end this paragraph with a recent application of a result of M.B. Bekka and N. Louvet (see [BL97] theorem A) which provides a lot of non-trivial pairs with Kazhdan's property (T):

2.1.16. Theorem. *Let G_1 and G_2 be two groups and let $H_2 \subset G_2$ such that (G_2, H_2) has Kazhdan's property (T). If Γ is an irreducible lattice in $G_1 \times G_2$ (that is: Γ projects densely in $(G_1 \times G_2)/N$ for every normal and non-central subgroup N), then the pair $(\Gamma, \Gamma \cap (G_1 \times H_2))$ has Kazhdan's property (T).*

2.1.17. Corollary. *Let G_1 and G_2 be two semi-simple connected Lie groups such that G_1 has Kazhdan's property (T). Let Γ be an irreducible lattice in $G_1 \times G_2$ and let Δ be a subgroup of Γ such that the pair $(G_2, p_2(\Delta))$ has Kazhdan's property (T) (where p_2 denotes the projection on the second factor). Then the pair (Γ, Δ) has Kazhdan's property (T).*

2.1.18. Remark. If we suppose that G_2 does not have property (T) then the pair obtained is non-trivial as soon as Δ is not a Kazhdan subgroup. On the other hand, a result of Lubotzky and Zimmer asserts that such a Γ has property (τ) as soon as G_1 or G_2 is a Kazhdan group.

The sketch of the proof of the preceding theorem was pointed to me by M.B. Bekka:
If σ is an irreducible unitary representation of Γ which is close to 1_Γ in $\widehat{\Gamma}$ then the point is that (by theorem A in [BL97]) σ is of the form $(\pi \circ p_2)_{|\Gamma}$ where π is an irreducible representation of G_2. Moreover, if Γ is sufficiently close to 1_Γ, π is close to 1_{G_2} in $\widehat{G_2}$ and then by the property (T) of the pair (G_2, H_2) we obtain that $\sigma_{|H_2}$ contains a non-zero invariant vector.

2.1.19. Remark. It is natural to ask if there exist non-obvious pairs with Kazhdan's property (T) which cannot be deduced (using stability results as theorem 2.2.18 below) from pairs of the form (G, H) with $H \lhd G$ and the previous result allows us to give such examples (see [C04]). Indeed let $q_\pm$ be the quadratic forms defined by:

$$q_\pm(x) := x_1^2 + x_2^2 + x_3^2 - x_4^2 \pm \sqrt{2} x_5^2$$

Let G_+ denote the subgroup of $SL(5)$ preserving q_+. Then $G_2 := G_+(\mathbb{R})$ is isomorphic to $SO(4,1)$ and doesn't have Kazhdan's property (T). On the other hand, let G_- denote the subgroup of $SL(5)$ preserving q_-. Then $G_1 := G_-(\mathbb{R})$ is isomorphic to $SO(3,2)$ and hence G_1 has property (T). Now $\Gamma := G(\mathbb{Z}[\sqrt{2}])$ embeds as an irreducible lattice in $G_1 \times G_2$ (and this lattice is arithmetic, see [Z84a] theorem 6.1.2) and if Δ is a subgroup of Γ which has relatively compact projection in G_2, then by the preceding theorem the pair (Γ, Δ) has Kazhdan's property (T). By a result of rigidity due to Margulis, normal subgroups of such a lattice has to be of finite index (see for instance [M89] theorem 5.3, see also [Z84a] theorem 8.1.2), and then Δ cannot be a finite extension of a normal subgroup of Γ. Actually there exist subgroups Δ satisfying the preceding hypothesis. To see this it suffices to remark that the set E of elliptic elements in $SO(4,1)$ (that is: the set of elements generating a relatively compact subgroup of $SO(4,1)$) has non empty interior. Then by irreducibility of Γ, $\Gamma \cap E$ must be infinite and in particular for each $A \in (\Gamma \cap E) - \{1\}$, the pair $(\Gamma, < A >)$ has Kazhdan's property (T). Moreover, by the Selberg's lemma (see for instance [Ra72] corollary 6.13), Γ is virtually torsion-free, and so we can find an $A \in (\Gamma \cap E) - \{1\}$ of infinite order.

2.1.20. Remark. In opposition to the preceding remark, it can be proved (see [C05b]) that if G is a connected Lie group locally isomorphic to $SO(4,1)$ or $SU(2,1)$, for any pair of subgroups (viewed as discrete groups) (Γ, Λ) with Kazhdan's property (T), if $\Lambda \lhd \Gamma$ then Λ must be finite. In particular this shows the pertinence of relative property (T) when the subgroup is not normal.

2.2 Properties and stability results

Stability results allow us on one part to deduce simple obstructions to the property (T) of a given pair and on the other part to show easily the property (T) of certain pairs, while this is in general difficult to prove that a given pair has property (T). In this section we try to give some stability results generalizing some results in the framework of the usual property (T). We give also some simple but useful properties.

It is natural to begin with the problem of the compact generation. Originally Kazhdan introduced property (T) because he had discovered that groups with this property are necessary compactly generated (in particular finitely generated when the groups are discrete). Therefore one can hope that property (T) of a pair (G, H) impose one of the two groups to be compactly generated. This is not true. Actually it is easy to find a lot of pairs (G, H) which have property (T) with G and H non compactly generated, for instance if A is a Kazhdan group containing a non compactly generated subgroup B (take $A = SL_3(\mathbb{Z})$ and $B = F_\infty$ the free group of infinite rank) then the pair $(A \times B, B \times \{1\})$ has Kazhdan's property (T) (see remark 0.2.1). Nevertheless, while imitating the proof of theorem 1.3.1 in [BHV03], one can show the following:

2.2.1. Theorem. *If the pair (G, H) has Kazhdan's property (T) then there exists a compactly generated group K (open in G) such that $H \subset K \subset G$.*

2.2.2. Remark. The previous theorem gives a simple obstruction to the property (T) for certain pairs. For instance the pair $(SL_n(\mathbb{Q}) \ltimes \mathbb{Q}^n, \mathbb{Q}^n)$ does not have Kazhdan's property (T) for any $n \geq 1$.

In the same way, one can ask if given a pair (G, H) having the Kazhdan's property (T) there is always a compactly generated subgroup L of G containing H such that (L, H) also has the Kazhdan's property (T)? This is actually always the case, more precisely:

2.2.3. Theorem. *Let (G, H) be a pair with Kazhdan's property (T) and let $(Q, \epsilon) := (Q(1/2), \epsilon(1/2))$ a Kazhdan pair as in (ii) of proposition 1.2.4. Let $\widetilde{H}$ be a compactly generated and open subgroup* *of G containing H as in the previous theorem and let us denote by $\widetilde{Q}$ a compact generating set for $\widetilde{H}$. Then, if L denotes the subgroup of G generated by $Q \cup \widetilde{Q}$, the pair (L, H) has Kazhdan's property (T).*

2.2.4. Remark. The proof of the previous result was pointed to me by M.B. Bekka. It is based on the fact that the homogeneous space G/L is discrete, and in this case every unitary representation π of L satisfies $\pi \leq Ind_L^G(\pi)_{|_L}$. By this way this is not difficult to deduce the property (T) of the pair (L, H) from the property (T) of the pair (G, H).

Now we investigate the stability of the relative property (T) under homomorphic image:

2.2.5. Proposition. *Let $\phi : G_1 \to G_2$ be a continuous group homomorphism and let H_1 be a subgroup of G_1. If (G_1, H_1) has Kazhdan's property (T) then the same is true for the pair $(G_2, \phi(H_1))$.*

2.2.6. Remarks. In opposition to the preceding proposition, it is not true that, given two isomorphic subgroups H_1 and H_2 of the same group G, the property (T) of the pair (G, H_1) is equivalent to the property (T) of the pair (G, H_2). It is easy to see this by taking for example the group $A \times B$ where $A = SL_3(\mathbb{Z})$ and $B = F_\infty$. Clearly $\{1\} \times B$ and $B \times \{1\}$ are isomorphic, but the pair $(A \times B, B \times \{1\})$ has Kazhdan's property (T) while the pair $(A \times B, \{1\} \times B)$ doesn't (because of the previous proposition, by taking the image under the projection on the second factor we would get the property (T) of the pair (B, B) or equivalently the property (T) of B which is clearly not a Kazhdan group). The point is that the isomorphism which send $\{1\} \times B$ to $B \times \{1\}$ is not an automorphism of the whole group.

2.2.7. Corollary. *Let (G, H) be a pair having Kazhdan's property (T).*

- *(i) If N is a normal subgroup of G and if $p : G \to G/N$ is the projection map, then the pair $(G/N, p(H))$ has Kazhdan's property (T);*
- *(ii) For every $\phi \in Aut(G)$, the pair $(G, \phi(H))$ has Kazhdan's property (T) (in particular for every $g \in G$, (G, gHg^{-1}) has property (T)).*

2.2.8. Remark. By (i) in the previous corollary we obtain that a pair $(K \ltimes H, K)$ have not Kazhdan's property (T) unless K is a Kazhdan group. For example the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, SL_2(\mathbb{Z}))$ doesn't have property (T).

The following proposition is a converse to (i) in corollary 2.2.7 and it gives an idea of the behavior under short exact sequences of the relative property (T):

2.2.9. Proposition. *Let $H \subset K \subset G$ such that H is normal in G. Suppose that the pair (G, H) has Kazhdan's property (T), then: $(G/H, K/H)$ has Kazhdan's property (T) if and only if the same is true for the pair (G, K). In particular, taking $K = G$, we obtain a useful criterion for property (T), that is: "If H is a normal subgroup of G such that G/H and (G, H) has Kazhdan's property (T), then G has Kazhdan's property (T)".*

2.2.10. Remarks. The previous result allows us to give a proof of the property (T) of the group $SL_n(\mathbb{K}) \ltimes \mathbb{K}^n$ for $n \geq 3$. Indeed, using proposition 2.1.5, one can show that the pair $(SL_n(\mathbb{K}) \ltimes \mathbb{K}^n, \mathbb{K}^n)$ has Kazhdan's property (T) for every $n \geq 3$ (see also proposition 2 in chapter 2 of [HV89]) and we already know that $SL_n(\mathbb{K}) \ltimes \mathbb{K}^n/\mathbb{K}^n \simeq SL_n(\mathbb{K})$ is a Kazhdan group when $n \geq 3$. Another application is the property (T) of $(SL_p(\mathbb{Z}) \times SL_q(\mathbb{Z})) \ltimes \mathbb{Z}^{pq}$ when $p, q \geq 3$ (see examples 1.1.10 above).

On the other hand, in the case of discrete groups, the above proposition shows that if $[K : H]$ is finite then the property (T) of the pair (G, H) is equivalent to the property (T) of the pair (G, K). For instance this shows that the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, F \ltimes \mathbb{Z}^2)$ has Kazhdan's property (T) for every finite subgroup F of $SL_2(\mathbb{Z})$. Theorem 2.2.19 below generalizes this remark.

Another application of the above proposition (which appears as lemma 6 in [Fe04]) is the following:

2.2.11. Corollary. *Let $1 \to B \to A \to C \to 1$ is a short exact sequence and let G be a group acting by automorphisms on A, B being invariant. If the pair $(G \ltimes B, B)$ has property (T), then $(G \ltimes C, C)$ has property (T) if and only if $(G \ltimes A, A)$ has property (T)*

2.2.12. Definition. Let H and K be two subgroups of the same countable group G. We say that H and K are commensurable if there exists an element g in G such that $gHg^{-1} \cap K$ is of finite index in both gHg^{-1} and K.

2.2.13. Corollary. *Let H and K be two commensurable subgroups of the same countable group G, then the pair (G, H) has Kazhdan' property (T) if and only if the same is true for the pair (G, K).*

In the same way it is natural to expect that relative property (T) is stable under direct product. Actually we have the following:

2.2.14. Proposition. *If $H_1 \subset G_1$ and $H_2 \subset G_2$, then (G_1, H_1) and (G_2, H_2) have Kazhdan's property (T) if and only if the pair $(G_1 \times G_2, H_1 \times H_2)$ has Kazhdan's property (T).*

The following result has to be compared with theorem 4.1.4 in [CCJJV01] (see also theorem 1.5.11 in [BHV03] or proposition 11 p.26 in [HV89]):

2.2.15. Proposition. *Let H be a subgroup of G such that $\overline{[H, H]} \subset \mathcal{Z}(G)$, then the pair $(G/\overline{[H, H]}, H/\overline{[H, H]})$ has Kazhdan's property (T) if and only if the pair (G, H) has Kazhdan's property (T).*

2.2.16. Remark. An application of the preceding result (which can be found in [C05b]), is the property (T) of the pair $(SL_2(\mathbb{K}) \ltimes H_n(\mathbb{K}), H_n(\mathbb{K}))$ for every $n \geq 1$, where $H_n(\mathbb{K})$ denote the $(2n+1)$-dimensional Heisenberg group. This follows from the fact that $SL_2(\mathbb{K}) \ltimes H_n(\mathbb{K})$ modulo its center, $[H_n(\mathbb{K}), H_n(\mathbb{K})]$, is isomorphic to $SL_2(\mathbb{K}) \ltimes V_{2n-1}(\mathbb{K})$ and from *(iii)* in 2.1.9. Actually it is proved in [C04] that if a Lie group G has a Lie subgroup H being locally isomorphic to $SL_2(\mathbb{K}) \ltimes H_n(\mathbb{K})$ or to $SL_2(\mathbb{K}) \ltimes V_n(\mathbb{K})$ for $n \geq 1$ then the pair $(G, \overline{\mathrm{Rad}(H)})$ has Kazhdan's property (T) ($\mathrm{Rad}(H)$ denoting the radical of H, that is the maximal solvable normal subgroup of H).

The last proposition admits the following generalization:

2.2.17. Theorem. *Let N and Z be two closed subgroups of a group G such that $Z \subset \overline{[N, N]} \cap \mathcal{Z}(G)$, and such that N has abelian image in any compact Lie group, then the pair $(G/Z, H/Z)$ has Kazhdan's property (T) if and only if the pair (G, H) has Kazhdan's property (T).*

We give now the most useful stability results which are some kind of analogue of the results originally due to Kazhdan and Wang (see [BHV03] theorem 1.5.1, see also [HV89] p.33 and 39). Proof of the following statement can be found in [J04] and [J00]:

2.2.18. Theorem. *Let $H \subset L \subset K \subset G$. Suppose that the pair (G, H) has Kazhdan's property (T).*

(*i*) *If H is normal in G and if G/K is amenable in the Eymard sense (that is: $1_G \prec \lambda_{G/H}$) then the pair (K, H) has Kazhdan's property (T);*

(*ii*) *If there exists a L-invariant probability measure on L/H then the pair (G, L) has Kazhdan's property (T).*

In particular, if there exists a L-invariant probability measure on L/H, if G/K is amenable in the Eymard sense and if H is normal in G then the pair (K, L) has Kazhdan's property (T).

2.2.19. Remark. Assertion (i) gives an immediate proof of the property (T) of the pair $(\mathbb{F}_2 \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ ($\mathbb{F}_2$ viewed as a subgroup of $SL_2(\mathbb{Z})$).
We don't know if the conclusion of (i) holds if we drop the hypothesis that H is normal in G. Actually the fact that the pair $(\Gamma, \Gamma \cap H)$ has Kazhdan's property (T) if (G, H) has Kazhdan's property (T), Γ being a lattice in G, is not proved in general. From a theoretical point of view this kind of result is very convenient because in order to prove the property (T) of a given pair it enables us to bring back ourselves in many cases to the proof of the property (T) within the framework of Lie groups, algebraic groups or at least non-discrete groups and thus allows us to have a whole additional machinery.
Nevertheless, in the normal case, we have the following which actually implies (i) in the theorem (taking $\Gamma = K$):

2.2.20. Proposition. *If H is a normal subgroup of G such that (G, H) has property (T) and if Γ is a subgroup of G such that G/Γ is amenable in the Eymard sense, then the pair $(\Gamma, \Gamma \cap H)$ has property (T)*

Using the preceding theorem taking either $K = H$ or $L = G$ we obtain:

2.2.21. Corollary. *If there exists a G-invariant probability measure on G/H then the following are equivalent:*

(*i*) *(G, H) has Kazhdan's property (T);*

(*ii*) *G is a Kazhdan group;*

(*iii*) *H is a Kazhdan group.*

In the case of discrete and finitely generated group an analogue of (i) in theorem 2.2.18 is true even in the non-normal case (see theorem 2.3 in [BR95]):

2.2.22. Theorem. *If G is a finitely generated group, if $H < K < G$ such that (G, H) has Kazhdan's property (T) and if G/K is finite then the pair (K, H) has Kazhdan's property (T).*

2.2.23. Remark. It must be noted that there is no proof of the preceding result in the non-discrete case (replacing "G/K is finite" by " there is an invariant probability measure on G/K").
In [BR95] the author affirms that the statement is true (just saying that the proof is the same that in the case of the usual property (T)) without any proof.

We end this paragraph by a remarkable stability result due to Y. Shalom (see [Sh99a] theorem 3.1):

2.2.24. Theorem. *Let R be a commutative discrete ring with unit. If the pair $(SL_2(R) \ltimes R^2, R^2)$ has Kazhdan's property (T) then the same is true for the pair $(SL_2(R[t]) \ltimes R[t]^2, R[t]^2)$. More precisely, if (Q, ϵ) is a Kazhdan pair for $(SL_2(R) \ltimes R^2, R^2)$, if $0 < \delta < \epsilon$ satisfies $\frac{\delta+2\delta/\epsilon}{1-\delta/\epsilon} \leq \frac{1}{10}$ and if $Q_t := Q \cup F_t$ (where F_t denote the four elementary matrices in $SL_2(R[t])$ having ± 1 off the diagonal), then (Q_t, δ) is a Kazhdan pair for $(SL_2(R[t]) \ltimes R[t]^2, R[t]^2)$.*

Combining property (T) of our preferred pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ with the previous theorem we get:

2.2.25. Corollary. *For every $m \geq 1$, the pair*

$$(SL_2(\mathbb{Z}[x_1, \ldots, x_m]) \ltimes \mathbb{Z}[x_1, \ldots, x_m]^2, \mathbb{Z}[x_1, \ldots, x_m]^2)$$

has Kazhdan's property (T).

2.2.26. Remarks. In the previous corollary one can give an estimate of the Kazhdan constant, actually if S_m denote the subset of
$SL_2(\mathbb{Z}[x_1, \ldots, x_m]) \ltimes \mathbb{Z}[x_1, \ldots, x_m]^2$ consisting of the four elements $(\pm 1, 0)$, $(0, \pm 1)$ and the $4(m+1)$ elementary matrices of $SL_2(\mathbb{Z}[x_1, \ldots, x_m])$ with $\pm 1, \pm x_1, \ldots, \pm x_m$ off the diagonal then:

$$\mathcal{K}((SL_2(\mathbb{Z}[x_1, \ldots, x_m]) \ltimes \mathbb{Z}[x_1, \ldots, x_m]^2, \mathbb{Z}[x_1, \ldots, x_m]^2), S_m) \geq \frac{2}{22^{m+1}}$$

The property (T) of the pairs of the form $(SL_2(R) \ltimes R^2, R^2)$ is the main step in the proof of the property (T) of $SL_n(R)$ for $n \geq 3$ when R is a commutative topological ring with unit which is boundedly generated and such that R contain a finitely generated dense subring (see [Sh99a], main theorem).

3 Relative property (T) from a dynamical viewpoint

The first two parts of this section are devoted to the study of results due to A. Connes, B. Weiss and K. Schmidt about the link between relative rigidities of measure preserving actions on probability spaces and relative property (T). The aim of the part 3.3 is to give a characterization of relative property (T) in terms of relatively rigid measure preserving actions on other types of measured spaces which are due on one part to T. Steger and G. Robertson for general measured spaces and on other part to P.A. Cherix, F. Martin and A. Valette. The last part uses some ideas developed in part 3.1 and relate relative property (T) and relative properties (T_{L^p}) recently introduced by U. Bader, A. Furman, T. Gelander and N. Monod.

3.1 Strong ergodicity

In this section, we give the characterization of relative property (T) in terms of ergodic actions which is originally due to A. Connes and B. Weiss for the usual property (T) (see [CW80]). We begin with some prerequisites:

3.1.1. Definition. A standard Borel space is a topological space X, endowed with the σ-algebra generated by its open subsets, which is Borel-isomorphic to a subset of a complete separable metric space. Then by a "standard probability space" $(X, \mathcal{B}, \mu)$ we mean that X is a standard Borel space and μ is a probability measure defined on $\mathcal{B}$.

In what follows we will need the following results (see for instance [H50] theorem C p. 173, and [P67] § I.1, I.2):

3.1.2. Proposition. *A countable product of standard Borel spaces endowed with the product topology is a standard Borel space.*

3.1.3. Proposition. *For every standard Borel Space X and for every probability measure μ on X without atom, there is a Borel isomorphism $\phi : X \to [0,1]$ such that $\phi_*(\mu)$ is the Lebesgue measure.*

3.1.4. Definition. Given a measure preserving action of G on a standard probability space $(X, \mathcal{B}, \mu)$, we will say that this action is H-ergodic if every $B \in \mathcal{B}$ which is H-invariant satisfies $\mu(B)(1-\mu(B)) = 0$.

3.1.5. Definition. Given a measure preserving action of G on a standard probability space $(X, \mathcal{B}, \mu)$, a sequence $(B_n)_{n\geq 1}$ of Borel subsets of X is said to be asymptotically invariant if for every compact subset K of G one has:
$\sup_{g\in K} \mu(gB_n \triangle B_n) \to 0$ as $n \to \infty$. Such a sequence is said to be non-trivial if moreover $\liminf_{n\to\infty} \mu(B_n)(1-\mu(B_n)) > 0$. The action of G is said to be strongly ergodic if there is no non-trivial asymptotically invariant sequence.

The link between strong ergodicity and relative property (T) is based on the following lemmas:

3.1.6. Lemma. *Let G acting H-ergodically by measure preserving automorphisms on a standard probability space $(X, \mathcal{B}, \mu)$. If $\phi : X \to \mathbb{R}$ is an H-invariant mesurable function then ϕ equal a constant μ-a.e.*

3.1.7. Lemma. *With the same notations, let π_X denote the orthogonal representation associated to this action, that is $\pi_X : G \to \mathcal{U}(L^2(X, \mathcal{B}, \mu))$ is defined by:*

$$[\pi_X(g)(\phi)](x) := \phi(g^{-1}x)$$

for every $g \in G$, $x \in X$ and $\phi \in L^2(X, \mathcal{B}, \mu)$. Let $L^2_0(X, \mathcal{B}, \mu)$ denote the orthogonal of the constants in $L^2(X, \mathcal{B}, \mu)$ and denote by π^0_X the restriction of π_X to this invariant subspace. Then, if the action is not strongly ergodic, one has $1_G \prec \pi^0_X$ (in particular (G, H) doesn't have property (T)).

And now we are in position to state:

3.1.8. Theorem. *The following two assertions are equivalent:*

(*i*) *The pair (G, H) has Kazhdan's property (T);*

(*ii*) *Every H-ergodic measure preserving action of G on a standard non-atomic probability space $(X, \mathcal{B}, \mu)$ is strongly ergodic.*

By proposition 3.1.3 we deduce:

3.1.9. Corollary. *Given a fixed standard non-atomic probability space $(X, \mathcal{B}, \mu)$, the following are equivalent:*

(*i*) *The pair (G, H) has Kazhdan's property (T);*

(*ii*) *Every H-ergodic measure preserving action of G on $(X, \mathcal{B}, \mu)$ is strongly ergodic.*

3.1.10. Remark. The deep part of theorem 3.1.8 is $(ii) \Rightarrow (i)$ and this is exactly the subject of [CW80] (see also [G03] theorem 13.3). Actually they construct an explicit action of G which is H-ergodic (even H-weakly mixing, using corollary 1.5.9 above) but not strongly ergodic. By "H-weakly mixing" we mean that if π_X denotes the corresponding unitary representation, $\pi^0_{X|_H}$ does not contain any finite dimensional sub-representation. The point is that a product of ergodic actions is not necessary ergodic but a product of weakly mixing actions is also weakly mixing (and then in particular is H-ergodic).

3.2 Uniqueness of invariant means

In this section the group G is supposed to be countable. Given an action of G by measure preserving automorphisms on a probability space $(X, \mathcal{B}, \mu)$, there is always at least one G-invariant mean, that is the mean I defined by: $I(\phi) := \int_X \phi d\mu$. Hence, the question of the uniqueness of a G-invariant mean on $L^\infty(X, \mathcal{B}, \mu)$ is whether I is the only G-invariant mean.

3.2.1. Definition. Given a measure preserving action of G on a standard probability space $(X, \mathcal{B}, \mu)$, then G acts in a natural way on $L^\infty(X, \mathcal{B}, \mu)$ ($g.\phi(x) = \phi(g^{-1}x)$ for every $g \in G$, $\phi \in L^\infty(X, \mathcal{B}, \mu)$ and $x \in X$). A mean on $L^\infty(X, \mathcal{B}, \mu)$ is a linear functional $m \in (L^\infty(X, \mathcal{B}, \mu))^*$ satisfying: $m(1_X) = 1_X$ and $m(\phi) \geq 0$ for every $\phi \geq 0$. A mean m on $L^\infty(X, \mathcal{B}, \mu)$ is said to be G-invariant if $m(g.\phi) = m(\phi)$ for every $\phi \in L^\infty(X, \mathcal{B}, \mu)$.

The link between the uniqueness of a G-invariant mean and strong ergodicity of the corresponding action of G is based on the following two results (see [R81] theorem 1.4 and [Sc81] §2):

3.2.2. Proposition. *Let G acting H-ergodically by measure preserving automorphisms on a standard non-atomic probability space $(X, \mathcal{B}, \mu)$, then there is a G-invariant mean $m \neq I$ on $L^\infty(X, \mathcal{B}, \mu)$ if and only if there exists a sequence $(F_n)_{n\geq 1}$ of Borel subsets of X satisfying:*

(i) $\mu(F_n) > 0$ *for all* $n \geq 1$ *and* $\mu(F_n) \to 0$ *as* $n \to \infty$,

(ii) $\mu(gF_n \triangle F_n)/\mu(F_n) \to 0$ *as* $n \to \infty$ *for every* $g \in G$.

The sequence $(F_n)_{n\geq 1}$ is called a Folner sequence.

The following lemma appears in [Sc81] without proof. The proof can be deduced from lemma 2.3 in [JS87] and also holds in the non-countable case:

3.2.3. Lemma. *Given an H-ergodic action of G by measure preserving automorphisms on a non-atomic probability space $(X, \mathcal{B}, \mu)$, then the existence of a non-trivial asymptotically invariant sequence implies the existence of asymptotically invariant sequences with measures prescribed in* $]0, 1[$. *In particular this proves that the existence of a non-trivial asymptotically invariant sequence in X implies the existence of a Folner sequence.*

And now, using the preceding two results and one more time the theorem 3.1.8, one can show $(ii) \Rightarrow (i)$ of the following generalization of theorem 2.5 in [Sc81] (the proof of $(i) \Rightarrow (ii)$ being identic to the proof of lemma 3.1.7 above):

3.2.4. Theorem. *Suppose that G is countable. Given a fixed standard non-atomic probability space $(X, \mathcal{B}, \mu)$, the following are equivalent:*

(i) *The pair (G, H) has Kazhdan's property (T);*

(*ii*) *For every H-ergodic measure preserving action of G on $(X, \mathcal{B}, \mu)$ there is a unique G-invariant mean on $L^\infty(X, \mathcal{B}, \mu)$.*

3.2.5. Remarks. The characterization of the property (T) in terms of uniqueness of invariant mean does not hold in the non-countable case. Actually, given an ergodic action of G by measure preserving automorphisms on a non-atomic probability space $(X, \mathcal{B}, \mu)$, there is always a unique $[G]$-invariant mean on $L^\infty(X, \mathcal{B}, \mu)$ (where $[G]$ denotes the full group of G, that is the group of all Borel automorphisms of $(X, \mathcal{B}, \mu)$ preserving the G-orbits, see [Sc81] theorem 2.6). Moreover, the strong ergodicity of a fixed ergodic measure preserving action of G on a fixed standard non-atomic Borel space and the uniqueness of an invariant mean on $L^\infty(X, \mathcal{B}, \mu)$ are not equivalent in general, even in the countable case (see example 2.7 in [Sc81]).
On other hand, one can show that as soon as there is strictly more than one invariant mean on $L^\infty(X, \mathcal{B}, \mu)$ then the cardinality of the set of invariant means on $L^\infty(X, \mathcal{B}, \mu)$ is at least $2^{\aleph_1}$ (see [C87]).

3.3 Actions on arbitrary measured spaces and spaces with walls

In this section the groups are supposed to be countable.
What follows is based on the paper of G. Robertson and T. Steger (see [RS98]) and the point is the link between conditionally of negative type functions on a group G and *measure definite kernels* which arise from measure preserving actions of G.

3.3.1. Definition. A function $f : G \times G \to \mathbb{R}_+$ is a measure definite kernel if there exists a measured space $(\Omega, \mathcal{B}, \mu)$ and if there exists a family $(B_g)_{g \in G}$ in $\mathcal{B}$ such that:

$$f(g_1, g_2) = \mu(B_{g_2} \triangle B_{g_2}),\ \forall\ g_1, g_2 \in G$$

3.3.2. Proposition. *One has the following correspondence:*

(*i*) *A measure definite kernel is a conditionally of negative type one;*

(*ii*) *The square root of a conditionally of negative type kernel is a measure definite one.*

Now the relative version of theorem 2.1 in [RS98] is:

3.3.3. Theorem. *The two conditions below are equivalent:*

(*i*) *The pair (G, H) has property (T);*

(*ii*) *For every measure preserving action of G on a measured space $(\Omega, \mathcal{B}, \mu)$, if $B \in \mathcal{B}$, one has:*

$$\mu(B \triangle gB) < \infty \quad \forall\ g \in G \quad \Rightarrow \quad \sup_{h \in H} \mu(B \triangle hB) < \infty$$

Recently P.A. Cherix, F. Martin and A. Valette used the same idea to relate property (T) and rigidity property of actions on spaces with measured walls (see theorem 2 in [CMV04]).

First, we recall some definitions.

3.3.4. Definition. Let X be a set endowed with a subset $\mathcal{W} \subset \mathcal{P}(X)$ ("walls" are realized by partitions $\{W, X \smallsetminus W\}$ for $W \in \mathcal{W}$). For every $x, y \in X$, we define:

$$\mathcal{W}(x, y) := \{W \in \mathcal{W}\ /:\ W \text{ separates x and y}\}$$

A "space with measured walls" is given by a 4-tuple $(X, \mathcal{W}, \mathcal{B}, \mu)$, where $\mathcal{B}$ is a σ-algebra of sets in $\mathcal{W}$ and μ is a measure on $\mathcal{B}$, in such a way that $\mathcal{W}(x, y) \in \mathcal{B}$ and $\mu(\mathcal{W}(x, y)) < \infty$ for every $x, y \in X$.

3.3.5. Definition. Let X be a space with measured walls. An "action" of a group G on X is given by a measure-preserving action on $(\mathcal{W}, \mathcal{B}, \mu)$ by $\mathcal{B}$-measurable automorphisms. If H is a subgroup of G, the H-orbit of a point $x_0 \in X$ is said to be bounded if $\sup_{h \in H} \mu(\mathcal{W}(x_0, gx_0)) < \infty$.

Now we can state:

3.3.6. Theorem. *The following are equivalent:*

(*i*) *The pair (G, H) has Kazhdan's property (T);*

(*ii*) *For every action of G on a space with measured walls, the H-orbits are bounded.*

3.3.7. Remarks. This result is based on proposition 3.4.2, and on the correspondence between measure definite kernels introduced in [RS98] and spaces with measured walls (see proposition 2 in [CMV04]). Theorem 3.4.6 shows in particular that a group with property (TR) cannot act properly on a space with measured walls. By a result of M. Sageev (see [Sa95]), any CAT(0) cubical complex carries a structure of space with measured walls. Hence, as a consequence of the preceding theorem, we can deduce the following:

3.3.8. Corollary. *If a group G acts properly on a CAT(0) cubical complex then G doesn't have property (TR).*

By the so-called "center lemma" this implies the existence of a global fixed point. This result was already obtained in 1997 by G. Niblo and L. Reeves (see [NR97]).

3.4 Relative properties (T_{L^p}) and (F_{L^p})

Recently U. Bader, A. Furman, T. Gelander and N. Monod, following ideas developped in [FM04], have studied analogs of property (T) and property (FH) for actions on (uniformly convex and uniformly smooth) Banach spaces like L^p spaces $(1 < p < \infty)$. One of the main result of [BFGM05] is the surprising equivalence between properties (T_{L^p}) for $1 < p < \infty$ and usual property (T), and a trivial adaptation gives the same result in the relative case.

3.4.1. Definition. Let B be a Banach space and let $\mathcal{O}(B)$ denote the group of invertible linear isometries of B. A pair (G, H) is said to have property (T_B) if for any continuous linear isometric representation of G, $\rho : G \mapsto \mathcal{O}(B)$, the quotient representation $\rho' : G \mapsto \mathcal{O}(B/B^{\rho(H)})$ doesn't have almost invariant vectors ($B^{\rho(H)}$ denoting the closed subspace of $\rho(H)$-invariant vectors).

3.4.2. Remarks. If B is uniformly convex and uniformly smooth (ie:B^* uniformly convex) , for any continuous linear isometric representation of G, $\rho : G \mapsto \mathcal{O}(B)$, there exits a natural complement B' for $B^{\rho(H)}$ (see proposition 2.6 in [BFGM05]) and then property (T_B) can be rephrased as the non-existence of almost invariant vectors for $\rho_{|_{B'}}$ for any ρ. When $B = \mathcal{H}$ is an Hilbert space, we obtain exactly the definition of Kazhdan's property (T).

3.4.3. Definition. Let B be a Banach space. A pair (G, H) is said to have property (F_B) if any continuous action of G on B by isometries has a H-fixed point.

3.4.4. Remark. If $B = \mathcal{H}$ is an Hilbert space, this is exactly Serre's property (FH). By an adaptation of a result due to Guichardet (see [G72] and 3.1 of [BFGM05]), one can show that property (F_B) implies property (T_B) for any Banach space B. As it is shown in [BFGM05] (example 2.22), the converse is false in general.

Now we can state results appearing in [BFGM05] showing in particular that property (T) is totally encoded by linear isometric representations in L^p for some fixed $1 < p < \infty$. More precisely:

3.4.5. Theorem. *Let $L^p(\mu)$ denote the usual space of equivalence classes (up to null sets) of measurable p-integrable functions, μ being any σ-finite measure on some standard Borel space. $L^p([0,1])$ will denote $L^p(\mu)$ when μ is the Lebesgue measure on $[0,1]$.*

(i) If $1 \leq p < \infty$, $(T) \Rightarrow (T_{L^p(\mu)})$;

(ii) If $1 < p < \infty$, $(T_{L^p([0,1])}) \Rightarrow (T)$;

(iii) If a pair (G, H) has property (T), there exists a constant $\epsilon(G, H) > 0$ such that (G, H) has property (F_{L^p}) for any $1 \leq p < 2 + \epsilon(G, H)$.

3.4.6. Remark. As for the characterization of property (T) by strong ergodicity (see 3.1), point (ii) of the preceding theorem is based on ideas of A.Connes and B.Weiss (see [CW80]) giving a systematic way to construct a unitary representation in $L^2_0(\mu)$ having almost invariant vectors for a pair (G, H) which doesn't have Kazhdan's property. Actually the same arguments give a linear isometric representation in any $L^p_0(\mu)$ having almost invariant vectors.

4 Property (TR)

4.1 A strong negation for the Haagerup property

Recall that a group G is said to have the Haagerup property if there exists a function on G which is proper and conditionally of negative type, or equivalently if G admits a metrically proper isometric action on some affine Hilbert space (for a good introduction to this property see [CCJJV01]). It is well known that the property (T) and the Haagerup property are incompatible for non-compact groups but the more general property (TR) is also incompatible with the Haagerup property:

4.1.1. Proposition. *If (G, H) has Kazhdan's property (T) and if G has the Haagerup property then H is compact or equivalently: $(TR) \Rightarrow$ "no Haagerup".*

The incompatibility between Haagerup property and the property (T) relative to non-compact subgroups will be highlighted in section 4.2.

4.1.2. Remarks. This last proposition shows that the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ doesn't have the haagerup property, in particular this shows that a semi-direct product of two groups having the Haagerup property need not to have the same property.
On the other hand, using the same kind of arguments, one can answer to the following naive question: "is there a non trivial pair (G, H) having property (T) such that G contains no infinite Kazhdan subgroup ?" The answer is yes and the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ will do. Indeed, if K is a Kazhdan subgroup of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and if

$$\phi : SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \to SL_2(\mathbb{Z})$$

is the quotient map, then $\phi(K)$ is a Kazhdan group which has also the Haagerup property since it is a subgroup of $SL_2(\mathbb{Z})$, and therefore $\phi(K)$ has to be finite. Moreover, while $K/(\mathbb{Z}^2 \cap K)$ is isomorphic to $\phi(K)$, $\mathbb{Z}^2 \cap K$ is a Kazhdan group (being a finite index subgroup in a Kazhdan group) and is also amenable, so $\mathbb{Z}^2 \cap K$ is a finite subgroup of $\mathbb{Z}^2$ but the only one in $\mathbb{Z}^2$ is $\{1\}$. Finally we conclude that K is actually isomorphic to $\phi(K)$ and therefore is finite. The same proof shows that for any discrete groups A and B with the Haagerup property, a Kazhdan subgroup of the semi-direct product $A \ltimes B$ (or of any extension of A and B) has to be finite.

A non-obvious generalization for Lie groups of the preceding remark is the following (see [C04]):

4.1.3. Theorem. *Let G be a connected Lie group. The following conditions are equivalent:*

(*i*) *G is locally isomorphic to (ie: has the same Lie algebra than) $(SO_3(\mathbb{R}))^l \times (SL_2(\mathbb{C}))^m \times (SL_2(\mathbb{C}))^n \times R$ for some solvable Lie group R and integers l, m, n;*

(*ii*) *Every countable subgroup of G has the Haagerup property.*

4.1.4. Remark. This result becomes false if we replace condition (ii) by "No countable subgroup of G has property (TR)" (see remark 4.10 of [C04]).

Now we give a result due to P. Jolissaint (see [J00] theorem 3.2) which improves a result of R.J. Zimmer (see [Z84a] theorem 9.1.1). But before to state the result let us recall some definitions:

4.1.5. Definition. Let G acting on a Borel space $(X, \mathcal{B}, \mu)$. Given another group G', a Borel function $\alpha : X \times G \to G'$ is called a cocycle if $\alpha(x, g_1 g_2) = \alpha(x, g_1)\alpha(xg_1, g_2)$ for all $g_1, g_2 \in G$ and for almost every $x \in X$.
Two cocycles $\alpha : X \times G \to G'$ and $\beta : X \times G \to G'$ are said to be equivalent (or cohomologous) if there exists a Borel function $\phi : X \to G'$ such that: $\beta(x, g) = \phi(x)\alpha(x, g)\phi(xg)^{-1}$ for all $g \in G$ and for almost every $x \in X$.

4.1.6. Theorem. *Let (G, H) be a pair with Kazhdan's property (T). Assume that G acts on a standard probability space $(X, \mathcal{B}, \mu)$ by measure preserving automorphisms in such a way that the action is H-ergodic. Let G' be a group with the Haagerup property and let $\alpha : X \times G \to G'$ be a Borel cocycle. Then $\alpha_{|X \times H}$ is equivalent to a cocycle with range in a compact subgroup of G'.*

4.1.7. Remarks. In particular, if $G' = \mathbb{R}^n \times \mathbb{Z}^m$ then $\alpha_{|X \times H}$ has to be trivial.
This result traduces a relative rigidity of ergodic actions on probability spaces and highlights a little more the incompatibility between property (TR) and the Haagerup property. Actually, in the case $G = H$, Zimmer used the previous result to show that a Kazhdan group does not have any weakly hyperbolic volume preserving ergodic action on a compact manifold (see [Z84a] proposition 9.1.4, see also [Z84b]).

4.2 Property (TR) in Lie groups and algebraic groups

In comparison with the known examples, it is natural to think that Haagerup property is indeed the only obstruction to property (TR). This is actually the case in the framework of connected Lie groups or of linear algebraic groups and until little time ago the problem remained open for locally compact (σ-compact) groups in general.
The following surprising classification result is due to P.A. Cherix, M. Cowling and A. Valette and can be found in [CCJJV01] (chapter 4):

4.2.1. Theorem. *Let G be a connected Lie group, then the following conditions are equivalent:*

(*i*) *G has the Haagerup property;*

(*ii*) *G has not the property (TR);*

(*iii*) *G is locally isomorphic to a direct product of the form:*

$$M \times SO(n_1, 1) \times \ldots \times SO(n_k, 1) \times SU(m_1, 1) \times \ldots \times SU(m_l, 1)$$

where M is an amenable Lie group ($SO(n,1)$ and $SU(n,1)$ denoting respectively the isometry group of the n-dimensional real and complex hyperbolic spaces).

The class of groups for which the equivalence between (i) and (ii) in the preceding theorem holds was recently widened by Y. de Cornulier in [C04]:

4.2.2. Theorem. *Let G be a linear algebraic group over a local field of characteristic zero, then the following conditions are equivalent:*

(*i*) *G has the Haagerup property;*

(*ii*) *G has not the property (TR);*

(*iii*) *G has no simple factor of rank 1 with property (T), and G has no Zariski closed subgroup isomorphic to a direct product of the form $SL_2(\mathbb{K}) \ltimes V_n(\mathbb{K})$ (or perhaps $PSL_2(\mathbb{K}) \ltimes V_n(\mathbb{K})$ if n is even), or to $SL_2(\mathbb{K}) \ltimes H_n(\mathbb{K})$ for any $n \geq 1$.*

Now we give a recent complete characterization of the presence of relative property (T) in Lie groups and algebraic groups over local fields which is due to Y. de Cornulier (see [C05b]).

We need to introduce some notations:

4.2.3. Definition. Let G denote a connected Lie group or an algebraic group over a local field, let $\mathrm{Rad}(G)$ de note the radical of G and let $S(G)$ denote a Levi factor. Then we define $S_{nc}(G)$ as the sum of all non-compact factor of $S(G)$, and $S_{nh}(G)$ as the sum of all factors of $S_{nc}(G)$ with property (T) (or equivalently "no Haagerup" factors). Hence we can define the "T-radical" of G as $R_T(G) := \overline{S_{nh}(G)[S_{nc}(G), \mathrm{Rad}(G)]}$.

Following result generalize theorems 1.9 and 2.10 of [Wa82], and also 4.1.1 and 4.1.2 in [CCJJV01].

4.2.4. Theorem. *Let G be a connected Lie group or an algebraic group over a local field. Then $G/R_T(G)$ has the Haagerup property and $(G, R_T(G))$ has property (T). In particular, given a subgroup H of G, the pair (G, H) has property (T) if and only if the projection of H in $G/R_T(G)$ is relatively compact.*

Theorems above can be used to provides examples of groups without Haagerup property and also without property (TR) (see [C04]):

4.2.5. Theorem. *The following groups have neither the Haagerup property nor the property (TR)*

(*i*) $SO_3(\mathbb{Z}[2^{1/3}]) \ltimes \mathbb{Z}[2^{1/3}]^3$ *for any odd prime number p;*

(*ii*) $SO_3(\mathbb{Z}[1/p]) \ltimes \mathbb{Z}[1/p]^3$, *for any odd prime number p;*

(*iii*) $SO(q)(\mathbb{Z}[\sqrt{2}]) \ltimes \mathbb{Z}[\sqrt{2}]^n$, *for any $n \geq 3$, where q denotes the quadratic form $\sqrt{2}x_1^2 + x_2^2 + \ldots + x_n^2$.*

4.2.6. Remark. This result shows that, in the framework of discrete groups, the property (TR) is actually not the only obstruction to the Haagerup property. It would be interesting to find counter-examples which does not arise from linear algebraic groups.

4.3 Wreath products

In this section we present ideas developed by M. Neuhauser to explore the presence of relative property (T) in wreath products.

4.3.1. Definition. Let Γ and H be two discrete groups, and let X be a set on which Γ acts. Then we define the restricted wreath product of H and Γ relative to X as $H \wr_X \Gamma := (\bigoplus_X H) \rtimes \Gamma$ where the action of Γ on $\bigoplus_X H$ is given by the shift (ie: $(\gamma \cdot n)_x = n_{\gamma^{-1}x}$ for every $\gamma \in \Gamma, n \in \bigoplus_X H$ and $x \in X$). When $X = \Gamma$, this is just the classical wreath product denoted by $H \wr \Gamma$.

The principal result of [N04] is the following:

4.3.2. Theorem. *If $H \wr_X \Gamma$ has property (TR), then Γ has property (TR) or H does not have the Haagerup property.*

The proof of this theorem is based on some analysis of positive definite functions on wreath products and the structure of some of their subgroups.

One of the key points of the proof is certainly this one:

4.3.3. Proposition. *Let $s : H \wr_X \Gamma \to \mathbb{N}, n \mapsto |supp(n)|$. Then for any subgroup U of $\bigoplus_X H$, we have the following dichotomy: $s_{|U}$ is unbounded or $supp(U)$ is finite.*

Given a subgroup U of $H \wr_X \Gamma$, if the projection of U in Γ is infinite then this implies property (TR) for Γ. When U projects finitely in Γ, $N \cap U$ must be infinite and then the previous proposition allows us to distinguish only two cases which are recovered by the following two theorems (proposition 3.2 and 3.3 in [N04]):

4.3.4. Theorem. *If $H \wr_X \Gamma$ has property (TR) relatively to an infinite subgroup U of $\bigoplus_X H$, then $s_{|U}$ must be bounded.*

4.3.5. Theorem. *If $H \wr_X \Gamma$ has property (TR) relatively to an infinite subgroup U of $\bigoplus_X H$ such that $supp(U)$ is finite, then H cannot have the Haagerup property.*

4.4 Small cancellation property

In this section, using a result of section 3.4 and a result due to D. T. Wise, we emphasize the incompatibility between small cancellations conditions and property (TR).

4.4.1. Definition. Let G be a finitely presentable group and let $P :=< X, R >$ be a fixed finite presentation for G. Let R^* denote the set of cyclic permutations of elements in $R \cup R^{-1}$. Then for a given $\lambda \in]0, 1[$, the presentation P satisfies the condition $C'(\lambda)$ if for every $r, r' \in R^*$, one has ($r = uv$ and $r' = uv'$) $\Rightarrow |u| < \lambda \min\{|r|, |r'|\}$ ($|\cdot|$ denoting the length function on G associated to X). We say that a group G is a small cancellation group if G admits a finite presentation satisfying a condition $C'(\lambda)$ for some $\lambda \leq 1/6$.

One of the results obtained by D. T. Wise in [Wi03] is the following:

4.4.2. Theorem. *Every small cancellation group acts properly discontinuously and cocompactly on some CAT(0) cube complex.*

Hence by corollary 3.4.8 above, we can deduce the following:

4.4.3. Proposition. *Let G be a small cancellation group and let H be a subgroup. If (G, H) has property (T), then H must be finite. In particular, $(TR) \Rightarrow$ "no small cancellation".*

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