Torsion groups of subexponential growth cannot act on finite-dimensional CAT(0)-spaces without a fixed point

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Abstract

We show that finitely generated groups which are Liouville and without infinite finite-dimensional linear representations must have a global fixed point whenever they act by isometry on a finite-dimensional complete CAT(0)-space. This provides a partial answer to an old question in geometric group theory and proves partly a conjecture formulated in [NOP22]. It applies in particular to Grigorchuk's groups of intermediate growth and other branch groups as well as to simple groups with the Liouville property such as those found by Matte Bon and by Nekrashevych. The method of proof uses ultralimits, equivariant harmonic maps, subharmonic functions, horofunctions and random walks.

1 Introduction

The existence of infinite, finitely generated, torsion groups has a long and rich history which started with a problem posed by Burnside around the year 1900. The discovery that such groups actually could exist came in the 1960s by work of Golod and Shafarevich. Since then other infinite torsion groups with additional important features appeared, notably the papers of Novikov-Adian on infinite Burnside groups and the Grigorchuk groups having intermediate growth, see [CSD21, Os22] for recent discussions.

From the viewpoint of geometric group theory it is a natural question as to whether such groups can act on CAT(0)-spaces, indeed this topic already appeared in the foundational essay by Gromov [Gr87, sect. 4.5.C]. By a theorem of Schur [S11] (or by Selberg's lemma) it is known that any linear finitely generated torsion group must be finite, thus one can say that every action by a finitely generated torsion group on the classical symmetric spaces of nonpositive curvature must fix a point by the Cartan fixed point theorem. Another instance is the elegant argument in Serre's book [Se80, sect. 6.5] that finitely generated torsion groups acting on a tree must have a global fixed-point. By the end of the 1990s it was a well-known open problem formulated as follows (see [Sw99, Be00]):

Can a finitely generated group that acts properly discontinuously (and cocompactly) by isometry on a proper CAT(0)-space contain an infinite torsion group?

At least at that time the expected answer depended on whether the cocompact assumption is included or not. It was again referred to as one of the key open problems in [Ca14].

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Grigorchuk constructed a continuum of finitely generated torsion groups of growth strictly between polynomial and exponential (thus amenable) in the early 1980s, see [G80, G84, dlH00, CSD21] and references therein. Amenable groups admit proper actions on Hilbert spaces ([BCV95]). Both Grigorchuk groups and Burnside groups, that is, infinite torsion groups of bounded exponent, can act without a global fixed point on infinite-dimensional CAT(0) cubical complexes (see [Sa95, Os18, Sc22]). Also note that [Se80, Theorem 25] shows that every infinitely generated group acts on a tree without a global fixed-point, see also [BH99, II.7.11] for another infinitely generated torsion group example. We refer to the recent paper [HO22] for a historical survey on this problem, its variants and with references to the many previous partial results. In [NOP22, HO22] the following conjecture is stated:

Conjecture 1. *([NOP22]) Every finitely generated group acting without a global fixed point on a finite-dimensional CAT(0) complex contains an element of infinite order.*

For CAT(0) cubical complexes this is known from a paper by Sageev [Sa95] (see [LV20] for a discussion), and recently extended to CAT(0) cubical complexes without infinite cubes in [GLU24]. Norin, Osajda, and Przytycki recently proved the conjecture for two-dimensional CAT(0) complexes (with certain additional minor assumptions) as the main result of [NOP22]. We say that a finitely generated group is *weakly Liouville* if it admits a symmetric measure μ of finite support for which the drift ℓ of the corresponding random walk is 0:

$$\ell = \lim_{n\to\infty} \frac{1}{n} \sum_{g\in\Gamma} \|g\| \, \mu^{*n}(g) = 0.$$

It implies amenability and the property gets its name because it is known that this is equivalent to the non-existence of non-constant bounded harmonic functions for the Laplacian defined by μ, see [La23]. This class of groups contains all groups of subexponential growth due to Avez ([La23]), many groups acting on rooted trees [AOMV16] and a rich collection of groups coming from dynamical systems [M14, MNZ23]. Our main result is:

Theorem 2. *Let Γ be a finitely generated group which is weakly Liouville and such that every homomorphism $\Gamma \to GL_N(\mathbb{R})$ has finite image for any finite N. Whenever Γ acts by isometries on a complete CAT(0)-space Y of finite dimension, it must have a global fixed point in Y.*

Branch groups, which are special types of groups acting on rooted trees, are not linear as shown by Delzant-Grigorchuk and a more general theorem of Abért [A06], thus they have no faithful linear representation, and at the same time many such groups are just-infinite, meaning that every proper quotient is finite [BGZ03], thus whenever they are Liouville, see [AOMV16], the theorem applies.

In view of Schur's theorem, Conjecture 1 is true for any finitely generated group admitting an infinite linear representation, so Theorem 2 implies:

Corollary 3. *Conjecture 1 is true for groups which are weakly Liouville.*

The following special case applies in particular to Grigorchuk's torsion groups of intermediate growth:

Corollary 4. *Let Γ be a finitely generated torsion group of subexponential growth. Whenever Γ acts by isometries on a complete CAT(0)-space Y of finite dimension, it must have a global fixed point in Y.*

As pointed out above, the finite generation and finite-dimensionality assumptions are necessary, and let us also emphasize that our result is more general in the sense that:

- the space Y is not necessarily proper or a CAT(0)-complex, and it is not assumed to have a large isometry group.
- no assumption on the action is made, such as being non-elementary (in the sense of assuming no fixed-point in ∂Y), properly discontinuous, or being part of a cocompact action.

It was previously shown by Caprace-Monod in [CM13, Corollary E] that finitely generated groups of intermediate growth cannot be a *discrete* subgroup of $Isom(Y)$ under the assumption that the space Y is a proper CAT(0)-space having a *cocompact* isometry group. On the other hand, Grigorchuk groups are by construction subgroups of the automorphism groups of rooted binary trees, thus they are subgroups of the isometry groups of certain proper CAT(0)-spaces possessing a global fixed-point, the root. One should also keep in mind that Caprace-Lytchak, [CL10, Theorem 1.6], extended theorems of Burger-Schroeder and Adams-Ballmann to CAT(0) spaces of finite telescoping dimension showing that amenable groups either fix a point at infinity or leave a flat invariant. Our theorem by-passes the trouble with that there might be a fixed point on the visual boundary at infinity. In this connection, we can mention a paper of Papasoglu-Swenson [PS18, Theorem 3.17] which shows the following: Assume that G acts properly and cocompactly on a proper CAT(0)-space. If Γ is an infinite torsion subgroup of G then it cannot fix a point in the limit set of Γ. (Note that in our setting, say assuming finite generation and subexponential growth but no properness or cocompactness, the limit set is empty since by Theorem 4 any orbit is bounded.)

Groups which have a global fixed point whenever they act on simplicial trees are said to have *Serre's property FA*. This is the 1-dimensional version of *property* FA_n in [F09], that is, having a global fixed point whenever acting by isometry on n-dimensional CAT(0) cell complexes. The literature on these topics is vast, and we cannot survey this here. For property FA and the groups of the type of most immediate relevance for our paper, we can refer to [DG08, GS23] for results and more information.

We may also formulate:

Corollary 5. *Let Γ be a finitely generated simple group which is weakly Liouville. Whenever Γ acts by isometries on a complete CAT(0)-space Y of finite dimension, it must have a global fixed point in Y.*

This applies to the uncountable family of such groups established by Matte Bon [M14] as well as to the simple groups of intermediate growth that Nekrashevych constructed in [Ne18].

Our proof uses the technique of ultralimits of spaces and equivariant harmonic maps, heavily inspired by Bourdon's paper [Bo12], combined with substantial arguments and results from the first author's paper [Iz23] (much of what is not repeated here, making our paper shorter than what it would be if all needed details from [Iz23] were reproduced). The use of random walks might be somewhat surprising. We think that the present method, in particular the combination of ultralimits, equivariant harmonic maps, subharmonic functions on the group, horofunctions and random walks, could have further applicability in geometric group theory.

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2 Ultralimits of metric spaces

For a detailed exposition on this topic we refer to [CSD21, Chapter 11] or [Bo12, Iz23, Ly05], and for basics on CAT(0)-spaces we refer to [BH99]. Let ω denote a non-principal ultrafilter on $\mathbb{N}$. Let Y_j be metric spaces and p_j a point in each Y_j. The associated ultralimit metric space (Y_∞, d) is defined as follows. First, let Y^∞ be all (one-sided) sequences $y_j \in Y_j$ such that $d(y_j, p_j)$ is bounded in j. Via the triangle inequality, one can therefore define the following function on Y^∞,

$$d((x_j),(y_j)) := \omega - \lim d(x_j, y_j).$$

This function is clearly symmetric and satisfies the triangle inequality. Now we identify all sequences for which $d((x_j),(y_j)) = 0$. The set of equivalence classes, denoted by Y_∞, is thus a metric space with the same d well-defined on the quotient. It is rather immediate that if each Y_j is a complete CAT(0)-space, then so is (Y_∞, d).

The dimension of a CAT(0)-space Y is defined as follows (see [Kl99] for various equivalent definitions of dimension for CAT(0)-spaces). Given a set of points $x_0, x_1, ..., x_n$ in Y, one has a barycentric simplex $\sigma : \Delta_n \to Y$ defined by mapping the point $(\alpha_0, ..., \alpha_n)$ in the standard simplex to the unique minimizer (the *barycenter*), which exists by standard CAT(0)-space theory, see [BH99, Ch. II.2], of the uniformly convex function

$$f(y) = \sum \alpha_i d(x_i, y)^2.$$

The dimension of Y is the greatest number D such that for some points $x_0, x_1, ..., x_D$ the corresponding barycentric simplex is non-degenerate. A simplex σ is degenerate if $\sigma(\Delta) = \sigma(\partial\Delta)$, see [Kl99, Definition 4.7]. We repeat the argument by Lytchak in [Ly05, Lemma 11.1] (alternatively see [Iz23, Proposition 6.10]) to show:

Proposition 6. *If the dimensions of Y_j is at most $D < \infty$, then the dimension of Y_∞ is at most D.*

Proof. Let $x_0, x_1, ..., x_n$ be points in Y_∞ such that the associated barycentric simplex σ is not degenerate. Represent these classes by actual sequences of points in Y, denoted $z_{i,j}$, here $j \to \infty$. By the continuity of barycenters in CAT(0)-spaces and the definition of barycentric simplices implies that σ is the ultralimit of the simplices defined by $z_{i,j}$ in Y_j. If all of these simplices would be degenerate then so would σ. □

3 Existence of equivariant harmonic maps with positive energy when passing to certain ultralimits

Let Γ be a group generated by a finite set S. Assume that Γ acts by isometry on a complete metric space Y, that is, there is a homomorphism $\rho : \Gamma \to Isom(Y)$. A *($\rho$-)equivariant map* is a map $f : \Gamma \to Y$ such that

$$f(g) = \rho(g) f(e)$$

for all $g \in \Gamma$. These maps are determined by $f(e)$ and hence are just the orbit maps, $g \mapsto \rho(g) f(e)$.

Let μ be a symmetric probability measure on Γ whose support contains S. We further assume that μ has finite second moment with respect to ρ:

$$\sum_\Gamma d(f(e), f(g))^2 \mu(g) < \infty.$$

This property does not depend on the choice of an equivariant map f. Consider the following *(μ-)energy* of maps $f : \Gamma \to Y$,

$$E(f) = \sum_{\Gamma} d(f(e), f(g))^2 \mu(g),$$

which clearly is finite by the finite second moment assumption. We are trying to find a map with minimal energy among the maps which are equivariant. Such maps are called *harmonic.* Intuitively we want to place our orbit in a minimal position, which is a generalization of the consideration of the infimal displacement for single isometries.

When Y is a complete CAT(0)-spaces it is well-known, and already used in the previous section about barycenters, that a convex function such as

$$x \mapsto \sum_{\Gamma} d(x, \rho(g)x)^2 \mu(g),$$

which is the energy of the map defined by $f(e) = x$, either has a minimum in Y or a minimizing sequence p_j that escapes to infinity, that is $d(x, p_j) \to \infty$ for any $x \in Y$.

In order to deal with the second scenario, we assume that Γ acts on a finite-dimensional CAT(0)-space Y without a global fixed point. We now follow Bourdon's techniques for L^p-spaces in [Bo12]. Let

$$\delta(x) = \max_{s \in S} d(x, s(x)),$$

where we from now on suppress ρ in the notation. Later on, we want $\inf_{x \in Y} \delta(x) > 0$, which clearly would imply that

$$\inf E(f) \geq \inf_{x \in Y} \delta(x)^2 \cdot \min_{s \in S} \mu(s) > 0$$

over all equivariant maps f, see below. Since S is a generating set, the action has a global fixed point in Y if and only if there exists an equivariant map f such that $E(f) = 0$.

Suppose instead that $\inf_{x \in Y} \delta(x) = 0$. We will now do the initial scaling of the action using ultralimits.

First, we rewrite a lemma that Bourdon attributes to Shalom [Sh00, Lemma 6.3].

Lemma 7. *Assume that $\inf_{x \in Y} \delta(x) = 0$ but there is no global fixed point. Then for any integer $n > 0$, there exists $r_n > 0$ and $v_n \in Y$ such that $\delta(v_n) \leq r_n/n$ and for all v with $d(v_n, v) < r_n$, it holds that*

$$\delta(v) \geq r_n/2n.$$

Proof. Let n be a positive integer. Take $w_1 \in Y$ such that $\delta(w_1) \leq 1/2n$. If $\delta(v) \geq 1/4n$ for every v with $d(v, w_1) < 1/2$, we let $v_n = w_1$ and $r_n = 1/2$. Otherwise, there is w_2 with $d(w_1, w_2) < 1/2$ and $\delta(w_2) \leq 1/4n$. If $\delta(v) \geq 1/8n$ for every v with $d(v, w_2) < 1/4$, then we let $v_n = w_2$ and $r_n = 1/4$. This procedure terminates, because otherwise we would get a sequence w_k such that $d(w_k, w_{k+1}) < 2^{-k}$. By completeness of Y this Cauchy sequence converges to a point $w \in Y$. But note that

$$|d(w, s(w)) - d(w_k, s(w_k))| \leq 2d(w, w_k),$$

which implies that $\delta(w) = 0$ since $\delta(w_k) \leq 1/(2^k n)$. This contradicts the assumption that the orbit did not have a global fixed point. $\square$

Using this lemma we can now show, what essentially is part of [Bo12, Prop. 3.1]:

Proposition 8. *Let Γ be a group generated by a finite set S. Assume that it acts on a finite-dimensional complete CAT(0)-space Y, with*

$$\inf_{x \in Y} \max_{s \in S} d(x, s(x)) = 0$$

but there is no global fixed point. Then by passing to a certain rescaled ultralimit action, there is a finite-dimensional complete CAT(0)-space Y_∞ on which Γ acts without a global fixed point and

$$\inf_{x \in Y_\infty} \max_{s \in S} d(x, s(x)) > 0.$$

Proof. Let $\lambda_n = n/r_n$ with r_n and v_n from Lemma 7 with $\delta(x) = \max_{s \in S} d(x, s(x))$ whose infimum is zero. Consider the metric spaces $\lambda_n Y = (Y, \lambda_n d)$, which of course again is a complete CAT(0) on which Γ acts by isometry via the same action. Take the ultralimit space Y_∞ of the sequence of pointed spaces $(\lambda_n Y, v_n)$ as in the previous section. Thanks to Proposition 6 it is still finite-dimensional. Let

$$\delta_n(x) = \max_{s \in S} \lambda_n d(x, s(x)),$$

so $\delta_n(v_n) \leq 1$ and $\delta_n(v) \geq 1/2$ for all v in the $\lambda_n d$-ball of radius n around v_n. Thanks to the condition $\delta_n(v_n) \leq 1$ we have that Γ still acts (by isometry) on the ultralimit and thanks to the inequality $\delta_n(v) \geq 1/2$ on larger and larger balls around v_n we have that this new action of Γ on Y_∞ satisfy $\inf_{x \in Y_\infty} \delta(x) > 0$. □

In summary, possibly by replacing Y with an ultralimit we have that Γ acts on a finite dimensional CAT(0)-space Y such that

$$\delta(x) = \max_{s \in S} d(x, s(x))$$

is bounded away from 0, which of course in particular excludes the existence of a global fixed-point. This implies that the energy of any equivariant map f is also bounded away from 0, since

$$E(f) = \sum_\Gamma d(f(e), f(g))^2 \mu(g) = \sum_\Gamma d(f(e), gf(e))^2 \mu(g) \geq \delta(f(e))^2 \min_{s \in S} \mu(s).$$

Now we pass to the other part of [Bo12, Prop. 3.1]:

Proposition 9. *Let Γ be a group generated by a finite set S, and μ a symmetric probability measure with finite second moment whose support contains S. Assume that it acts on a finite-dimensional complete CAT(0)-space Y without a global fixed point. Then there is a finite-dimensional complete CAT(0)-space Z on which Γ acts by isometry, such that there is a harmonic equivariant map $h : \Gamma \to Z$ with strictly positive energy.*

Proof. In view of Proposition 8, by passing to a certain ultralimit if necessary, we may assume that $\delta(x) = \max d(x, s(x))$ bounded away from 0 and hence the energy $E(f)$ of equivariant maps f is also bounded away from 0 in view of the above displayed inequality.

Let p_j be a sequence such that the equivariant maps with $f_j(e) := p_j$ have energy approaching the minimum. As remarked above in case the the p_j stay bounded there is an equivariant harmonic map because of the uniform convexity property of complete CAT(0)-spaces. In any case, we can argue as follows, consider the ω-ultralimit of (Y, p_j). By the above displayed inequality we have $\delta(p_j)$ is bounded from above, say by twice the square root of the minimal energy. This guarantees that the Γ action extends to this limit space by isometry, and as before this limit is a complete finite dimensional CAT(0)-space (in particular in view of Proposition 6). Likewise we may also define the map $h : \Gamma \to Y_\infty$ by $h(g) = \omega - \lim f_j(g)$. Since for each j this is equivariant so is g. We now claim it is moreover harmonic. Clearly, $E(h) = \omega - \lim E(f_j) = \inf_f E(f)$. To show that this value of the energy is also minimal in the limit space, take h_1 an equivariant map $\Gamma \to Y_\infty$ and consider its energy. Take a sequence w_j representing $h_1(e)$. Define the corresponding equivariant maps $f_{1,j}(e) = w_j$. For a given $g \in \Gamma$ let z_j be a sequence representing $g(w_j)$, by equivariance we now have

$$0 = \omega - \lim d(gw_j, z_j) = \omega - \lim d(f_{1,j}(g), z_j)$$

which shows that we can represent h_1 by the maps $f_{1,j}$, but then it follows that

$$E(h_1) = \omega - \lim E(f_{1,j}) \geq \inf_f E(f) = E(h)$$

. □

4 Proofs of the main theorems

Let Γ be a weakly Liouville group. Then, by definition, Γ admits a symmetric probability measure of finite support S that generates Γ and having zero drift. For any isometric action $\rho : \Gamma \to Isom(X, d)$, the orbit map $g \mapsto \rho(g)x$ from the group with its word metric to the metric space (X, d) is Lipschitz. Taken together this implies that

$$\ell = \lim_{n\to\infty} \frac{1}{n} \sum_{g\in\Gamma} d(x, \rho(g)x)\mu^{*n}(g) = 0,$$

where μ^{*n} are the convolution powers giving the distribution of the random walk at time n. We emphasize that this holds for any isometric action of Γ on a metric space.

Let Γ act by isometry on a finite-dimensional complete CAT(0)-space Y. Assume that Γ has no global fixed point in Y, and take any measure with finite second moment so that its support contains the set S. Then the ultralimits, in two steps as explained in the previous section, Propositions 8 and 9, give another finite-dimensional complete CAT(0) space Z on which Γ acts by isometry and with the existence of an equivariant harmonic map with positive energy. Since the energy is strictly positive there is no fixed point.

As is explained in the appendix we may modify the measure μ so that we can assume that its support contains all non-identity elements of Γ keeping the drift 0. The notion of horofunctions b_ξ with respect to a base point o also extends to ultralimits in natural way ([Iz23, Lemma 6.6]). We recall the important Proposition 6.9 in [Iz23] slightly tailored to fit the present notations:

Proposition 10. *([Iz23, Prop. 6.9]) Let Z be a complete CAT(0)-space. Take $o \in Z$ and set $Z_\infty = \omega - \lim(Z, o)$. Let Γ be a countable group equipped with a symmetric probability measure μ whose support contains $\Gamma \setminus \{e\}$. Suppose that Γ acts by isometry on Z and that there exists an equivariant harmonic map $f : \Gamma \to Z$. If the drift $\ell = 0$, then for almost all random walk trajectories σ, there exists $\xi(\sigma) \in Z_\infty \cup \partial Z_\infty$ such that a function $\varphi_\sigma : \Gamma \to \mathbb{R}$ defined by*

$$\varphi_\sigma(\xi) = \begin{cases} d(\xi(\sigma), f(g)) - d(\xi(\sigma), f(e)), & \text{if } \xi(\sigma) \in Z_\infty \\ b_{\xi(\sigma)}(f(g), f(e)), & \text{if } \xi(\sigma) \in \partial Z_\infty \end{cases} \tag{1}$$

is μ-harmonic.

The last notion of μ-harmonic means that the Laplacian defined by μ annihilates φ_σ. The proof requires a few pages and since it can be used in a form directly applicable in the present paper, we refer to [Iz23] for the proof.

Using this proposition one can show the following:

Proposition 11. *Let Z be a complete CAT(0)-space of finite dimension and Γ be a countable group equipped with a symmetric probability measure μ whose support contains $\Gamma\setminus\{e\}$. Suppose that Γ acts on Z by isometry and that there exists an equivariant harmonic map $f : \Gamma \to Z$. If the drift $\ell = 0$, then there is a Γ-invariant convex subset F isometric to a finite dimensional (possibly zero-dimensional) Euclidean space.*

Proof. Although this assertion is not explicitly stated in [Iz23], it can be proved by following the arguments in [Iz23, §6.4] appropriately in our setting. We sketch the outline here.

Suppose $\xi(\sigma) \in Z_\infty$ in Proposition 10. Then, as in the first part of [Iz23, §6.4], there is a point in Z_∞ fixed by Γ, and hence every orbit of Γ in Z_∞ is bounded. In particular, any orbit of Γ in Z is bounded, and this implies the existence of a global fixed point, an invariant zero-dimensional Euclidean space, in Z via the Bruhat-Tits fixed point theorem, [BH99, II.2.8].

If $\xi(\sigma) \in \partial Z_\infty$ in Proposition 10, then the horofunction $\varphi_\sigma : g \mapsto b_{\xi(\sigma)}(f(g), f(e))$ is μ-harmonic. Since horofunctions on CAT(0) spaces are convex, the pull-back of them by a harmonic map are μ-subharmonic functions on Γ in general (see for example [Iz23, Prop. 2.17]). Here a function u on Γ is μ-subharmonic if the Laplacian of u defined by μ is nonpositive. If u is the pull-back of a convex function on a CAT(0) space by a harmonic map, then the absolute value of its Laplacian measures the strength of the convexity of the function around the image of the harmonic map. Thus the fact $\varphi_\sigma : g \mapsto b_{\xi(\sigma)}(f(g), f(e))$ is μ-harmonic means that the horofunction has the weakest convexity around the image $f(\Gamma)$ of f. This suggests that the convex hull of $f(\Gamma)$ is flat, and it can be proved as follows.

Since we assume that Z is finite-dimensional, a compactness principle [Iz23, Theorem 6.12] tells us that $\xi(\sigma)$ actually lies in ∂Z. The compactness principle and [Iz23, Lemma 5.1] imply that there exist $\xi \in \partial Z$ and a bi-infinite geodesic c_g passing through $f(g)$ and terminating at ξ for each $g \in \Gamma$, and all the c_g geodesics are parallel to each other (in order to prove this, we use the weakest convexity of the horofunction). Then an argument involving the splitting theorem, see [BH99, Ch. II.2], leads us to see that there is a splitting of the convex hull of $f(\Gamma)$ one of whose factor is a Hilbert space, and the action of Γ also splits along this splitting. Taking a maximal flat factor in this splitting, and applying the argument above to the other factor, we see that the action of Γ on the other factor must fix a point; the action leaves the product F of the maximal flat factor and the fixed point in the other factor invariant. Since Z is finite-dimensional, so is F, and it becomes the desired flat subset in Z. □

Recall now that we have excluded the existence of a global fixed point in Z in the preceding section. Therefore, by Proposition 11, we obtain a representation of Γ into the group of affine isometries of a Euclidean space of finite and strictly positive dimension. Thus we obtain a linear representation of Γ over the real numbers. By the assumption of Theorem 2, the image of this representation is finite. Hence by the Cartan-Bruhat-Tits fixed point theorem it must fix a point in this flat. This leads to a contradiction to the construction of the space Z from Proposition 9. The conclusion is then that there must be a global fixed point for the original action, $\rho : \Gamma \to Isom(Y)$ which concludes the proof of Theorem 2.

In the case of a finitely generated torsion group, by Schur's theorem the linear representation above must have a finite image. Hence by the Cartan-Bruhat-Tits fixed point theorem it must fix a point in this flat. This leads to a contradiction to the construction of the space Z from Proposition 9. The conclusion is then that there must be a global fixed point for the original action, $\rho : \Gamma \to Isom(Y)$ which concludes the proof of Theorem 4.

In the case Γ is a simple group that is weakly Liouville, then the linear representation above must either be trivial or faithful. The former scenario directly contradicts the lack of fixed points, and in the latter scenario we can invoke the Tits alternative theorem and conclude that Γ must be virtually solvable (since nonamenable groups always have positive drift of random walks). Let H be a finite index solvable subgroup of Γ, then by considering the kernel of the action of Γ on the coset space Γ/H we must have, by simplicity, that $H = \Gamma$. As a both solvable and simple group, Γ must be the trivial group or a cyclic group of prime order, and again we have a fixed point in Z, leading to the desired contradiction, which allows us to conclude that there must be a global fixed point for the original action, $\rho : \Gamma \to Isom(Y)$.

Remark 12. Suppose $\Gamma = \mathbb{Z}$ then the drift is surely 0 for the simple symmetric random walk. This group may act without a fixed point, precisely when being a parabolic or hyperbolic isometry of Y. Note that the proof here applies all the way until the point that it preserves a flat in Z so it has an axis (and the hypothesis of only possessing finite linear representations is obviously not satisfied). This clearly does not contradict the no fixed-point assumption, and is also consistent with the case when Γ acts parabolically (i.e. without an axis) on the original space Y.

Appendix: the drift of random walks for convex combinations

Let Γ be a countable group and d a left-invariant metric on Γ. Given a probability measure μ on Γ with finite first moment, the *drift* of the random walk defined by μ and its convolution powers μ^{*n} is

$$l_d(\Gamma, \mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{g\in\Gamma} d(e,g)\mu^{*n}(g)$$

which exists by the well-known subadditivity of

$$L^n_d(\Gamma, \mu) = \sum_{g\in\Gamma} d(e,g)\mu^{*n}(g).$$

See for example [La23, Ch. 3] for details. We will establish the following inequality:

Proposition 13. *Consider a convex combination of μ^{*n}, that is, $\sum_{n=1}^{\infty} a_n\mu^{*n}$, where $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n = 1$. Then we have*

$$l_d\left(\Gamma, \sum_{n=1}^{\infty} a_n\mu^{*n}\right) \leq \left(1 + \sum_{n=1}^{\infty} na_n\right) l_d(\Gamma, \mu).$$

Remark 14. We should mention that Forghani gave an exact expression of $l_d(\Gamma, \sum_{n=1}^{\infty} a_n\mu^{*n})$ in [Fo17, Theorem 4.5]. However, we do not need this exact expression. Also the first author gave the proof of Proposition 13 in [Iz23] under certain additional assumptions; but it turns out to be overcomplicated. As we will see below, the proof of Proposition 13 presented here is quite elementary.

Note that Proposition 13 tells us that $l_d(\Gamma,\mu) = 0$ implies $l_d(\Gamma, \sum_{n=1}^{\infty} a_n\mu^{*n}) = 0$ as long as $\sum_{n=1}^{\infty} na_n < \infty$. Furthermore, assuming $\operatorname{supp}\mu$ generates Γ and taking $a_n \neq 0$ for each $n \in \mathbb{Z}_{>0}$, we obtain such a measure with $\operatorname{supp} \sum_{n=1}^{\infty} a_n\mu^{*n} \supset \Gamma \setminus \{e\}$. Now suppose that Γ is weakly Liouville. Then there exists a symmetric probability measure μ with finite support S that generates Γ, and having $l_{d_S}(\Gamma,\mu) = 0$, where d_S denotes the word metric on Γ with respect to S. Since the identity map $\mathrm{id}\colon (\Gamma, d_S) \to (\Gamma, d)$ is Lipschitz for any left invariant metric d on Γ, we see that $l_d(\Gamma,\mu) = 0$ for any left invariant metric d on Γ. Moreover, since the support of μ^{*n} is finite for each n, $\sum_{\Gamma} d(e,g)^2 d\mu^{*n}(g) < \infty$ for each n. Thus, by taking $\{a_n\}_{n\in\mathbb{Z}_{>0}}$ suitably, we can make a measure $\sum_{n=1}^{\infty} a_n\mu^{*n}$ to have finite second moment. Letting ν to be $\sum_{n=1}^{\infty} a_n\mu^{*n}$, we obtain the following:

Corollary 15. *Let Γ be a weakly Liouville group. Then Γ admits a probability measure ν satisfying*

(i) *ν is symmetric with finite second moment,*

(ii) $\operatorname{supp}\nu \supset \Gamma \setminus \{e\}$, *and*

(iii) $l_d(\Gamma,\nu) = 0$ *for any left invariant metric d.*

Proof of Proposition 13. In what follows, we fix a left invariant metric d on Γ, drop d from $l_d(\Gamma,\mu)$ and $L_d^n(\Gamma,\mu)$, and denote them simply by $l(\Gamma,\mu)$ and $L^n(\Gamma,\mu)$ respectively.
Set
$$\tilde{L}^n := \max\{L^m(\Gamma,\mu) \mid m \leq n\}.$$
Then $\{\tilde{L}^n\}_{n\in\mathbb{Z}_{>0}}$ is a nondecreasing sequence, and satisfies $\tilde{L}^n \geq L^n(\Gamma,\mu)$ for each $n \in \mathbb{Z}_{>0}$. Furthermore it is subadditive as shown as follows: Let k_n be the smallest positive integer satisfying $\tilde{L}^{k_n} = \tilde{L}^n$, then, for any $m < k_n$, we have $L^m(\Gamma,\mu) < \tilde{L}^{k_n}$, and hence $\tilde{L}^{k_n} = \max\{L^m(\Gamma,\mu) \mid m \leq k_n\} = L^{k_n}(\Gamma,\mu)$. Thus, for $m < k_n$, we have
$$\tilde{L}^n = \tilde{L}^{k_n} = L^{k_n}(\Gamma,\mu) \leq L^{k_n-m}(\Gamma,\mu) + L^m(\Gamma,\mu) \leq \tilde{L}^{k_n-m} + \tilde{L}^m \leq \tilde{L}^{n-m} + \tilde{L}^m$$
by the subadditivity of $\{L^n(\Gamma,\mu)\}_{n\in\mathbb{Z}_{>0}}$, the definition and the nondecreasing property of $\{\tilde{L}^n\}_{n\in\mathbb{Z}_{>0}}$. For $m \geq k_n$, assuming $m < n$, we see that
$$\tilde{L}^n = \tilde{L}^{k_n} \leq \tilde{L}^m \leq \tilde{L}^m + \tilde{L}^{n-m}$$
by the nondecreasing property of $\{\tilde{L}^n\}_{n\in\mathbb{Z}_{>0}}$. In any case, we have shown that $\{\tilde{L}^n\}_{n\in\mathbb{Z}_{>0}}$ is subadditive. In particular, we have
$$\tilde{L}^{ik} \leq i\tilde{L}^k \tag{2}$$
for any positive integers i and k. As already remaked above it is well-known that the subadditivity of $\{L^n\}_{n\in\mathbb{Z}_{>0}}$ and $\{L^n(\Gamma,\mu)\}_{n\in\mathbb{Z}_{>0}}$ implies the existence of the following limits:
$$\lim_{n\to\infty} \frac{\tilde{L}^n}{n} = \inf_{n\in\mathbb{Z}_{>0}} \frac{\tilde{L}^n}{n}, \quad \lim_{n\to\infty} \frac{L^n(\Gamma,\mu)}{n} = \inf_{n\in\mathbb{Z}_{>0}} \frac{L^n(\Gamma,\mu)}{n}. \tag{3}$$
If $\{L^n(\Gamma,\mu)\}_{n\in\mathbb{Z}_{>0}}$ is bounded, then so is $\{\tilde{L}^n\}$, and the both limit above are equal to 0. If $\{L(\Gamma,\mu)\}_{n\in\mathbb{Z}_{>0}}$ is unbounded, then there exists an unbounded subsequence $\{k_n\} \subset \mathbb{Z}_{>0}$ that satisfies $\tilde{L}^{k_n} = L^{k_n}(\Gamma,\mu)$ for each k_n; indeed, for any k_l with $\tilde{L}^{k_l} = L^{k_l}(\Gamma,\mu)$, by the unboundedness of $\{L^n(\Gamma,\mu)\}_{n\in\mathbb{Z}_{>0}}$, there exists the smallest k_m satisfying $L^{k_m}(\Gamma,\mu) > L^{k_l}(\Gamma,\mu)$, and we have $\tilde{L}^{k_m} = L^{k_m}(\Gamma,\mu)$. Thus we have
$$\lim_{n\to\infty} \frac{\tilde{L}^{k_n}}{k_n} = \lim_{n\to\infty} \frac{L^{k_n}(\Gamma,\mu)}{k_n}.$$

The limits on the both sides exist, since these limits are those of subsequences of the convergent sequences in (3). Hence these limits must coincide with those of the original sequences. This means

$$\lim_{n\to\infty} \frac{\tilde{L}^n}{n} = \lim_{n\to\infty} \frac{L^n(\Gamma,\mu)}{n} = l(\Gamma,\mu). \tag{4}$$

Now consider the measure $\sum_{n=1}^{\infty} a_n\mu^{*n}$. Take $k \in \mathbb{Z}_{>0}$ and fix it for a while. Note that $(\sum_{n=1}^{\infty} a_n\mu^{*n})^{*k}$ is again a convex combination of μ^{*j}'s; we can express as $(\sum_{i=n}^{\infty} a_n\mu^{*n})^{*k} = \sum_{j=1}^{\infty} a'_j\mu^{*j}$, where $a'_j = \sum_{n_1+\cdots+n_k=j} a_{n_1}\dots a_{n_k}$. Furthermore, we have $\sum_{j=1}^{\infty} a'_j = 1$, since $(\sum_{n=1}^{\infty} a_n\mu^{*n})^k$ is a probability measure. Recalling $\sum_{n=1}^{\infty} a_n = 1$, we get

$$\begin{aligned}
\sum_{j=1}^{\infty} ja'_j &= \sum_{n_1,\dots,n_k} (n_1+\cdots+n_k)a_{n_1}\dots a_{n_k} \\
&= \sum_{n_2,\dots,n_k} \left(\sum_{n_1}(n_1+\cdots+n_k)a_{n_1}\right) a_{n_2}\dots a_{nk} \\
&= \sum_{n_2,\dots,n_k} \left(\left(\sum_{n_1} n_1a_{n_1}\right) + n_2+\cdots+n_k\right) a_{n_2}\dots a_{n_k} \\
&= \sum_{i_1} n_1a_{n_1} + \cdots + \sum_{n_k} n_ka_{n_k} = k\sum_n na_n.
\end{aligned}$$

Now set $b_i = \sum_{j=k(i-1)+1}^{ki} a'_j$ $(i \in \mathbb{Z}_{>0})$. Then we get

$$\sum_{i=1}^{\infty} ib_i = \sum_{j=1}^{\infty} \left\lceil \frac{j}{k} \right\rceil a'_j \le \frac{1}{k}\left(k + \sum_{j=1}^{\infty} ja'_j\right) = 1 + \sum_{n=1}^{\infty} na_n,$$

where $\lceil s \rceil$ denotes the smallest integer greater than or equal to $s \in \mathbb{R}$, and we have used $\lceil j/k \rceil \le 1 + (j/k)$ and $\sum_{j=1}^{\infty} a'_j = 1$. Together with (2) and the nondecreasing property of $\{\tilde{L}^n\}_{n\in\mathbb{Z}_{>0}}$, we obtain

$$\left(1 + \sum_{n=1}^{\infty} na_n\right)\tilde{L}^k \ge \sum_{i=1}^{\infty} b_i\left(i\tilde{L}^k\right) \ge \sum_{i=1}^{\infty} b_i\tilde{L}^{ik} \ge \sum_{j=1}^{\infty} a'_j\tilde{L}^j.$$

On the other hand, recalling the definitions of $\tilde{L}^n$ and $L^n(\Gamma,\mu)$, and that $\sum_{j=1}^{\infty} a'_j\mu^{*j} = (\sum_{n=1}^{\infty} a_n\mu^{*n})^{*k}$, we have

$$\begin{aligned}
\sum_{j=1}^{\infty} a'_j\tilde{L}^j &\ge \sum_{j=1}^{\infty} a'_jL^j(\Gamma,\mu) = \sum_{j=1}^{\infty}\sum_{\Gamma} d(e,g)\ a'_j\mu^{*j}(g) \\
&= \sum_{\Gamma} d(e,g)\ \left(\sum_{n=1}^{\infty} a_n\mu^{*n}(g)\right)^{*k} = L^k\left(\Gamma, \sum_{n=1}^{\infty} a_n\mu^{*n}\right)
\end{aligned}$$

Therefore we get

$$\left(1 + \sum_{n=1}^{\infty} na_n\right)\frac{\tilde{L}^k}{k} \ge \frac{L^k(\Gamma, \sum_{n=1}^{\infty} a_n\mu^{*n})}{k}.$$

Letting $k \to \infty$ and recalling (4) completes the proof of Proposition 13.

□

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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