LCFT Exercises

February 10, 2020

Updated version of this material may or may not appear in http://www.helsinki.fi/∼jooi/ at some point.

The material here was written rather quickly, so there most likely are typos and/or inconsistencies.

1 Regularity of the GFF

Let Σ be a compact Riemann surface and g a smooth metric. Then the Laplace-Beltrami operator $-\Delta_g$ is positive and self-adjoint on $L^2(\Sigma, dv_g)$. It has a complete set of smooth eigenfunctions $e_{n,g}$

$$-\Delta_g e_{n,g} = e_{n,g}\,, \quad n \geq 0\,,$$

where $\lambda_{0,g} = 0$, $e_{g,0} =$ constant, $\lambda_{n,g} > 0$ for $n \geq 1$.

We denote by $H_0^s(\Sigma)$ the space of functions $f : \Sigma \to \mathbb{C}$ satisfying

$$\|f\|^2_{H_0^s(\Sigma,g)} := \sum_{n=1}^{\infty} |(f, e_{n,g})_g|^2 \lambda_{n,g}^s < \infty\,,$$

where

$$(f,h)_g := \int_\Sigma f(z)h(z)\,dv_g(z)\,.$$

The Gaussian Free Field is defined as the random series

$$X_g(z) = \sqrt{2\pi} \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{\lambda_{n,g}}} e_{n,g}(z)\,,$$

where $(x_n)_n$ are independent and identically distributed standard Gaussians. Show that the series converges almost surely in the Sobolev space $H^{-s}(\Sigma, g)$ for any $s > 0$, that is, show that almost surely for all $f \in H_0^1(\Sigma, g)$ the series

$$\sum_{n=1}^{\infty} \frac{x_n}{\sqrt{\lambda_{n,g}}} f_n$$

converges, where $f_n = (f, e_{n,g})_g$.

Solution: We have (we denote $f_n = (e_{n,g}, f)_g$)

$$
\begin{aligned}
|(X_g, f)_g)| &= \sqrt{2\pi} \left| \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{\lambda_{n,g}}} (e_{n,g}, f)_g \right| \\
&\leq \sqrt{2\pi} \sum_{n=1}^{\infty} |x_n \lambda_{n,g}^{-1/2-s/2} f_n \lambda_{n,g}^{s/2}| \\
&\leq \sqrt{2\pi} (\sum_{n=1}^{\infty} x_n^2 \lambda_{n,g}^{-1-s})^{1/2} (\sum_{n=1}^{\infty} f_n^2 \lambda_{n,g}^{s})^{1/2} \\
&= \sqrt{2\pi} \|f\|_{H_0^1(\Sigma,g)} (\sum_{n=1}^{\infty} x_n^2 \lambda_{n,g}^{-1-s})^{1/2} .
\end{aligned}
$$

The random series

$$
\sum_{n=1}^{\infty} x_n^2 \lambda_{n,g}^{-1-s}
$$

almost surely by noting that $\lambda_{n,g} \sim n$ as $n \to \infty$ and then applying the Kolmogorov two-series theorem, which gives almost sure convergence, and the set of probability 0 is independent of the choice of f.

2 Girsanov transform of the GFF

(a): Show that

$$
\mathbb{E}[F(X) e^{(X,f)_g - \frac{1}{2}\mathbb{E}(X,f)_g^2}] = \mathbb{E}[F(X + G_g f)] ,
$$

where $G_g f$ denotes the function

$$
G_g f(x) = \int G_g(x,y) f(y) \, dv_g(y) .
$$

Hint: Take $F(X) = e^{(X,h)}$, $h \in C_c^\infty(\Sigma)$, it suffices to prove the claim for these.

(b): Show that

$$
\mathbb{E}[(X,f)_g F(X)] = \int f(x) G_g(x,y) \mathbb{E}[\tfrac{\delta}{\delta X(y)} F(X)] \, d^2y \, dv_g(x) .
$$

The functional derivative $\frac{\delta}{\delta X(x)}$ is defined by setting

$$
\int f(x) \frac{\delta F(X)}{\delta X(x)} \, d^2 x := \frac{d}{d\varepsilon}|_{\varepsilon=0} F(X + \varepsilon f)
$$

for $f \in C_c^\infty(\Sigma)$.

Hint: use (a) with f replaced with αf. Then take the derivative $\frac{d}{d\alpha}|_{\alpha=0}$.

Solution: (a): We take $F(X) = e^{(X,h)}$ for $h \in C_c^\infty(\Sigma)$. Then

$$\begin{aligned}\mathbb{E}[F(X)e^{(X,f)_g - \frac{1}{2}\mathbb{E}(X,f)_g^2}] &= e^{-\frac{1}{2}\mathbb{E}(X,f)_g^2}\mathbb{E}e^{(X,f+g)_g} \\ &= e^{-\frac{1}{2}\mathbb{E}(X,f)_g^2}e^{\frac{1}{2}(f+h,G_g(f+h))_g} \\ &= e^{-\frac{1}{2}(f,G_gf)_g}e^{\frac{1}{2}(f+h,G_g(f+h))_g}\end{aligned}$$

Here the second last equality comes from the formula for the Laplace transform of a Gaussian measure (i.e. the infinite dimensional generalisation of the usual Gaussian integral formula). The last equality follows from the fact that G_g is the covariance operator of X, i.e. $\mathbb{E}(X,f)_g(X,h)_g = (f, G_g h)_g$.

On the other hand, we have

$$\begin{aligned}\mathbb{E}F(X + G_g f) &= \mathbb{E}e^{(X,h)_g + (G_g f,h)_g} \\ &= e^{\frac{1}{2}(h,G_g g)_g + (G_g f,h)}\,.\end{aligned}$$

Now the result follows since G_g is self-adjoint: $(f, G_g h) = (G_g f, h)$ (since Δ_g is self-adjoint and $G_g = \Delta_G^{-1}$).

Remark on why it suffices to consider $F = e^{(X,h)}$: The following is true for all Gaussian measures, but we work with the GFF. Let $\mathbb{E}_1$ denote the expectation with respect to the GFF X and $\mathbb{E}_2$ the expectation

$$\mathbb{E}_2 F(X) = \mathbb{E}_1 F(X) e^{(X,f) - \frac{1}{2}\mathbb{E}(X,f)^2}\,.$$

$\mathbb{E}_2$ has the following characteristic function (recall $\mathbb{E}(X,f)^2 = (f, G_g f)$)

$$\begin{aligned}\mathbb{E}_2 e^{(X,h)} &= \mathbb{E}_1 e^{(X,f+h) - \frac{1}{2}\mathbb{E}(X,f)^2} = e^{-\frac{1}{2}\mathbb{E}(X,f)^2} e^{\frac{1}{2}(f+h,G_g(f+h))} \\ &= e^{\frac{1}{2}(h,G_g h) + (h,G_g f)}\,.\end{aligned}$$

This is the characteristic function of a Gaussian measure with covariance operator G_g and mean $G_g f$. On the other hand, consider the probability law of the random field $X + G_g f$. This is described by the expectations

$$\mathbb{E}_1 F(X + G_g f)\,.$$

$X + G_g f$ is a Gaussian field and the characteristic function is given by

$$\mathbb{E}_1 e^{(X+G_g f,h)} = e^{\frac{1}{2}(h,G_g h)} e^{(h,G_g f)} = \mathbb{E}_2 e^{(X,h)}\,.$$

The characteristic function determines the Gaussian measure (Bochner–Minlos theorem), and thus the field $X + G_g f$ under $\mathbb{E}_1$ equals X under $\mathbb{E}_2$. This statement is equivalent with the Girsanov Transform formula and thus provides a full proof.

(b): Proceeding as the hint suggests, we get

$$\frac{d}{d\alpha}|_{\alpha=0}\mathbb{E}F(X)e^{\alpha(X,f)_g - \frac{\alpha^2}{2}\mathbb{E}(X,f)_g^2} = \mathbb{E}[(X,f)_g F(X)]$$

and by definition of the functional derivative we have

$$\begin{aligned}\frac{d}{d\alpha}|_{\alpha=0}\mathbb{E}F(X + \alpha G_g f) &= \int (G_g f)(y)\mathbb{E}\frac{\delta F(X)}{\delta X(y)} d^2 y \\ &= \int G_g(x,y) f(y) \mathbb{E}\frac{\delta F(X)}{\delta X(y)}\, d^2 y\, dv_g(x)\end{aligned}$$

Now the claim follows from (a).

3 Kahane

Let $(X_i)_{i=1}^N$, $(Y_i)_{i=1}^N$ be Gaussian random variables with mean zero. Assume

$$\mathbb{E}[X_iX_j] \leq E[Y_iY_j] \quad \forall i,j$$

Let $p_i \geq 0$ and let $F : \mathbb{R}_+ \to \mathbb{R}$ be a convex function with $|F(x)| \leq C(1+|x|)^k$ for some $k \in \mathbb{N}$. Prove

$$\mathbb{E}F(\sum_i p_i e^{X_i - \frac{1}{2}\mathbb{E}X_i^2}) \leq \mathbb{E}F(\sum_i p_i e^{Y_i - \frac{1}{2}\mathbb{E}Y_i^2}) .$$

Hint: Let $z_i(t) := \sqrt{t}\tilde{X}_i + \sqrt{1-t}\tilde{Y}_i$ where $\tilde{X}_i \overset{law}{=} X_i$ and $\tilde{Y}_i \overset{law}{=} Y_i$, $\tilde{X}_i, \tilde{Y}_j$ independent. Show that $\frac{d}{dt}\mathbb{E}(\sum p_i e^{Z_i(t) - \frac{1}{2}\mathbb{E}Z_i(t)^2}) \leq 0$.

Solution: We have

$$\frac{d}{dt}\mathbb{E}F(\sum_i p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2}) = \mathbb{E}\sum_i p_i(Z_i'(t) - \frac{1}{2}\frac{d}{dt}\mathbb{E}Z_i(t)^2)e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2}F'(\sum_i p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2}) .$$

We have

$$\begin{aligned}\frac{d}{dt}\mathbb{E}Z_i(t)^2 &= \frac{d}{dt}(t\mathbb{E}X_i^2 + (1-t)\mathbb{E}Y_i^2) \\ &= \mathbb{E}X_i^2 - \mathbb{E}Y_i^2 , \\ Z_i'(t) &= \frac{1}{2\sqrt{t}}\tilde{X}_i - \frac{1}{2\sqrt{1-t}}\tilde{Y}_i .\end{aligned}$$

Thus we get

$$\begin{aligned}&\mathbb{E}\sum_i p_i(Z_i'(t) - \frac{1}{2}\frac{d}{dt}\mathbb{E}Z_i(t)^2 e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2}F'(\sum_i p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2}) \\ &= \mathbb{E}\sum_i p_i(\frac{1}{2\sqrt{t}}\tilde{X}_i - \frac{1}{2\sqrt{1-t}}\tilde{Y}_i - \frac{1}{2}\mathbb{E}X_i^2 + \frac{1}{2}\mathbb{E}Y_i^2)e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2}F'(\sum_i p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2})\end{aligned}$$

Denote $P_i(t) = p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}Z_i(t)^2}$. Gaussian integration by parts implies

$$\begin{aligned}\mathbb{E}\tilde{X}_iP_i(t)F'(\sum_i P_i(t)) &= \sum_j \mathbb{E}[X_iX_j]\mathbb{E}\partial_{X_j}(P_i(t)F'(\sum_i P_i(t))) \\ &= \sum_j \mathbb{E}[X_iX_j]\mathbb{E}P_i(t)\partial_{X_j}F'(\sum_i P_i(t)) \\ &\quad + \mathbb{E}X_i^2\mathbb{E}(\partial_{X_i}P_i(t))F'(\sum_i P_i(t)) .\end{aligned}$$

We have $\partial_{X_i}P_i(t) = \sqrt{t}P_i(t)$ and

$$\partial_{X_j}F'(\sum_i P_i(t)) = \partial_{X_j}P_j(t)F''(\sum_i P_i(t)) = \sqrt{t}P_j(t)F''(\sum_i P_i(t)) .$$

We get

$$\sum_j \mathbb{E}[X_i X_j]\mathbb{E}P_i(t)\partial_{X_j}F'(\sum_i P_i(t)) + \mathbb{E}X_i^2\mathbb{E}(\partial_{X_i}P_i(t))F'(\sum_i P_i(t))$$
$$= \sum_j \mathbb{E}[X_i X_j]\mathbb{E}P_i(t)\sqrt{t}P_j(t)F''(\sum_i P_i(t)) + \mathbb{E}X_i^2\mathbb{E}\sqrt{t}P_i(t)F'(\sum_i P_i(t))\,.$$

By collecting together all the above formulae we get (we drop the $\sim$ from the notation)

$$\begin{aligned}\mathbb{E}\left[\sum_i P_i'(t)F'(\sum_i P_i(t))\right] = &-\frac{1}{2}\sum_{i,j}\mathbb{E}[Y_iY_j]\mathbb{E}\left[P_i(t)P_j(t)F''(\sum_i P_i(t))\right] \\ &+\frac{1}{2}\sum_{i,j}\mathbb{E}[X_iX_j]\mathbb{E}\left[P_i(t)P_j(t)F''(\sum_i P_i(t))\right] \\ &-\frac{1}{2}\mathbb{E}[Y_i^2]\mathbb{E}\left[P_i(t)F'(\sum_i P_i(t))\right] \\ &+\frac{1}{2}\mathbb{E}[X_i^2]\mathbb{E}\left[P_i(t)F'(\sum_i P_i(t))\right] \\ &+\frac{1}{2}\mathbb{E}[Y_i^2]\mathbb{E}\left[P_i(t)F'(\sum_i P_i(t))\right] \\ &-\frac{1}{2}\mathbb{E}[X_i^2]\mathbb{E}\left[P_i(t)F'(\sum_i P_i(t))\right]\,.\end{aligned}$$

The last 4 terms cancel. The difference of the first two is negative since $F'', P_i(t), P_j(t) \geq 0$ and $\mathbb{E}[X_iX_j] - \mathbb{E}[Y_iY_j] \leq 0$.

4 GMC Annulus integral

Let $m_{g,\gamma}$ bet the GMC measure of the GFF.

Show that

$$\mathbb{E}[(\int_{B(0,r)} |z|^{-\gamma\alpha} dm_{g,\gamma}(z))^p] \leq Cr^{\xi_\alpha(p)}\,,$$

where $\xi_\alpha(p) = \gamma(Q-\alpha)p - \frac{1}{2}\gamma^2 p^2$ and $B(0,r)$ is a ball of radius r at the origin.

As a corollary show that

$$\int_{B(0,r)} \frac{1}{|z|^{\gamma\alpha}} dm_{g,\gamma}(dz) < \infty$$

almost surely if $\gamma\alpha < 2 + \frac{\gamma^2}{2}$, i.e. $\alpha < Q$. Compare this to the $\gamma = 0$ case.

Solution: Assume first $\mathbb{E}[X(x)X(y)] = \ln\frac{1}{|x-y|}$. Then

$$\mathbb{E}X(2^{-n}x)X(2^{-n}y) = \ln\frac{1}{2^{-n}|x-y|} = \ln\frac{1}{|x-y|} + \ln 2^n\,.$$

Thus

$$X_g(2^{-n}\cdot) \overset{law}{=} X_g(\cdot) + x_n\,,$$

where x_n is an independent centered gaussian with variance $\ln 2^n$. Denote by A_n the annulus with radii 2^{-n} and 2^{-n-1}. Now

$$\begin{aligned}
I_n &:= \int_{A_n} |z|^{-\gamma\alpha} dm_{g,\gamma}(dz) \\
&= \int_{A_0} |2^{-n}z|^{-\gamma\alpha} e^{\gamma X_g(2^{-n}z) - \frac{\gamma^2}{2}\mathbb{E}X_g(2^{-n}z)^2} 2^{-2n} d^2z \\
&\overset{law}{=} 2^{n(\gamma\alpha-2)} e^{\gamma x_n - \frac{\gamma^2}{2}\mathbb{E}x_n^2} \int_{A_0} |z|^{-\gamma\alpha} e^{\gamma X_g(z)}\, d^2z \\
&= e^{\gamma x_n - \frac{1}{2}\gamma^2 \ln 2^n} 2^{n(\gamma\alpha-2)} I_0 \\
&= e^{\gamma x_n} 2^{\gamma(\alpha-Q)n} I_0\,.
\end{aligned}$$

For a ball $B(0,r)$ of radius $r = 2^{-N}$ we can write $B_r = \cup_{n=N}^{\infty} A_n$ and we get for any $p < 1$

$$\begin{aligned}
\mathbb{E}\Big(\int_{B(0,r)} |z|^{-\gamma\alpha} dm_{g,\gamma}(z)\Big)^p &= \mathbb{E}\Big(\sum_{n=N}^{\infty} I_n\Big)^p \\
&\le \sum_{n=N}^{\infty} \mathbb{E}I_n^p \\
&= \sum_{n=N}^{\infty} 2^{\gamma(\alpha-Q)np}\mathbb{E}e^{\gamma p x_n}\mathbb{E}I_0^p \\
&= \sum_{n=N}^{\infty} 2^{\gamma(\alpha-Q)np} e^{\frac{1}{2}(\gamma p)^2 \ln 2^n}\mathbb{E}I_0^p \\
&= C\sum_{n=N}^{\infty} 2^{\gamma(\alpha-Q)np + \frac{1}{2}\gamma^2 p^2 n} \\
&\le C(2^{-N})^{\xi_\alpha(p)}\,.
\end{aligned}$$

In the lectures it was shown that $\mathbb{E}I_0^p < \infty$ for $p < \frac{4}{\gamma^2}$.

The second claim follows by taking p small but positive, since then in $\xi_\alpha(p) = \gamma(Q-\alpha)p - \frac{1}{2}\gamma^2p^2$ the linear term dominates, so $\xi_\alpha(p) > 0$ for $0 < p \ll 1$.

In the case $\gamma = 0$ the singularity is integrable iff $\gamma\alpha < 2$, so with $\gamma > 0$ we get better integrability for the measure.

5 Radial decomposition of the GFF

Define

$$X_r(0) = \frac{1}{2\pi}\int X(z + re^{i\theta})d\theta\,,$$

the circle-average of the GFF. Show that the process

$$t \mapsto X_{e^{-t}}(0)$$

is the standard Brownian motion.

Furthermore, show that we have

$$X(z) = X_{|z|}(0) + Y(z)\,,$$

where $X_{|z|}$ and Y are independent Gaussian processes.

Solution: We have

$$\begin{aligned}\mathbb{E}X_{e^{-t}}(0)X_{e^{-s}}(0) &= \frac{1}{(2\pi)^2}\iint \mathbb{E}X(e^{-t+i\theta})X(e^{-s+i\theta'})\,d\theta\,d\theta' \\ &= \frac{1}{(2\pi)^2}\iint \ln\frac{1}{|e^{-t+i\theta}-e^{-s+i\theta'}|}\,d\theta\,d\theta'\,.\end{aligned}$$

One simple way to compute this integral is to note that $z \mapsto \ln|z|$ is harmonic on $\mathbb{C}\setminus\{0\}$ and use the mean value property of harmonic functions.

Another simple way is to just Taylor expand, which we will present here. Let $t < s$. Then

$$\begin{aligned}\iint \ln\frac{1}{|e^{-t+i\theta}-e^{-s+i\theta'}|}d\theta d\theta' &= \iint \ln\frac{1}{e^{-t}|1-e^{(t-s)+i(\theta'-\theta)}|} \\ &= (2\pi)^2 t + \iint \ln\frac{1}{|1-e^{t-s+i(\theta'-\theta)}|}\,.\end{aligned}$$

The latter integral vanishes: denote $z = e^{t-z}$. Then $|z| < 1$ and we can write

$$\begin{aligned}\ln\frac{1}{|1-z|} &= \frac{1}{2}\ln\frac{1}{|1-z|^2} \\ &= \frac{1}{2}\ln\frac{1}{1-z} + \frac{1}{2}\ln\frac{1}{1-\bar{z}} \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{z^n}{n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\bar{z}^n}{n}\end{aligned}$$

This implies

$$\begin{aligned}\int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \ln\frac{1}{|1-ze^{i(\theta'-\theta)}|} &= \sum_{n=1}^{\infty}\frac{z^n}{n}\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta' e^{in(\theta'-\theta)} + \sum_{n=1}^{\infty}\frac{\bar{z}^n}{n}\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta' e^{-in(\theta'-\theta)} \\ &= 0\,.\end{aligned}$$

Thus $\mathbb{E}X_{e^{-t}}(0)X_{e^{-s}}(0) = t = \min\{t, s\}$. Also,

$$X_{e^{-0}} = X_1 = \frac{1}{2\pi}\int_0^{2\pi} X(e^{i\theta})d\theta\,.$$

This random variable is almost surely equal to 0 since it has vanishing mean and the variance is

$$\mathbb{E}X_1^2 = \frac{1}{4\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \ln\frac{1}{|e^{i\theta}-e^{i\theta'}|} = 0\,.$$

Thus $t \mapsto X_{e^{-t}}(0)$ is a Gaussian process with the covariance kernel $\mathbb{E}X_{e^{-t}}(0)X_{e^{-s}}(0) = \min\{t,s\}$ and initial condition almost surely 0, thus it is the standard Brownian motion.

Define $Y(z) = X(z) - X_{|z|}(0)$. To show that Y is independet of $X_{|z|}(0)$ we compute (assume $|z| \geq |w|$)

$$\begin{aligned}
\mathbb{E}Y(z)X_{|w|}(0) &= \mathbb{E}X(z)X_{|w|}(0) - \mathbb{E}X_{|z|}X_{|w|} \\
&= \frac{1}{2\pi}\int \ln \frac{1}{|z-|w|e^{i\theta}|}d\theta - \mathbb{E}X_{-\exp \ln |z|^{-1}}(0)X_{-\exp(\ln|w|^{-1})}(0) \\
&= \frac{1}{2\pi}\int \ln \frac{1}{|z|}d\theta + \frac{1}{2\pi}\int \ln \frac{1}{|1-\frac{|w|}{|z|}e^{i\theta}|}d\theta - \ln \frac{1}{\max\{|z|,|w|\}} \\
&= -\ln|z| + 0 + \ln \max\{|z|,|w|\} \\
&= 0\,.
\end{aligned}$$

As a corollary we get

$$\mathbb{E}Y(z)Y(w) = \mathbb{E}X(z)X(w) - \mathbb{E}X_{|z|}(0)X_{|w|}(0) = \ln \frac{\max\{|z|,|w|\}}{|z-w|}\,.$$

6 Fusion estimate

(a): By using Exercise 4 and the definition of the Liouville expectation, show that

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i)\rangle \leq C|z_1 - z_2|^{-\alpha_1\alpha_2}\,,$$

as $|z_1 - z_2| \to 0$, where $\alpha_1 + \alpha_2 < Q$, $\sum_i \alpha_i > 2Q$ and $\alpha_i < Q$ for all i.

(b): By using the radial decomposition of the GFF, show that for $\alpha = Q$

$$\mathbb{E}[(\int_{A_T} |z|^{-\alpha\gamma} dm_\gamma(z))^{-s}] \leq CT^{-1/2}\,,$$

where A_T is an annulus centered at the origin and with radii e^{-T} and 1, and $s > 0$.

(c): Show that for $\alpha_1 + \alpha_2 = Q$, $\sum_i \alpha_i > 2Q$, $\alpha_i < Q$ $\forall i$, we have

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i)\rangle \leq C|z_1 - z_2|^{-\alpha_1\alpha_2}|\ln|z_1-z_2||^{-1/2}|\ln|z_1-z_2||^{-1/2}\,,$$

as $|z_1 - z_2| \to 0$, where $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

Solution: We work with the metric

$$\begin{aligned}
g &= e^\sigma(dz \otimes d\bar{z} + d\bar{z} \otimes dz)\,, \\
e^\sigma &= \mathbf{1}_{|z|\leq 1} + |z|^{-4}\mathbf{1}_{|z|\geq 1}
\end{aligned}$$

(a): In the lectures it was shown that

$$\langle \prod_{i=1}^N V_{\alpha_i}(z_i)\rangle_g = 2\mu^{-s}\gamma^{-1}\Gamma(s)\prod_{i<j}\frac{1}{|z_i - z_j|^{\alpha_i\alpha_j}}\mathbb{E}\left[\left(\int_{\mathbb{C}}\prod_{i=1}^N(\frac{|u|_+}{|u-z_i|})^{\gamma\alpha_i}dM_{g,\gamma}(u)\right)^{-s}\right]\,, \tag{6.1}$$

where $s = \frac{\sum_i \alpha_i - 2Q}{\gamma} > 0$. Note that (essentially) by Exercies 4 this is finite. We investigate what happens to the expected value in the above expression as $|z_1 - z_2| \to 0$. Denote by A the annulus with center z_1 and radii $2|z_1 - z_2|$ and r, where $2|z_1 - z_2| < r < \min_{j \neq 1}\{|z_j - z_1|\}$ (assume $|z_1 - z_2|$ small). Then, because $A \subset \mathbb{C}$, we get

$$\left(\int_{\mathbb{C}} \prod_{i=1}^{N} (\frac{|u|_+}{|u-z_i|})^{\gamma\alpha_i} dM_{g,\gamma}(u)\right)^{-s} \leq \left(\int_{A} \prod_{i=1}^{N} (\frac{|u|_+}{|u-z_i|})^{\gamma\alpha_i} dM_{g,\gamma}(u)\right)^{-s} .$$

Furthermore, on A we can bound

$$\left(\int_{A} \prod_{i=1}^{N} (\frac{|u|_+}{|u-z_i|})^{\gamma\alpha_i} dM_{g,\gamma}(u)\right)^{-s} \leq C \left(\int_{A} \frac{1}{|u-z_1|^{\gamma\alpha_1}|u-z_2|^{\gamma\alpha_2}} dM_{g,\gamma}(u)\right)^{-s} .$$

For $u \in A$ we have $|u - z_2| \leq |u - z_1| + |z_1 - z_2| \leq |u - z_1| + \frac{1}{2}|u - z_1| = \frac{3}{2}|u - z_1|$. This leads to

$$\left(\int_{A} \frac{1}{|u-z_1|^{\gamma\alpha_1}|u-z_2|^{\gamma\alpha_2}} dM_{g,\gamma}(u)\right)^{-s} \leq C \left(\int_{A} \frac{1}{|u-z_1|^{\gamma(\alpha_1+\alpha_2)}} dM_{g,\gamma}(u)\right)^{-s} .$$

For $\alpha_1 + \alpha_2 < Q$ this is finite, and stays finite even if $|z_1 - z_2| \to 0$ (follows essentially from Exercise 4). Now the claim follows from the prefactor $|z_1 - z_2|^{-\alpha_1\alpha_2}$ in (6.1).

(b): We sketch the main parts of the argument. Recall that formally

$$dm_{g,\gamma}(z) = e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}X(z)^2} .$$

The radial decomposition of the GFF (Exercise 5) $X(z) = X_{|z|}(0) + Y(z)$ where $X_{|z|}(0)$ and $Y(z)$ are independent Gaussian processes. Thus

$$dm_{g,\gamma}(z) = e^{\gamma X_{|z|}(0) - \frac{\gamma^2}{2}\mathbb{E}X_{|z|}(0)^2} e^{\gamma Y(z) - \frac{\gamma^2}{2}\mathbb{E}Y(z)^2} d^2 z .$$

We make the change of variables $z = e^{-t}e^{i\theta}$. Then

$$\begin{aligned} dm_{g,\gamma}(z) &\stackrel{law}{=} e^{\gamma B_t - \frac{\gamma^2}{2}t} e^{\gamma Y(t,\theta) - \frac{\gamma^2}{2}\mathbb{E}Y(t,\theta)^2} e^{-2t} dt d\theta \\ &= e^{\gamma(B_t - Qt)} dM_Y(t,\theta) . \end{aligned}$$

where B_t denotes the Brownian motion (because $t \mapsto X_{e^{-t}}(0)$ is the Brownian motion, see Exercise 5) and $dM_Y(t,\theta) = e^{\gamma Y(t,\theta) - \frac{\gamma^2}{2}\mathbb{E}Y(t,\theta)^2} dt d\theta$. Note that now the measure is an exponential of Brownian motion with a drift times a multiplicative chaos measure of Y.

Now

$$\int_{A_T} |z|^{-\gamma\alpha} dm_{g,\gamma}(z) = \int_0^T e^{\gamma y_t} d\mu(t) =: I_T ,$$

where A_T is the annulus centered at 0 with radii e^{-T} and 1, $d\mu(t) = \int_0^{2\pi} dM_Y(t,\theta)$ and

$$y_t = B_t - (Q - \alpha)t$$

is a Brownian motion with drift $-(Q-\alpha)t$. Thus we expect $I_T \to \infty$ if the drift is ≥ 0 (i.e. $\alpha \geq Q$) and we expect I_T to have a finite limit if $\alpha < Q$.

For the following we set $\alpha = Q$ so that $y_t = B_t$ is just the Brownian motion.

Let E_n denote the event $\{\sup_{t\leq T} y_t \in [n-1,n]\}$. Then

$$\mathbb{E} I_T^{-s} = \sum_{n=1}^{\infty} \mathbb{E}[\mathbf{1}_{E_n} I_T^{-s}] \,.$$

Let T_n be the first time y_t hits $n-1$:

$$T_n = \inf\{t \geq 0 : y_t \geq n-1\} \,.$$

Then

$$\mathbb{E}[\mathbf{1}_{E_n} I_T^{-s}] = \mathbb{E}[\mathbf{1}_{E_n}\mathbf{1}_{T_n\leq T-1} I_T^{-s}] + \mathbb{E}[\mathbf{1}_{E_n}\mathbf{1}_{T_n> T-1} I_T^{-s}] \,.$$

We consider the first term (the second can be worked with the same way). Under $\mathbf{1}_{E_n}\mathbf{1}_{T_n\leq T-1}$ we have

$$I_T = \int_0^T e^{\gamma B_t} d\mu(t) \geq \int_{T_n}^{T_n+1} e^{\gamma B_t} d\mu(t) = e^{\gamma(n-1)} \int_0^1 e^{\gamma B'_t} d\mu(T_n+t) \,,$$

where $B'_t = B_{T_n+t} - B_{T_n}$. Now we get

$$\mathbb{E}[\mathbf{1}_{E_n}\mathbf{1}_{T_n\leq T-1} I_T^{-s}] \leq e^{-s\gamma(n-1)} \mathbb{E}\left[\frac{\mathbf{1}_{\sup_{t\in[0,T-T_n]} B'_t \leq 1}\mathbf{1}_{T_n\leq T-1}}{(\int_0^1 e^{\gamma B'_t} d\mu(T_n+t))^s}\right] .$$

We can replace B' by an independent Brownian motion by the strong Markov property of the Brownian motion. Also, μ is independent of B and stationary: $\mu(T_n+t) \overset{law}{=} \mu(t)$ so that the above expression is bounded by

$$\leq \mathbb{E}[F(T_n)\mathbf{1}_{T_n\leq T-1}]\mathbb{E}[(\int_0^1 d\mu(t))^{-s}] \,,$$

where

$$F(\tau) := \mathbb{E}[\mathbf{1}_{\sup_{t\in[0,T-\tau]} B'_t\leq 1} e^{-s\gamma \inf_{t\in[0,1]} B'_t}] \,.$$

We have $\mathbb{P}(E_n) \leq \mathbb{P}(\sup_{t\leq T} B_t \leq n)$. It is well-known that

$$\mathbb{P}(\sup_{t\leq T} B_t \leq n) \leq \frac{2n}{\sqrt{2\pi T}}$$

which leads to

$$F(\tau) \leq C(T-\tau)^{-1/2} \,.$$

The probability density of the random variable T_n is

$$p(\tau) = (2\pi\tau)^{-3/2}(n-1)e^{-\frac{(n-1)^2}{2\tau}} \,.$$

The GMC measure μ has negative moments, so we get

$$\mathbb{E}[F(T_n)\mathbf{1}_{T_n\le T-1}]\mathbb{E}[(\int_0^1 d\mu(t))^{-s}] \le C\int p(\tau)(T-\tau)^{-1/2}d\tau \le CnT^{-1/2}\,,$$

and furthermore

$$\mathbb{E}I_T^{-s} = \sum_{n=1}^{\infty}\mathbb{E}[\mathbf{1}_{E_n}I_T^{-s}] \le C\sum_{n=1}^{\infty}e^{-s\gamma(n-1)}nT^{-1/2}\,,$$

which proves the claim.

(c): Proceeding as in **(a)** and using the argument in **(b)** we see that in addition to the $|z_1 - z_2|^{-\alpha_1\alpha_2}$ factor we get (from the analysis of I_T above)

$$CT^{-1/2}$$

where $e^{-T} = |z_1 - z_2|$, i.e. $T = \ln\frac{1}{|z_1-z_2|}$, so that

$$\langle\prod_i V_{\alpha_i}(z_i)\rangle \le C|z_1 - z_2|^{-\alpha_1\alpha_2}$$

7 Conformal Ward identity

Let $g = e^{\sigma}\delta$ be a diagonal metric, where δ is the Euclidean metric. Set

$$\phi(z) = X_g(z) + \frac{Q}{2}\sigma(z)\,,$$

where X_g is the GFF. Define

$$T(z) := Q\partial_z^2\phi(z) - (\partial_z\phi(z))^2\,.$$

Show that

$$\langle T(z)\prod_{i=1}^{N}V_{\alpha_i}(x_i)\rangle_g = \sum_{i=1}^{N}(\frac{\Delta_{\alpha_i}}{(z-x_i)^2} + \frac{\partial_z}{z-x_i})\langle\prod_{i=1}^{N}V_{\alpha_i}(x_i)\rangle_g\,.$$

Solution:

We will denote $F(X) = \prod_i e^{\alpha_i X(z_i)}$. The functional derivative then takes the form

$$F_x := \frac{\delta F(X)}{\delta X(x)} = \sum_i \alpha_i\delta(x-z_i)F(X)\,, \tag{7.1}$$

and

$$F_{xy} := \frac{\delta^2 F(X)}{\delta X(x)\delta X(y)} = \sum_{i,j}\alpha_i\alpha_j\delta(x-z_i)\delta(y-z_j)F(X)\,. \tag{7.2}$$

The Ward identity then follows by Gaussian integration by parts:

$$
\begin{aligned}
Q\langle\partial_z^2 X\prod_i V_{\alpha_i}(z_i)\rangle &= \frac{Q}{2}\int\frac{1}{(z-x)^2}\langle F_x+V(x)F\rangle d^2x \\
&= \frac{Q}{2}\int\frac{1}{(z-x)^2}\langle F_x\rangle d^2x+\frac{\gamma}{4}\int\frac{1}{(z-x)^2}\langle V(x)F\rangle d^2x+\frac{1}{\gamma}\int\frac{1}{(z-x)^2}\langle V(x)F\rangle d^2x\,,
\end{aligned}
$$

where we used $Q=\frac{\gamma}{2}+\frac{2}{\gamma}$. Integrating by parts in the last term we get

$$
\begin{aligned}
\frac{1}{\gamma}\int\frac{1}{(z-x)^2}\langle V(x)F\rangle d^2x &= -\frac{1}{\gamma}\int\frac{1}{z-x}\langle\partial_x V(x)F\rangle d^2x \\
&= \frac{1}{2}\int\frac{1}{z-x}\frac{1}{x-y}\langle F_y V(x)\rangle d^2x+\frac{1}{2}\int\frac{1}{z-x}\frac{1}{x-y}\langle FV(x)V(y)\rangle d^2xd^2y\,.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
-\langle(\partial_z X)^2F\rangle &= -\frac{1}{4}\int\frac{1}{z-x}\frac{1}{z-y}\langle(\frac{\delta}{\delta X(x)}+V(x))(F_y+V(y)F)\rangle d^2xd^2y \\
&= -\frac{1}{4}\int\frac{1}{(z-x)(z-y)}\langle F_{xy}\rangle d^2xd^2y \\
&\quad -\frac{1}{2}\int\frac{1}{(z-x)(z-y)}\langle F_yV(x)\rangle d^2xd^2y \\
&\quad -\frac{1}{4}\int\frac{1}{(z-x)(z-y)}\langle FV(x)V(y)\rangle d^2xd^2y \\
&\quad -\frac{\gamma}{4}\int\frac{1}{(z-x)^2}\langle FV(x)\rangle d^2x
\end{aligned}
$$

All in all

$$
\begin{aligned}
\langle T(z)F\rangle = &-\frac{1}{4}\int\frac{1}{(z-x)(z-y)}\langle F_{xy}\rangle d^2xd^2y \\
&+\frac{Q}{2}\int\frac{1}{(z-x)^2}\langle F_x\rangle d^2x \\
&-\frac{1}{2}\int\frac{1}{(z-x)(z-y)}\langle F_yV(x)\rangle d^2xd^2y+\frac{1}{2}\int\frac{1}{z-x}\frac{1}{x-y}\langle F_yV(x)\rangle d^2x\,.
\end{aligned}
$$

The terms on the last line simplify to

$$
\begin{aligned}
&-\frac{1}{2}\int\frac{1}{(z-x)(z-y)}\langle F_yV(x)\rangle d^2xd^2y+\frac{1}{2}\int\frac{1}{z-x}\frac{1}{x-y}\langle F_yV(x)\rangle d^2x \\
&=\frac{1}{2}\int\frac{1}{z-x}(\frac{1}{x-y}-\frac{1}{z-y})\langle F_yV(x)\rangle d^2xd^2y \\
&=\frac{1}{2}\int\frac{1}{(x-y)(z-y)}\langle F_yV(x)\rangle d^2xd^2y
\end{aligned}
$$

Now by using (7.1) and (7.2) we get

$$\begin{aligned}\langle T(z)F\rangle &= -\frac{1}{4}\sum_{i,j}\alpha_i\alpha_j\frac{1}{(z-z_i)(z-z_j)}\langle\prod_i V_{\alpha_i}(z_i)\rangle \\ &\quad+\frac{Q}{2}\sum_i\alpha_i\frac{1}{(z-z_i)^2}\langle\prod_i V_{\alpha_i}(z_i)\rangle \\ &\quad+\frac{1}{2}\sum_i\frac{1}{z-z_i}\int\frac{1}{x-z_i}\langle V(x)\prod_i V_{\alpha_i}(z_i)\rangle d^2x \\ &=\sum_i\frac{\Delta_{\alpha_i}}{(z-z_i)^2}\langle\prod_i V_{\alpha_i}(z_i)\rangle \\ &\quad-\frac{1}{4}\sum_{i,j\ i\neq j}\alpha_i\alpha_j\frac{1}{(z-z_i)(z-z_j)}\langle\prod_i V_{\alpha_i}(z_i)\rangle+\frac{1}{2}\sum_i\frac{1}{z-z_i}\int\frac{1}{x-z_i}\langle V(x)\prod_i V_{\alpha_i}(z_i)\rangle d^2x\,,\end{aligned}$$

where $\Delta_\alpha=\frac{\alpha}{2}(Q-\frac{\alpha}{2})$.

Now we need to note that

$$\begin{aligned}\partial_{z_j}\langle\prod_i V_{\alpha_i}(z_i)\rangle &= \alpha_i\langle\partial_{z_j}\phi(z_j)\prod_i V_{\alpha_i}(z_i)\rangle \\ &= -\frac{\alpha_j}{2}\sum_{i\neq j}\frac{\alpha_i}{z_j-z_i}\langle\prod_i V_{\alpha_i}(z_i)\rangle+\frac{\alpha_j\mu\gamma}{2}\int\frac{1}{z_j-x}\langle V_\gamma(x)\prod_i V_{\alpha_i}(z_i)\rangle d^2x\end{aligned}$$

Then it follows that

$$\langle T(z)\prod_i V_{\alpha_i}(z_i)\rangle=\sum_i\left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2}+\frac{\partial_{z_i}}{z-z_i}\right)\langle\prod_i V_{\alpha_i}(z_i)\rangle\,.$$

Remark: Note that

$$\langle F\rangle_g=Z_{\Sigma,g}\int dc\,e^{-Q\chi(\Sigma)c}\mathbb{E}_g F(c+X_g)e^{-\frac{1}{4\pi}\int QR_gX_g dv_g-\mu e^{\gamma c}M_{g,\gamma}(\Sigma)}$$

If we vary (z,z)-component of the metric, we get

$$\begin{aligned}\frac{\delta}{\delta g^{zz}}\langle F\rangle_g &= \frac{\delta Z_{\Sigma,g}}{\delta g^{zz}}\langle F\rangle_g-\tfrac{1}{4\pi}\int Q\frac{\delta R_g(z)}{\delta g^{zz}}\langle X_g(z)F\rangle_g\,dv_g(z) \\ &\quad+Z_{\Sigma,g}\int dc\,e^{-Q\chi(\Sigma)c}\left(\tfrac{\delta}{\delta g^{zz}}\mathbb{E}_g\right)F(c+X_g)e^{-\frac{1}{4\pi}\int QR_gX_gdv_g-\mu e^{\gamma c}M_{g,\gamma}(\Sigma)}\end{aligned}$$

(note that $\frac{\delta}{\delta g^{zz}}M_{g,\gamma}=0$ since $\frac{\delta v_g}{\delta g^{zz}}=0$). If we specialise to the metric

$$\begin{aligned}g&=e^\sigma(dz\otimes d\bar z+d\bar z\otimes dz)\,,\\ e^\sigma&=\mathbf{1}_{|z|\leq 1}+|z|^{-4}\mathbf{1}_{|z|\geq 1}\end{aligned}$$

and vary the g^{zz} component inside the unit disk, we have

$$\begin{aligned}\frac{\delta R_g(z)}{\delta g^{zz}}&=-\partial_z^2\delta(z)\,,\\ \frac{\delta Z_{\Sigma,g}}{\delta g^{zz}}&=0\,.\end{aligned}$$

What is the variation of $\mathbb{E}_g$? There is a general principle on varying Gaussian measures: Let μ_C be a Gaussian measure with covariance operator C and let $(C_s)_{s\in\mathbb{R}}$ be a family of covariance operators depending smoothly on the parameter s. Then

$$\frac{d}{ds}\int F(\varphi)d\mu_{C_s}(\varphi) = \frac{1}{2}\int dx\int dy\frac{d}{ds}C_s(x,y)\int \frac{\partial^2 F}{\partial\varphi(x)\partial\varphi(y)}d\mu_{C_s}(\varphi)\,.$$

Here $C_s(x,y)$ is the kernel of the operator C_s. In practice for us this implies

$$\frac{\delta}{\delta g^{zz}}\mathbb{E}_g F = \frac{1}{2}\int \frac{\delta}{\delta g^{zz}}G_g(x,y)\mathbb{E}\frac{\delta^2 F}{\delta X_g(x)X_g(y)}\,d^2x\,d^2y\,,$$

where $G_g(x,y)$ is the Green function of $-\Delta_g$. One can show that

$$\frac{\delta}{\delta g^{zz}}G_g(x,y) = -\frac{1}{2\pi}\partial_x G_g(x,z)\partial_z G(x,z)\partial_z G(y,z)\,,$$

where z is the point where g^{zz} is varied. Thus, in the end together with the Gaussian Integration by Parts formula one can show that

$$\frac{\delta}{\delta g^{zz}}\mathbb{E}_g F = -\frac{1}{4\pi}\mathbb{E}_g[(\partial_z X_g(z))^2 F]$$

and all in all

$$4\pi\frac{\delta}{\delta g^{zz}}\langle F\rangle_g = \langle(Q\partial_z^2 X_g(z) - (\partial_z X_g(z))^2)F\rangle_g$$

8 BPZ equation

Show that

$$(\frac{1}{\alpha^2}\partial_z^2 + \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \sum_{i=1}^N \frac{\partial_{z_i}}{z-z_i})\langle V_\alpha(z)\prod_{i=1}^N V_{\alpha_i}(z_i)\rangle_g = 0\,,$$

where $\alpha = -\frac{\gamma}{2}$ or $\alpha = -\frac{2}{\gamma}$.

Solution:

$$\begin{aligned}\partial_z\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle &= -\tfrac{\gamma}{2}\langle\partial_z X(z)V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\\ &= \frac{\gamma}{4}\sum_i \alpha_i\frac{1}{z-z_i}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle\\ &\quad -\frac{\gamma}{4}\gamma\mu\int\frac{1}{z-x}\langle V_\gamma(x)V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle d^2x\,.\end{aligned}$$

$$\partial_z^2 \langle V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle = -\frac{\gamma}{4} \sum_i \alpha_i \frac{1}{(z-z_i)^2} \langle V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle$$
$$+ \frac{\gamma}{4} \sum_i \alpha_i \frac{1}{z-z_i} (-\frac{\gamma}{2}) \langle \partial_z X(z) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle$$
$$+ \frac{\gamma^2 \mu}{4} \int \frac{1}{(z-x)^2} \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$- \frac{\gamma^2 \mu}{4} \int \frac{1}{z-x} (-\frac{\gamma}{2}) \langle \partial_z X(z) V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x \, .$$

We apply Gaussian integration by parts to the second and the fourth terms.

$$\frac{\gamma}{4} \sum_i \alpha_i \frac{1}{z-z_i} (-\frac{\gamma}{2}) \langle \partial_z X(z) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle = \frac{\gamma^2}{16} \sum_{i,j} \alpha_i \alpha_j \frac{1}{(z-z_i)(z-z_j)} \langle V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle$$
$$- \frac{\gamma^3 \mu}{16} \sum_i \alpha_i \frac{1}{z-z_i} \int \frac{1}{z-x} \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x \, .$$

$$- \frac{\gamma^2 \mu}{4} \int \frac{1}{z-x} (-\frac{\gamma}{2}) \langle \partial_z X(z) V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$= -\frac{\gamma^3 \mu}{16} \sum_i \alpha_i \frac{1}{z-z_i} \int \frac{1}{z-x} \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$- \frac{\gamma^3 \mu}{16} (-\frac{\gamma}{2}) \int \frac{1}{(z-x)^2} \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$+ \frac{\gamma^4 \mu^2}{16} \int \frac{1}{(z-x)(z-y)} \langle V_\gamma(x) V_\gamma(y) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x d^2y \, .$$

The $\int \frac{1}{(z-x)^2} V_\gamma(x)$ from earlier term can be integrated by parts to the form

$$\frac{\gamma^2 \mu}{4} \int \frac{1}{(z-x)^2} \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x = -\frac{\gamma^2 \mu}{4} \int \frac{1}{z-x} \partial_x \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$= -\frac{\gamma^3 \mu}{4} \int \frac{1}{z-x} \langle \partial_x X(x) V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$= \frac{\gamma^3 \mu}{8} \int \frac{1}{z-x} (-\frac{\gamma}{2}) \frac{1}{x-z} \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$+ \frac{\gamma^3 \mu}{8} \int \frac{1}{z-x} \sum_i \alpha_i \frac{1}{x-z_i} \langle V_\gamma(x) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x$$
$$- \frac{\gamma^4 \mu^2}{8} \int \frac{1}{z-x} \frac{1}{x-y} \langle V_\gamma(x) V_\gamma(y) V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i) \rangle d^2x d^2y$$

All the $\int \frac{1}{(z-x)^2} V_\gamma(x)$ and $\int \frac{1}{(z-x)(z-y)} V_\gamma(x) V_\gamma(y)$ terms cancel and we are left with

$$\begin{aligned}
\partial_z^2 \langle V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i)\rangle = &-\frac{\gamma}{4}\sum_i \alpha_i \frac{1}{(z-z_i)^2}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&+\frac{\gamma^2}{16}\sum_{i,j}\alpha_i\alpha_j \frac{1}{(z-z_i)(z-z_j)}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&-\frac{\gamma^3\mu}{16}\sum_i \alpha_i \frac{1}{z-z_i}\int \frac{1}{z-x}\langle V_\gamma(x) V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle d^2x \\
&-\frac{\gamma^3\mu}{16}\sum_i \alpha_i \frac{1}{z-z_i}\int \frac{1}{z-x}\langle V_\gamma(x) V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle d^2x \\
&+\frac{\gamma^3\mu}{8}\int \frac{1}{z-x}\sum_i \alpha_i \frac{1}{x-z_i}\langle V_\gamma(x) V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle d^2x\,.
\end{aligned}$$

Note that the final three terms combine by the observation

$$\begin{aligned}
\frac{1}{(z-x)(x-z_i)} - \frac{1}{(z-z_i)(z-x)} &= \frac{z-z_i-(x-z_i)}{(z-x)(x-z_i)(z-z_i)} \\
&= \frac{1}{(x-z_i)(z-z_i)}
\end{aligned}$$

We obtain

$$\begin{aligned}
\partial_z^2 \langle V_{-\gamma/2}(z) \prod_i V_{\alpha_i}(z_i)\rangle = &-\frac{\gamma}{4}\sum_i \alpha_i \frac{1}{(z-z_i)^2}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&+\frac{\gamma^2}{16}\sum_{i,j}\alpha_i\alpha_j \frac{1}{(z-z_i)(z-z_j)}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&+\frac{\gamma^3\mu}{8}\sum_i \alpha_i \frac{1}{z-z_i}\int \frac{1}{x-z_i}\langle V_\gamma(x) V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle d^2x
\end{aligned} \tag{8.1}$$

We want to compare this to

$$\begin{aligned}
-\frac{\gamma^2}{4}\sum_i\left(\frac{\Delta_{\alpha_i}}{(z-z_i)^2}+\frac{\partial_{z_i}}{z-z_i}\right)\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle &= -\frac{\gamma^2}{4}\sum_i \frac{\Delta_{\alpha_i}}{(z-z_i)^2}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&\quad -\frac{\gamma^2}{4}\sum_i \frac{\alpha_i}{z-z_i}\langle \partial_{z_i}X(z_i) V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&= -\frac{\gamma^2}{4}\sum_i \frac{\Delta_{\alpha_i}}{(z-z_i)^2}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&\quad -\frac{\gamma^3}{16}\sum_i \frac{\alpha_i}{z-z_i}\frac{1}{z_i-z}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&\quad +\frac{\gamma^2}{8}\sum_i\sum_{j\neq i} \frac{\alpha_i\alpha_j}{(z-z_i)(z_i-z_j)}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&\quad -\frac{\gamma^3\mu}{8}\sum_i \frac{\alpha_i}{z-z_i}\int \frac{1}{z_i-x}\langle V_\gamma(x) V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle\rangle d^2x
\end{aligned}$$

After simple manipulations this becomes

$$
\begin{aligned}
&-\frac{\gamma}{4}\sum_i \frac{\alpha_i}{(z-z_i)^2}\langle V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle + \frac{\gamma^2}{16}\sum_{i,j}\frac{\alpha_i^2}{(z-z_i)(z-z_j)}V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle \\
&-\frac{\gamma^3\mu}{8}\sum_i \frac{\alpha_i}{z-z_i}\int \frac{1}{z_i-x}\langle V_\gamma(x)V_{-\gamma/2}(z)\prod_i V_{\alpha_i}(z_i)\rangle\rangle d^2x
\end{aligned}
$$

The result follows by comparing with (8.1).