OBSTRUCTIONS TO APPROXIMATING TROPICAL CURVES IN SURFACES VIA INTERSECTION THEORY

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Abstract. We provide some new local obstructions to approximating tropical curves in smooth tropical surfaces. These obstructions are based on the relation between tropical and complex intersection theories which is also established here. We give two applications of the methods developed in this paper. First we classify all locally irreducible approximable 3-valent fan tropical curves in a non-singular fan tropical plane. Secondly, we prove that a generic non-singular tropical surface in tropical projective 3-space contains finitely many approximable tropical lines if it is of degree 3, and contains no approximable tropical lines if it is of degree 4 or more.

Dedicated to the memory of Mikael Passare

Contents

1. Introduction/Main results

Tropical geometry is a recent field of mathematics which provides powerful tools to study classical algebraic varieties. The most striking example is undoubtedly the use of tropical methods in real and complex enumerative geometry initiated by Mikhalkin in [Mik05]. Tropical varieties are piecewise polyhedral objects and satisfy the so-called balancing condition (see [Mik06] or [MS] for a precise definition). One possible way, among others, to relate tropical geometry to classical algebraic

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geometry is via **amoebas** of complex varieties. Given a family $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ of algebraic subvarieties of the complex torus $(\mathbb{C}^*)^N$, one may consider the corresponding family of amoebas $\text{Log}_t(\mathcal{X}_t)$ in $\mathbb{R}^N$ where Log_t is the map defined by

$$\begin{array}{cccc} \text{Log}_t: & (\mathbb{C}^*)^N & \longrightarrow & \mathbb{R}^N \\ & (z_i) & \longmapsto & (\frac{\log|z_i|}{\log t}) \end{array}.$$

When the family $\text{Log}_t(\mathcal{X}_t)$ converges (in the sense of the Hausdorff metric on compact sets of $\mathbb{R}^N$) as t goes to infinity, it is known that the limit set X is a rational polyhedral complex (see [BG84]); moreover the facets of X come naturally equipped with positive integer weights making X balanced (see [Spea] or [MS]).

In this paper, we call a positively weighted, balanced, rational, polyhedral complex a **tropical variety**. We say a tropical variety is **approximable** when it is the limit of amoebas of a family of complex algebraic varieties in the above sense. Not all tropical varieties are approximable. The first example was given by Mikhalkin who constructed in [Mik05] a spatial elliptic tropical cubic C which is not tropically planar: by the Riemann-Roch Theorem any classical spatial elliptic cubic is planar, therefore the tropical curve C cannot be approximable. One of the challenging problems in tropical geometry is to understand which tropical varieties are approximable. It follows from the works of Viro, Mikhalkin, and Rullgård (see [Vir01], [Mik04], [Rul01], see also [Kap00]) that any tropical hypersurface in $\mathbb{R}^N$ is approximable. In addition, many nice partial results about approximation of tropical curves in $\mathbb{R}^N$ have been proved by different authors (see [Mik05], [Mik06], [Speb], [NS06], [Mik], [Nis], [Tyo], [Kat], [BM], [BBMa]).

Tropical varieties in $\mathbb{R}^N$ are related to classical subvarieties of toric varieties. When considering non-toric varieties, or when working in tropical models of the torus different from $\mathbb{R}^N$, one is naturally led to the approximation problem for pairs. That is to say, given $X \subset Y$ two tropical varieties in $\mathbb{R}^N$, does there exist two families $\mathcal{X}_t \subset \mathcal{Y}_t \subset (\mathbb{C}^*)^N$ of complex varieties approximating respectively X and Y?

Non-approximable pairs of tropical objects show up even in very simple situations. Some well known pathological examples of such pairs were given by Vigeland, who constructed in [Vig09] examples of generic non-singular tropical surfaces in $\mathbb{R}^3$ of any degree $d \geq 3$ containing infinitely many tropical lines (called Vigeland lines throughout this text). Moreover, the surfaces constructed by Vigeland form an *open* subset of the space of all tropical surfaces of the given degree d, which means that these families of lines survive when perturbing the coefficients of a tropical equation of the surface. Vigeland's construction dramatically contrasts with Segre's Theorem (see [Seg43]) asserting that any non-singular complex surface of degree $d \geq 3$ in $\mathbb{C}P^3$ can contain only finitely many lines.

A very important feature in tropical geometry is the so-called **initial degeneration** or **localization** property (see [MS]): let X be a tropical variety approximated by a family of amoebas $\text{Log}_t(\mathcal{X}_t)$; then given any point p of X we may also produce from $\mathcal{X}_t$ an approximation of $Star_p(X)$ by a *constant* family of complex algebraic varieties. Recall that the **star** $Star_p(X)$ of X at p is the fan composed of all vectors $v \in \mathbb{R}^N$ such that $p + \varepsilon v$ is contained in X for ε a small enough positive real number. In other words, any approximable tropical variety is locally approximable by constant families. This already produces non-trivial obstructions to globally approximating tropical subvarieties of dimension and codimension greater than one in $\mathbb{R}^N$. For example, tropical linear fans are in correspondence with matroids (see [Spe08]), and it is well known that there exists matroids which are not realisable over any field.

As in the case of a single tropical variety, a globally approximable pair is locally approximable by constant families and there are local obstructions to global approximations. The main motivation for this paper is to provide combinatorial local obstructions in the case of curves in surfaces. Results in this direction were previously obtained by the first author and Mikhalkin (see [BM] and Theorem 1.9 below), and by Bogart and Katz (see [BK]). As an application of this latter work, Bogart and Katz proved that none of the pairs (S, L), where L is a 3-valent Vigeland line in a non-singular tropical surface S of degree $d \geq 3$, is approximable. Vigeland lines are a particular instance of a 1-parameter family of tropical lines in a non-singular tropical surface. Beyond providing these particular examples, Vigeland also completely classified in [Vig] the combinatorial types of tropical lines contained in a generic non-singular tropical surface in $\mathbb{R}^3$.

Thanks to the methods developed here, we prove in Theorem 1.8 the two following statements for a generic non-singular tropical surface $S \subset \mathbb{R}^3$: if S is of degree 3 there are only finitely many tropical lines $L \subset S$ such that the pair (S, L) is approximable; if S is of degree at least 4 and $L \subset S$ is a tropical line, the pair (S, L) is never approximable. It should be noted that in order to demonstrate Theorem 1.8 it suffices to apply only the local obstructions developed here. This solves the problem raised in [Vig09] of generic tropical surfaces of degree $d \geq 4$ containing tropical lines, and tropical surfaces of degree $d = 3$ containing infinitely many tropical lines.

Our strategy in this paper is to relate tropical and complex intersection theories in order to translate classical results (e.g. adjunction formula) into combinatorial formulas involving only tropical data. The relation between tropical intersection theory from [Sha] and intersections of complex algebraic curves in complex algebraic surfaces is established in Theorem 3.7. In the case of a finite intersection of subvarieties of $(\mathbb{C}^*)^n$, such a relation has been previously obtained in [Rab], [BLdM], and [OR].

Let us now describe more precisely the main results of this paper. As mentioned above, our main goal is to provide local obstructions to the global approximability of a pair (S, C) where C is a tropical curve contained in a non-singular tropical surface S. By definition, a non-singular tropical variety is locally a tropical linear space, and a tropical curve is locally a fan. This motivates the following problem.

Question 1.1. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a linear space, and $C \subset Trop(\mathcal{P}) \subset \mathbb{R}^N$ be a fan tropical curve. Does there exist a complex algebraic curve $\mathcal{C} \subset \mathcal{P} \subset (\mathbb{C}^*)^N$ such that $Trop(\mathcal{C}) = C$?*

Precise definitions of fan tropical curves and of **tropicalisations** are given in Section 2. For the moment, given an algebraic subvariety $\mathcal{X}$ of $(\mathbb{C}^*)^N$, one can think of $\text{Trop}(\mathcal{X})$ as $\lim_{t\to\infty} \text{Log}_t(\mathcal{X})$.

When the answer to Question 1.1 is positive, we say that the tropical curve C is **coarsely approximable** in $\mathcal{P}$. If in addition the curve $\mathcal{C}$ is irreducible and reduced, we say that C is **finely approximable**.

As it appears in many works by different authors, it is more natural to consider the approximation problem from the point of view of parameterised tropical curves instead of embedded tropical curves. Hence we refine Question 1.1 as follows.

Question 1.2. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a linear space, and $f : C \to Trop(\mathcal{P})$ be a tropical morphism from an abstract fan tropical curve C. Does there exist a non-singular Riemann surface $\mathcal{C}$ and a proper algebraic map $\mathcal{F} : \mathcal{C} \to \mathcal{P}$ such that $Trop(\mathcal{F}) = f$?*

When the answer is positive, we say that the tropical morphism $f : C \to \text{Trop}(\mathcal{P})$ is **coarsely approximable** in $\mathcal{P}$. If in addition the morphism $\mathcal{F}$ is irreducible (i.e. does not factor through a holomorphic map of degree at least 2 between Riemann surfaces $\mathcal{F}' : \mathcal{C} \to \mathcal{C}'$), we say that f is **finely approximable** in $\mathcal{P}$.

Our results provide combinatorial obstructions to the approximation problems posed in Questions 1.1 and 1.2, when $\mathcal{P}$ is a plane. As mentioned above, they are based on the relation between tropical and complex intersection theories established in Section 3. In particular, in this section we define the tropical intersection product of fan curves in a fan tropical plane.

In Section 4, we combine Theorem 3.7 with the adjunction formula for algebraic curves in surfaces. The following theorem is a weak but easy-to-state version of Theorem 4.1. A plane $\mathcal{P}$ in $(\mathbb{C}^*)^N$ is called **uniform** if its compactification $\overline{\mathcal{P}} \subset \mathbb{C}P^N$ as a projective linear subspace does not meet any $N-k$-coordinate linear spaces with $k \geq 3$. The geometric genus of a reduced algebraic curve $\mathcal{C}$ (i.e. the genus of its normalization) is denoted by $g(\mathcal{C})$. Underlying a tropical curve C is a 1-dimensional polyhedral fan equipped with positive integer weights on the edges. We denote by $\text{Edge}(C)$ the set of edges of C and w_e the weight of an edge $e \in \text{Edge}(C)$. Each curve C in a uniform plane has a well defined degree which is described in Definition 3.2.

Theorem 1.3. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a uniform plane, and $C \subset Trop(\mathcal{P}) \subset \mathbb{R}^N$ be a fan tropical curve of degree d. If there exists an irreducible and reduced complex curve $\mathcal{C} \subset \mathcal{P}$ such that $Trop(\mathcal{C}) = C$, then*

$$C^2 + (N-2)d - \sum_{e_i \subset Edge(C)} w_{e_i} + 2 \geq 2g(\mathcal{C}).$$

In particular, if the right hand side is negative then C is not finely approximable in $\mathcal{P}$.

In Section 5, we combine Theorem 3.7 and intersections of a plane algebraic curve with its Hessian curve. Since the statement is quite technical, we refer to Theorem 5.3 for the precise details. As an application we prove Corollary 5.4, which will be used in Section 7 to prohibit all 4-valent Vigeland lines in a degree $d \geq 4$ non-singular tropical surface.

It is worth stressing that Theorem 4.1 is not contained in Theorem 5.3 and vice versa. This is not surprising since it is already the case in complex geometry: the adjunction formula prohibits the existence of an irreducible quartic with 4 nodes while intersection with the Hessian curve does not; on the other hand, intersection with the Hessian curve prohibits the existence of an irreducible quintic with 6 cusps, while the adjunction formula does not. Note also that Theorem 4.1 provides an obstruction to approximate an embedded tropical curve, although Theorem 5.3 provides an obstruction to approximate a tropical morphism.

In Section 6, given a non-degenerate plane $\mathcal{P} \subset (\mathbb{C}^*)^N$, we classify all 2 or 3-valent fan tropical curves finely approximable in $\mathcal{P}$ (Theorem 6.9). Here we give two simple instances of this classification.

Theorem 1.4. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and let $C \subset Trop(\mathcal{P})$ be a reduced 2 or 3-valent fan tropical curve.*

1. *If $N = 3$, then C is finely approximable in $\mathcal{P}$ if and only if $C^2 = 0$ or $C^2 = -1$;*
2. *If $N \geq 6$ and C is of degree at least 2, then C is not approximable in $\mathcal{P}$.*

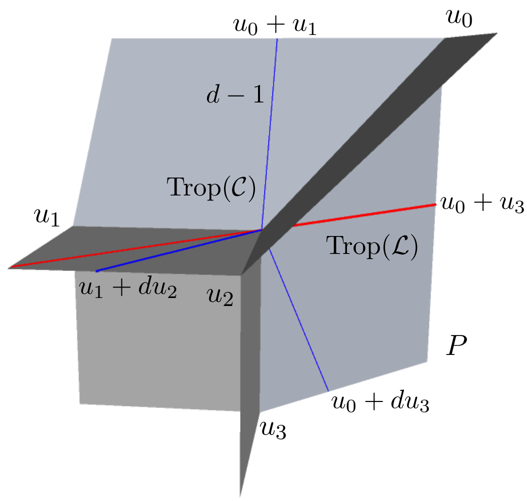
Proof. Point (1) is a consequence of Theorem 6.1 and Lemma 6.5. Point (2) is contained in Theorem 6.9. □

In Theorem 6.1, the finely approximable tropical morphisms from point (1) are described. The intermediate cases $N = 4$ and 5 are described in Lemma 6.8. The classification of degree 1 fan tropical curves is given in Lemma 6.7.

Example 1.5. A fan tropical curve from case (1) inside the standard tropical plane in $\mathbb{R}^3$ is depicted on the left side of Figure 1.

At this point, it is interesting to note that all fan tropical curves $C \subset \mathbb{R}^3$ known to us to be finely approximable in a plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ satisfy $C^2 \geq -1$ in $\text{Trop}(\mathcal{P})$. This leads us to the following open question.

Question 1.6. *Does there exist a fan tropical curve $C \subset \mathbb{R}^3$ which is finely approximable in a plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ and satisfies $C^2 \leq -2$ in $Trop(\mathcal{P})$?*



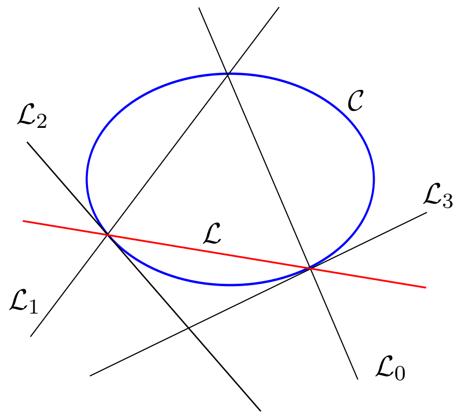


FIGURE 1. On the left are two tropical curves in the standard plane from Case (1) of Theorem 6.1. In red is the 2-valent curve in the case $d = 1$. On the right are the complex line and conic which approximate the tropical curves with their positions drawn relative to the four lines in $\mathbb{C}P^2 \backslash \mathcal{P}$ drawn in $\mathbb{R}P^2$.

Remark 1.7. If the ambient torus has dimension bigger than 3, then such fan tropical curves can exist: consider an arrangement of 6 lines $\mathcal{L}_1, \ldots, \mathcal{L}_6$ in $\mathbb{C}P^2$ such that the 3 points $\mathcal{L}_1 \cap \mathcal{L}_2$, $\mathcal{L}_3 \cap \mathcal{L}_4$, $\mathcal{L}_5 \cap \mathcal{L}_6$ lie on the same line $\mathcal{L}$; one can embed $\mathbb{C}P^2 \setminus \{\mathcal{L}_1, \ldots, \mathcal{L}_6\}$ as a plane $\mathcal{P}$ in $(\mathbb{C}^*)^5$ (see Section 2), and if we denote $L = \text{Trop}(\mathcal{L})$ we have $L^2 = -2$ in Trop($\mathcal{P}$). Note that if $\mathcal{P}' \subset (\mathbb{C}^*)^5$ is the complement in $\mathbb{C}P^2$ of 6 lines chosen generically, the tropical -2-line L is still in $\text{Trop}(\mathcal{P}') = \text{Trop}(\mathcal{P})$ yet it is no longer approximable in $\mathcal{P}'$. See [BK] for another example (though based on the same observation) of approximation problem for pairs $(\text{Trop}(\mathcal{P}), C)$ which cannot be resolved using solely the combinatorial information of Trop($\mathcal{P}$).

Finally in Section 7, we apply our methods to the study of tropical lines in tropical surfaces. In [Vig09] and [Vig], Vigeland exhibited generic non-singular tropical surfaces of degree $d \geq 4$ containing tropical lines, and generic non-singular tropical surfaces of degree $d = 3$ containing infinitely many tropical lines. The next theorem shows that when we restrict our attention to the tropical lines which are approximable in the surface, the situation turns out to be analogous to the case of complex algebraic surfaces.

Theorem 1.8. *Let S be a generic non-singular tropical surface in $\mathbb{T}P^3$ of degree d. If $d = 3$, then there exist finitely many tropical lines $L \subset S$ such that the pair (S, L) is approximable.*

If $d \geq 4$, then there exist no tropical lines $L \subset S$ such that the pair (S, L) is approximable.

Proof. According to [Vig], such a generic tropical surface contains finitely many isolated lines and finitely many 1-parametric families of tropical lines. Now the result follows from initial degeneration, Theorems 7.1, 7.3, 7.2, and 7.4. □

Except in the case of Vigeland lines, all 1-parametric families contained in a generic non-singular tropical surface S in $\mathbb{T}P^3$ are *singular* in S (see Section 7). By this we mean that according to the adjunction formula a line in a surface S of degree d should have self-intersection $2 - d$, which is not the case for such tropical lines. This raises a strange tropical phenomenon: L and S are both non-singular in $\mathbb{R}^3$, but L is singular as a subvariety of S. In other words, being non-singular does not seem to be an abstract property of tropical subvarieties.

To end this introduction, we would like to point out that the techniques presented in this paper should generalise to study tropical curves or morphisms in higher dimensional tropical varieties. At present, this problem is widely unexplored. Up to our knowledge, the following tropical Riemann-Hurwitz condition is the only general obstruction to the approximation of a tropical morphism to a tropical variety. Recall that $g(\mathcal{C})$ denotes the geometric genus of a reduced algebraic curve $\mathcal{C}$.

Theorem 1.9 (Brugallé-Mikahlkin see [BM] or [BBMb]). *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a linear space of dimension n such that Trop($\mathcal{P}$) is composed of k faces of dimension n and one face F of dimension $n-1$. Let $f : C \to$ Trop($\mathcal{P}$) be a tropical morphism from a fan tropical curve C, let d be the tropical intersection number of $f(C)$ and F in Trop($\mathcal{P}$), and suppose that C has exactly l edges which are not mapped entirely in F. Then if $f : C \to$ Trop($\mathcal{P}$) is coarsely approximable by an algebraic map $\mathcal{F} : \mathcal{C} \to \mathcal{P}$, one has*

$$g(\mathcal{C}) \geq \frac{d(k-2) - l + 2}{2}.$$

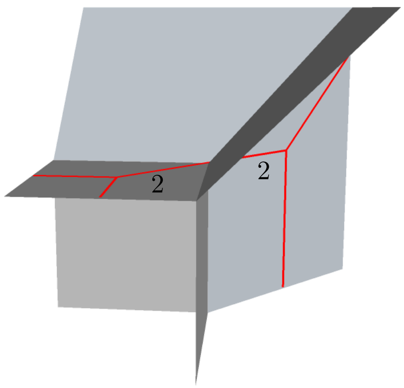


FIGURE 2. The above tropical curve is approximable in P by a rational curve with 4 punctures if and only if the vertex of the tropical plane is the midpoint of the edge of weight 2.

A tropical rational curve in $\mathbb{R}^N$ which is locally approximable at any point is globally approximable (see [Mik06], [Mik], [BBMa], [Speb], [NS06], [Tyo]). Using Theorem 1.9, it is proved in [BM] that this is no longer true for a global approximation of a rational curve in a non-singular tropical surface: the tropical curve C depicted in Figure 2 is approximable in the tropical hyperplane P by an algebraic curve of genus 0 with 4 punctures if and only if the vertex of P is the middle of the weight 2 edge of C.

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2. Preliminaries

2.1. **Linear spaces and their tropicalisations.** A **linear space** $\mathcal{P}$ in $(\mathbb{C}^*)^N$ is a subvariety which is given, up to the action of $Aut\left((\mathbb{C}^*)^N\right) = Gl_N(\mathbb{Z})$, by a system of equations of degree 1. Equivalently, a subvariety $\mathcal{P} \subset (\mathbb{C}^*)^N$ is a linear space if it is given by a system of equations with support contained in a primitive simplex Δ. Here, *primitive* indicates that Δ has the same volume as the standard simplex in $\mathbb{R}^N$. In particular the support of such a system of equations does not depend on the choice of the system, and we denote by $\Delta(\mathcal{P})$ its convex hull.

If $\mathcal{P} \subset (\mathbb{C}^*)^N$ is a linear space then there exists a toric compactification of $(\mathbb{C}^*)^N$ to $\mathbb{C}P^N$ such that $\mathcal{P}$ compactifies in a projective linear subspace $\overline{\mathcal{P}} = \mathbb{C}P^r$ The toric compactification of $(\mathbb{C}^*)^N$ to $\mathbb{C}P^N$ is induced by a primitive simplex Δ containing $\Delta(\mathcal{P})$. Of course, the simplex Δ determining a compactification of $(\mathbb{C}^*)^N$ such that $\mathcal{P}$ compactifies to $\mathbb{C}P^r$ may not be unique, see Example 3.3. However two pairs $(\mathbb{C}P^N, \overline{\mathcal{P}})$ obtained by compactifying $((\mathbb{C}^*)^N, \mathcal{P})$ using two different simplices are torically isomorphic. A linear space $\mathcal{P} \subset (\mathbb{C}^*)^N$ is said to be **non-degenerate** if it is not contained in any translation of a strict sub-torus of $(\mathbb{C}^*)^N$. If $\mathcal{P} \subset (\mathbb{C}^*)^N$ is a non-degenerate linear space, then we must have $\dim(\Delta(\mathcal{P})) \geq N + 1 - \dim(\mathcal{P})$ and it is only when $\dim(\Delta(\mathcal{P})) \leq N - 1$ that there is a choice of the simplex Δ. A **plane** is a linear space of dimension 2, and in this case $\dim(\Delta(\mathcal{P})) \geq N - 1$.

Definition 2.1. *The tropicalisation of a linear space $\mathcal{P}$ in $(\mathbb{C}^*)^N$, denoted by Trop$(\mathcal{P})$ and called a tropical linear fan, is defined as* $\lim_{t\to\infty} Log_t(\mathcal{P})$.

We say that $\mathcal{P}$ approximates Trop$(\mathcal{P})$. The tropicalisation Trop$(\mathcal{P})$ is a rational polyhedral fan of pure dimension $\dim \mathcal{P}$. Moreover, it is naturally equipped with a constant weight function equal to 1 on each face of maximal dimension making Trop$(\mathcal{P})$ into a balanced polyhedral fan (see [MS] and [AK06]). Note that if $\dim(\Delta(\mathcal{P})) = N - k$ then the fan Trop$(\mathcal{P})$ contains an affine space of dimension k.

The above mentioned compactification $\overline{\mathcal{P}} = \mathbb{C}P^r$ of a linear space $\mathcal{P} \subset (\mathbb{C}^*)^N$ by way of a simplex Δ defines a hyperplane arrangement $\mathcal{A} = \overline{\mathcal{P}} \setminus \mathcal{P}$ in $\overline{\mathcal{P}} = \mathbb{C}P^r$. Even in the case when the polytope Δ is not unique, the arrangement $\mathcal{A}$ is unique up to isomorphism. Two hyperplane arrangements $\mathcal{A}$ and $\mathcal{A}'$ in $\mathbb{C}P^r$ are isomorphic if there is an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}P^r)$ inducing a bijection $\mathcal{A} \to \mathcal{A}'$. In the case when $\mathcal{P}$ is a plane, the polytope Δ is not unique if and only if N of the $N + 1$ lines of $\mathcal{A}$ belong to the same pencil.

Conversely, a hyperplane arrangement $\mathcal{A} = \{\mathcal{H}_0, \dots, \mathcal{H}_N\}$ in $\mathbb{C}P^r$ satisfying $\cap_{i=0}^N \mathcal{H}_i = \emptyset$ defines an embedding

$$\begin{array}{cccc} \phi_{\mathcal{A}}: & \mathbb{C}P^r & \longrightarrow & \mathbb{C}P^N \\ & z & \longmapsto & [f_0(z) : \dots : f_N(z)] \end{array}$$

where f_i is a linear form defining the line $\mathcal{H}_i$. Up to a rescaling of each coordinate in $\mathbb{C}P^N$, the map $\phi_{\mathcal{A}}$ depends only on $\mathcal{A}$. The linear space $\mathcal{P} = \phi_{\mathcal{A}}(\mathbb{C}P^r) \cap (\mathbb{C}^*)^N$ is non-degenerate and is the complement $\mathbb{C}P^r \setminus \mathcal{A}$ embedded in $(\mathbb{C}^*)^N$. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}P^r$ is **uniform** if any m hyperplanes in $\mathcal{A}$ intersect in a codimension m linear space. So a linear space in $(\mathbb{C}^*)^N$ is uniform if its corresponding hyperplane arrangement is. In this text, all line arrangements $\mathcal{A} = \{\mathcal{L}_0, \dots, \mathcal{L}_N\}$ are assumed to contain a uniform sub-arrangement of three lines, i.e. $\cap_{i=0}^N \mathcal{L}_i = \emptyset$.

The tropicalisation Trop$(\mathcal{P})$ of a linear space $\mathcal{P} \subset (\mathbb{C}^*)^N$ is the Bergman fan of the matroid corresponding to $\mathcal{P}$, and has a very nice combinatorial construction as described in [AK06]. We recall now this construction in the case when $\mathcal{P}$ **is a plane**. Without loss of generality we may assume that $\mathcal{P}$ is non-degenerate.

As previously, let us consider the toric compactification of $(\mathbb{C}^*)^N$ in $\mathbb{C}P^N$ defined by a simplex Δ. Denote by $u_0, \dots, u_N \in \mathbb{Z}^N$ the outward primitive integer normal vectors to the faces of Δ. There is a natural correspondence between the vectors u_i, the hyperplanes of $\mathbb{C}P^N \setminus (\mathbb{C}^*)^N$, and the lines in the arrangement $\mathcal{A} = \overline{\mathcal{P}} \setminus \mathcal{P}$. In particular we can write $\mathcal{A} = \{\mathcal{L}_0, \dots, \mathcal{L}_N\}$ where $\mathcal{L}_i$ lies in the coordinate hyperplane corresponding to u_i. A **point** of an arrangement $\mathcal{A}$ is a point in $\overline{\mathcal{P}}$ contained in at least two lines of $\mathcal{A}$. To a point of $\mathcal{A}$ we associate the maximal subset $I \subset \{0, \dots, N\}$ such that $\mathbf{p} = \cap_{i\in I}\mathcal{L}_i$, as well as the vector $u_I = \sum_{i\in I} u_i$. Thus we may denote a point of $\mathcal{A}$ by $\mathbf{p}_I$, and denote the set of points of $\mathcal{A}$ by $\mathbf{p}(\mathcal{A})$. From the construction of the Bergman fan in [AK06], as a set $\text{Trop}(\mathcal{P})$ is the union of all cones

$$\{\lambda u_i + \mu u_I \mid \lambda, \mu \in \mathbb{R}_{\geq 0}\}$$

where i is contained in I and $I \subset \{0, \dots, N\}$ corresponds to a point of $\mathcal{A}$. An **edge** of $\text{Trop}(\mathcal{P})$ is an edge of the coarse polyhedral structure on $\text{Trop}(\mathcal{P})$ (i.e. it is a ray made of points where $\text{Trop}(\mathcal{P})$ is not locally homeomorphic to $\mathbb{R}^2$), see [AK06]. The coarse polyhedral structure may be obtained from the fine one by removing all rays in the direction u_i+u_j corresponding to points $\mathbf{p}_{i,j}$ and all rays in the direction u_k for all $\mathcal{L}_k$ which contain only two points $\mathbf{p}_I$, $\mathbf{p}_J$ of the arrangement. In particular the set of edges in the coarse polyhedral structure on $\text{Trop}(\mathcal{P})$ is contained in $\{u_0, \dots, u_N, u_I \mid p_I \in \mathbf{p}(\mathcal{A})\}$, this inclusion is usually strict. The tropicalisation $\text{Trop}(\mathcal{P})$ does not depend on the choice of polytope Δ used to compactify $(\mathbb{C}^*)^N$, therefore neither does this coarse fan structure. However, the fine polyhedral structure on $\text{Trop}(\mathcal{P})$ does depend on the choice of Δ, when a choice exists.

It follows from this construction that the tropical fan $\text{Trop}(\mathcal{P})$ depends only on the intersection lattice of the arrangement $\mathcal{A}$. Thus, non-isomorphic line arrangements on $\mathbb{C}P^2$ may have the same tropicalisations. This leads to the phenomena explained in Remark 1.7 and in [BK, section 7], where the approximation problem of a tropical curve in $\text{Trop}(\mathcal{P})$ depends on $\mathcal{P}$ and not just on $\text{Trop}(\mathcal{P})$.

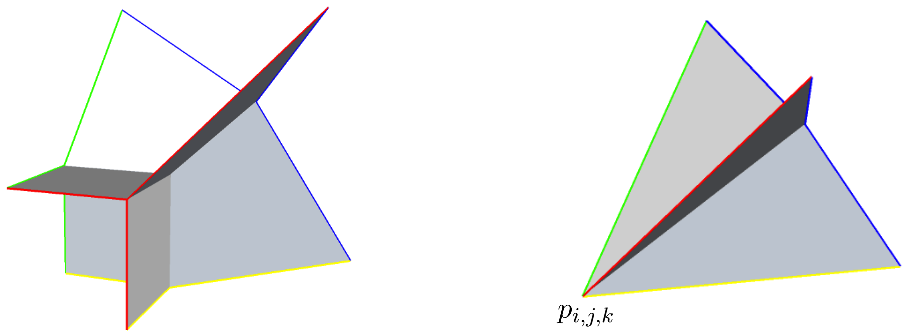

FIGURE 3. The compactifications of two tropical planes in $\mathbb{T}P^3$. On the right there is a corner point $p_{i,j,k}$ corresponding to a triple of lines.

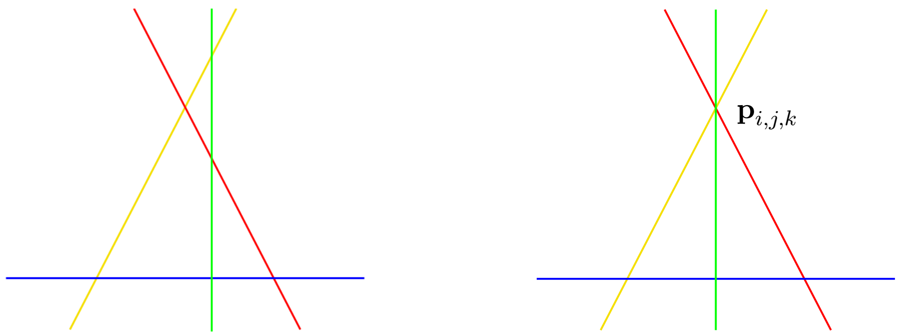

FIGURE 4. The line arrangements corresponding to the tropical planes from Figure 3.

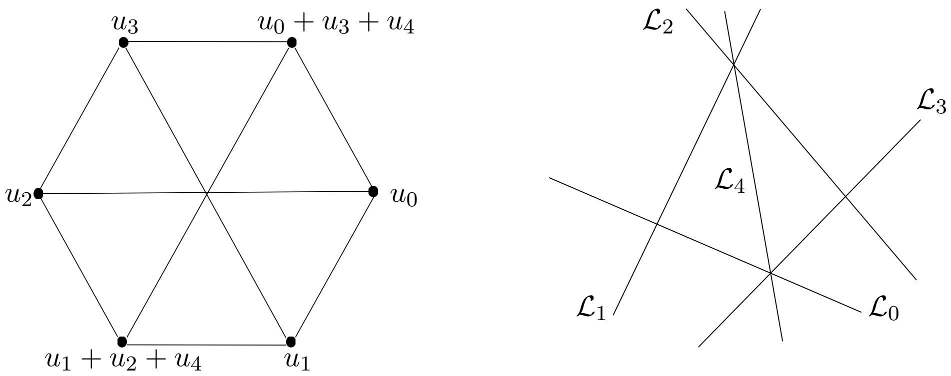

FIGURE 5. The link of singularity of a tropical plane in $\mathbb{R}^4$ and the corresponding line arrangement.

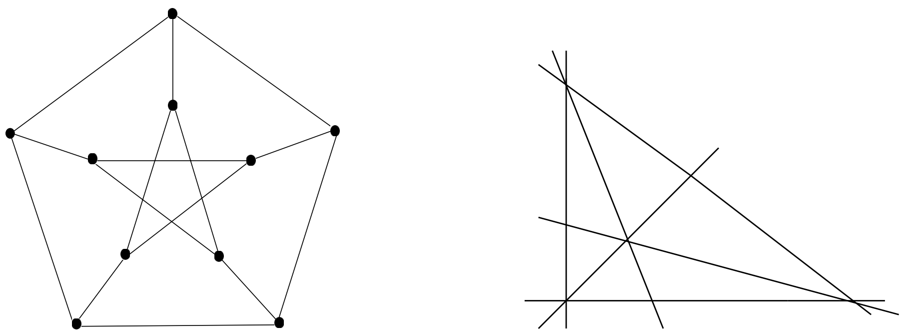

FIGURE 6. The Peterson graph is the link of singularity of tropical $\mathrm{Trop}(\mathcal{M}_{0,5}) \subset \mathbb{R}^5$, it corresponds to the braid arrangement of lines drawn on the right, the directions of the rays are omitted since they will not be used here. Since the line arrangement is symmetric we choose not to label it.

By declaring $\mathrm{Log}(0) = -\infty$ we can extend the map Trop continuously to varieties in $\mathbb{C}^N$ and even in $\mathbb{C}P^N$. The images of $\mathbb{C}^N$ and $\mathbb{C}P^N$ under the extended Log map being respectively tropical affine space, $\mathbb{T}^N = [-\infty, \infty)^N$, and tropical projective space $\mathbb{T}P^N$. Here we will use tropical projective space as it appears in [Mik06]. This space is compact and is obtained in accordance with classical geometry. It is equipped with tropical homogeneous coordinates

$$[x_0 : \cdots : x_N] \sim [x_0 + a : \cdots : x_N + a]$$

where $x_i \neq -\infty$ for at least one $0 \leq i \leq N$, and $a \in \mathbb{R}$. Moreover, it is covered by the affine charts

$$U_i = \{[x_0 : \cdots : x_N] \mid x_i = 0\} \cong \mathbb{T}^N.$$

Given a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$, its tropicalisation $P = \mathrm{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ may be compactified to $\overline{P} \subset \mathbb{T}P^N$, and by continuity $\mathrm{Trop}(\overline{\mathcal{P}}) = \overline{P}$. Clearly, any line $\mathcal{L}_i \in \overline{\mathcal{P}} \setminus \mathcal{P}$ tropicalises to a boundary component L_i of $\overline{P}$, and $\overline{P} \setminus P = \bigcup L_i$.

Example 2.2. Figure 3 shows the compactifications in $\mathbb{T}P^3$ of the standard tropical plane in $\mathbb{R}^3$, and another plane with only three faces. The standard plane corresponds to the complement of a uniform arrangement of four lines in $\mathbb{C}P^2$ whereas the other plane corresponds to an arrangement where three of the four lines belong to the same pencil (see Figure 4). Note that in either case, up to a change of coordinates, there exists a unique plane in $(\mathbb{C}^*)^3$ corresponding to the given line

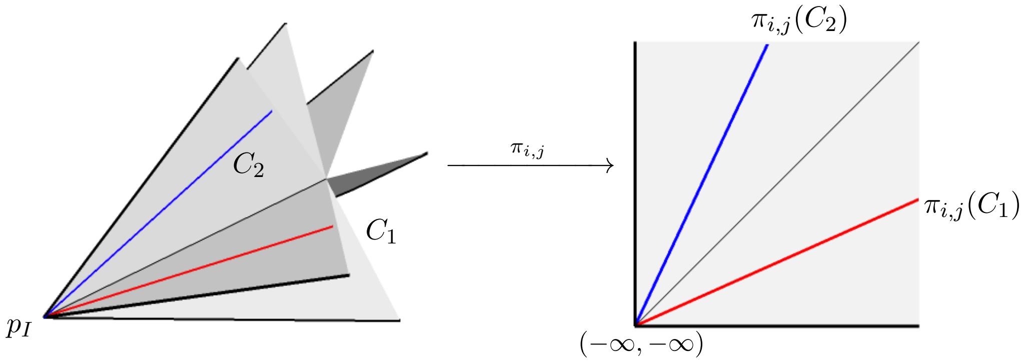

FIGURE 7. On the left is a neighborhood of a corner point $p_I \subset \overline{P}$ for $|I| = 6$, along with rays of two tropical curves $\overline{C}_1, \overline{C}_2 \subset \overline{P}$ passing through p_I. On the left is the image under the projection $\pi_{i,j} : U_l \longrightarrow \mathbb{T}^2$.

arrangement. See Example 3.3 for two different polytopes Δ and Δ' which compactify the second plane.

Example 2.3. Figures 5 and 6 are again examples of tropical planes and their corresponding arrangements. In both figures the graphs on the left are the link of singularity of the tropical fan, so that the fans are the cones over the graphs. The rays of the coarse polyhedral structure on the fans correspond to the thick vertices. The line arrangements corresponding to the two planes are drawn on the right in each figure. The arrangement in Figure 6 is known as the Braid arrangement. The complement of this arrangement in $\mathbb{C}P^2$ is $\mathcal{M}_{0,5}$, the moduli space of complex rational curves with 5 marked points. In [AK06], this is also shown to be the moduli space of 5-marked rational tropical curves. Since both arrangements are determined by a generic configuration of four points in $\mathbb{C}P^2$, in each case there is a unique arrangement up to automorphism of $\mathbb{C}P^2$ with the corresponding intersection lattice.

To a point $\mathbf{p}_I$ of $\mathbf{p}(\mathcal{A})$ corresponds a point $p_I = \text{Trop}(\mathbf{p}_I)$ in $\overline{P}$, called a **corner** of $\overline{P}$. Locally at such a point p_I the tropical plane $\overline{P}$ is determined by $|I|$. If $|I| = 2$, a neighborhood of $p_I \in \overline{P}$ is a neighborhood of $(-\infty, -\infty) \in \mathbb{T}^2$. If $|I| = k > 2$, a neighborhood of p_I is the cone over a k-valent vertex, with p_I the vertex of the cone, see Figure 7. At a point $\mathbf{p}_I \in \overline{\mathcal{P}} \subset \mathbb{C}P^N$ we may choose an affine chart $\mathcal{U}_l \cong \mathbb{C}^N$ for $l \notin I$, then for any pair $i, j \in I$ define the projection $\pi_{i,j} : \mathcal{U}_l \longrightarrow \mathbb{C}^2$ by $(z_0, \dots, z_{l-1}, z_{l+1}, \dots z_N) \mapsto (z_i, z_j)$. We use the same notation $\pi_{i,j}$ for the induced projection on $U_l \subset \mathbb{T}P^N$ since the projection commutes with tropicalisation. Then $\pi_{i,j}(\mathbf{p}_I) = (0, 0)$, and of course $\pi_{i,j}(p_I) = (-\infty, -\infty)$. Throughout the article these projections will be applied in order to work locally over $\mathbb{C}^2$ and $\mathbb{T}^2$.

2.2. **Tropical curves and morphisms.** Since we are mainly concerned with the local situation in this paper, for the sake of simplicity we restrict ourselves to *fan* tropical curves. Given a 1-dimensional polyhedral fan $C \subset \mathbb{R}^N$, we denote by Edge(C) the set of its edges.

Definition 2.4. *A fan tropical curve $C \subset \mathbb{R}^N$ is a 1-dimensional rational polyhedral fan such that each edge $e \in Edge(C)$ is equipped with positive integer weight w_e and satisfying the balancing condition*

$$\sum_{e \in Edge(C)} w_e v_e = 0$$

where v_e is the primitive integer direction of e pointing away from the vertex of C.

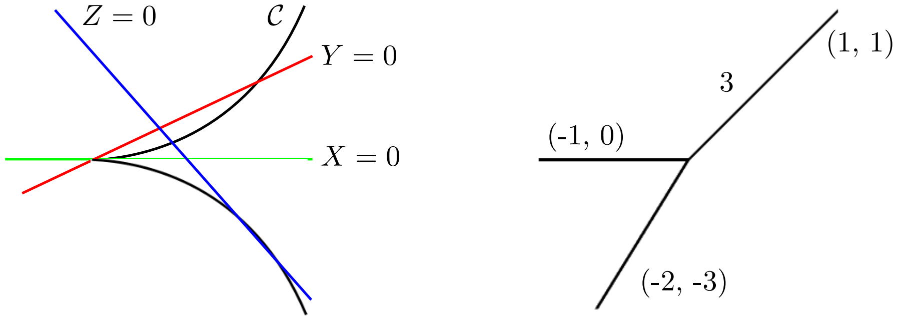

FIGURE 8. On the left is the real cubic curve from Example 2.6 with its position with respect to the three coordinate axes. On the right is the corresponding tropical curve in $\mathbb{R}^2$.

Note that contrary to some other definitions of tropical curves, the vertex of a fan tropical curve can be 2-valent.

An algebraic curve $\mathcal{C} \subset (\mathbb{C}^*)^N$ produces a fan tropical curve $\text{Trop}(\mathcal{C}) \subset \mathbb{R}^N$, the support of $\text{Trop}(\mathcal{C})$ being $\lim_{t\to\infty} \text{Log}_t(\mathcal{C})$. In order to define the weights on edges of $\text{Trop}(\mathcal{C})$, consider any toric compactification $\mathcal{X}$ of $(\mathbb{C}^*)^N$ such that the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ in $\mathcal{X}$ does not intersect any $\mathcal{D}_i \cap \mathcal{D}_j$ where $\mathcal{D}_i$ and $\mathcal{D}_j$ are any two toric divisors of $\mathcal{X}$. Such a compactification of $(\mathbb{C}^*)^N$ will be called **compatible** with $\mathcal{C}$. For every edge $e \subset \text{Trop}(\mathcal{C})$ there is a corresponding toric divisor $\mathcal{D}_e \subset \mathcal{X}$. The curve $\overline{\mathcal{C}}$ intersects $\mathcal{D}_e$ in a finite number of points and we define the weight w_e of e as the sum of the multiplicities of these intersection points. The fan $\lim_{t\to\infty} \text{Log}_t(\mathcal{C})$ enhanced with the weights w_e on its edges is a tropical curve $\text{Trop}(\mathcal{C})$.

Definition 2.5. *The tropicalisation of a curve $\mathcal{C} \subset (\mathbb{C}^*)^N$ is the tropical curve $Trop(\mathcal{C})$.*

Example 2.6. The tropicalisaton C of the curve $\mathcal{C}$ in $(\mathbb{C}^*)^2$ with equation $-Y^2 - X^3 + 4X^2Y - 5XY^2 + 2Y^3 = 0$ is depicted in Figure 8. The tropical curve C has 3 rays e_1, e_2, and e_3 with

$$w_{e_1}v_{e_1} = (-2,-3), \quad w_{e_2}v_{e_2} = (-1,0), \quad \text{and} \quad w_{e_3}v_{e_3} = (3,3).$$

Note that the presence of the edge e_1 of C is equivalent to the fact that the compactification of the curve $\mathcal{C}$ in $\mathbb{C}P^2$ has a cusp at $[0:0:1]$.

Let us now turn to tropical morphisms. For a graph Γ, let $l(\Gamma)$ denote its set of leaves. Recall that a star graph is a tree with a unique non-leaf vertex.

Definition 2.7. *A punctured abstract fan tropical curve C is $\Gamma \backslash l(\Gamma)$ equipped with a complete inner metric, where Γ is a star graph.*

A continuous map $f : C \to \mathbb{R}^n$ from a punctured abstract fan tropical curve C is a tropical morphism if

- *for any edge e of C with unit tangent vector u_e, the restriction $f_{|e}$ is a smooth map with $df(u_e) = w_{f,e}u_{f,e}$ where $u_{f,e} \in \mathbb{Z}^n$ is a primitive vector, and $w_{f,e}$ is a positive integer;*
- *it satifsies the balancing condition*

$$\sum_{e \in Edge(C)} w_{f,e}u_{f,e} = 0$$

where $u_{f,e}$ is chosen so that it points away from the vertex of C.

Note that there may be several rays of C having the same image in $\mathbb{R}^N$ by f. A tropical morphism $f : C \to \mathbb{R}^N$ induces a tropical curve in $\mathbb{R}^N$ in the sense of Definition 2.4: the set $f(C)$ is a 1-dimensional rational fan in $\mathbb{R}^N$, and an edge e of $f(C)$ is equipped with the weight

$$w_e = \sum_{\substack{e' \in \mathrm{Edge}(C) \\ f(e')=e}} w_{f,e'}.$$

Given a proper algebraic map $\mathcal{F} : \mathcal{C} \to (\mathbb{C}^*)^N$ from a punctured Riemann surface $\mathcal{C}$, we construct a tropical morphism $f : C \to \mathbb{R}^N$ as follows. Consider the punctured abstract fan tropical curve C such that the edges e of C are in one to one correspondence with punctures p_e of $\mathcal{C}$, and set $f(v) = 0$, where v is the vertex of C; for each edge e of C, consider a small punctured disc $D_e \subset \mathcal{C}$ around the corresponding puncture p_e of $\mathcal{C}$; the set $\lim_{t\to\infty} \mathrm{Log}_t(\mathcal{F}(D_p))$ is a half line in $\mathbb{R}^N$ with primitive integer direction v_e and we can define its weight $w_{f,e}$ as in the case of the tropicalisation of an algebraic curve in $(\mathbb{C}^*)^N$ (in this case $\mathcal{D}_e \cap \overline{\mathcal{F}(D_e)}$ is a single point and $w_{f,e}$ is the intersection multiplicity of $\mathcal{D}_e$ and $\overline{\mathcal{F}(D_e)}$ at that point); we define f on e by $df(u_e) = w_{f,e} v_e$ where u_e is a unit tangent vector on e pointing away from the vertex v.

Definition 2.8. *The tropical morphism $f : C \to \mathbb{R}^N$ is the tropicalisation of $\mathcal{F} : \mathcal{C} \to (\mathbb{C}^*)^N$, and is denoted by Trop($\mathcal{F}$).*

Note that the definitions of tropicalisations of morphisms and curves are consistent since we have

$$\mathrm{Trop}(\mathcal{F})(C) = \mathrm{Trop}(\mathcal{F}(\mathcal{C})).$$

Example 2.9. The map

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{C} = \mathbb{C}^* \setminus \{-1, 1\} & \longrightarrow & (\mathbb{C}^*)^2 \\ & z & \longmapsto & \left(\frac{z^2(z+1)}{z-1}, \frac{z^3}{z-1}\right) \end{array}.$$

tropicalises to the tropical morphism $f : C \to \mathbb{R}^2$ where C has 4 edges e_1, e_2, e_3, and e_4 with

$$w_{f,e_1} v_{f,e_1} = (-2, -3),\ w_{f,e_2} v_{f,e_2} = (-1, 0),\ w_{f,e_3} v_{f,e_3} = (1, 1),\ \text{and},\ w_{f,e_4} v_{f,e_4} = (2, 2).$$

The image $\mathcal{F}(\mathcal{C})$ is the algebraic curve in $(\mathbb{C}^*)^2$ of Example 2.6, (see Figure 8). The weights of e_3 and e_4 come from the factorization $-X^3 + 4X^2Y - 5XY^2 + 2Y^3 = -(X - 2Y)(X - Y)^2$.

Example 2.10. Consider the map

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{C} = \mathbb{C}^* \setminus \{1, 2, 3\} & \longrightarrow & (\mathbb{C}^*)^2 \\ & z & \longmapsto & \left(-\frac{z}{3(z-1)}, -\frac{(z-2)(z-3)}{6(z-1)}\right) \end{array}.$$

Figure 9 shows the curve $\mathcal{C}$ with respect to the 3 lines in $\mathbb{C}P^2 \setminus (\mathbb{C}^*)^2$. This morphism tropicalises to $f : C \longrightarrow \mathbb{R}^2$, where C is a tropical curve with 5 edges e_1, e_2, e_3, e_4 and e_5 where,

$$w_{f,e_1} v_{f,e_1} = (-1, 0), \quad w_{f,e_2} v_{f,e_2} = (0, 1), \quad w_{f,e_3} v_{f,e_3,} = (1, 1), \quad \text{and}$$

$$w_{f,e_4} v_{f,e_4} = w_{f,e_5} v_{f,e_5} = (0, -1).$$

Now consider another embedding of $\mathcal{C}$ into a torus of higher dimension given by

$$\begin{array}{rccc} \mathcal{F}': & \mathcal{C} = \mathbb{C}^* \setminus \{1, 2, 3\} & \longrightarrow & (\mathbb{C}^*)^3 \\ & z & \longmapsto & \left(-\frac{z}{3(z-1)}, -\frac{(z-2)(z-3)}{6(z-1)}, -\frac{z}{6}\right) \end{array}.$$

The curve $\mathcal{F}'(\mathcal{C})$ is contained in the plane $\mathcal{P}$ given by $X + Y + Z + 1 = 0$. In Figure 10 the curve $\mathcal{F}'(\mathcal{C})$ is drawn in $\overline{\mathcal{P}}$ with respect to the 4 lines in $\overline{\mathcal{P}} \setminus \mathcal{P}$.

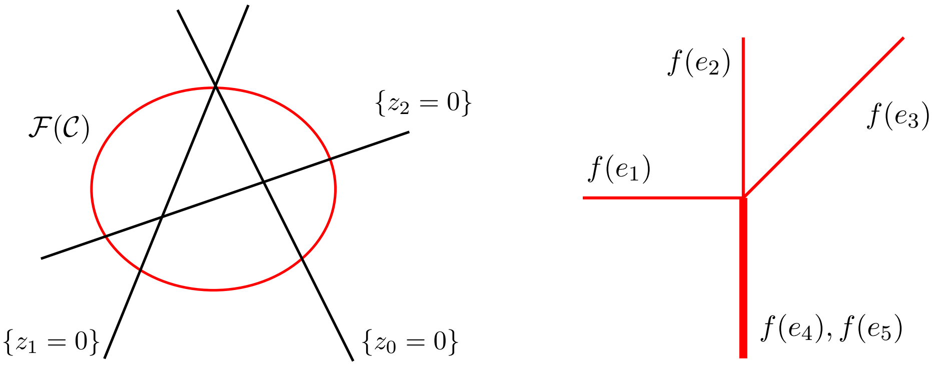

FIGURE 9. a) The image of the morphism $\mathcal{F}(\mathcal{C})$ from Example 2.10 drawn with respect to the three lines in $\overline{\mathcal{P}}\backslash\mathcal{P}$. b) The tropicalisation of the image $\mathcal{F}(\mathcal{C}) \subset \mathbb{R}^2$.

The map $\mathcal{F}'$ tropicalises to the tropical morphism $f' : C \to \mathbb{R}^3$ where C has 5 edges e_1, e_2, e_3, e_4 and e_5 with (see Figure 10)

$$w_{f,e_1} v_{f,e_1} = (-1, 0, -1), \quad w_{f,e_2} v_{f,e_2} = (0, 1, 1), \quad w_{f,e_3} v_{f,e_3}, = (1, 1, 0), \quad \text{and}$$

$$w_{f,e_4} v_{f,e_4} = w_{f,e_5} v_{f,e_5} = (0, -1, 0).$$

2.3. **Further notation.** We finish this section by fixing some conventions. Given $\mathcal{C}$ an algebraic curve in affine space $\mathbb{C}^2$ defined by a polynomial $P(z, w) = \sum a_{i,j} z^i w^j$, we denote by $\Delta(\mathcal{C}) = Conv\{(i, j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$ its Newton polygon, and we define (see Figure 13)

$$\Gamma(\mathcal{C}) = Conv(\Delta(\mathcal{C}) \cup \{(0, 0\}), \quad \text{and} \quad \Gamma^c(\mathcal{C}) = \Gamma(\mathcal{C}) \setminus \Delta(\mathcal{C}).$$

Once a coordinate system is fixed in $\mathbb{C}^2$, the equation of an algebraic curve is defined up to a non-zero multiplicative constant. In particular the polygons $\Delta(\mathcal{C})$, $\Gamma(\mathcal{C})$, and $\Gamma^c(\mathcal{C})$ do not depend on the particular choice of the defining polynomial $P(z, w)$.

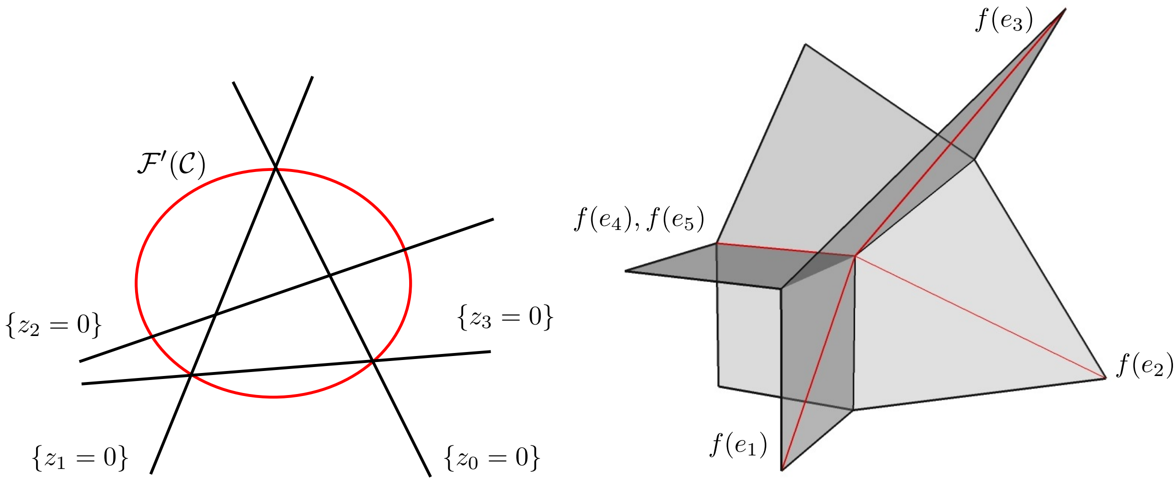

FIGURE 10. On the left is the conic $\mathcal{F}'(\mathcal{C})$ from Example 2.10 with its position with respect to the 4 lines in $\overline{\mathcal{P}}\backslash\mathcal{P}$. On the right the tropicalisation drawn in the compactification $\overline{P} \subset \mathbb{T}P^3$ of the standard tropical plane.

The latter definition translates literaly to tropical curves in $\mathbb{T}^2$. If C is the tropicalisation of a projective plane curve $\mathcal{C}$ in the coordinates (z, w), then we have

$$\Delta(\mathcal{C}) = \Delta(C), \quad \Gamma(\mathcal{C}) = \Gamma(C), \quad \text{and} \quad \Gamma^c(\mathcal{C}) = \Gamma^c(C).$$

Therefore, a tropical curve $C \subset \mathbb{T}^2$ determines the Newton polytope with respect to a fixed coordinate system of a complex curve $\mathcal{C}$ such that $\text{Trop}(\mathcal{C}) = C$. Our results are derived from exploiting this fact at the various different coordinate charts of $\mathbb{C}P^2$ provided by the line arrangement $\mathcal{A}$.

In the whole text, given a polygon Δ in $\mathbb{R}^2$ we will denote by $A(\Delta)$ its lattice area, i.e. one half of its euclidean area.

3. Intersections of tropical curves in planes.

Here we define the intersection of two fan tropical curves C_1, C_2 in a fan tropical plane $\text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ by defining first intersections at the corner points p_I of a compactification $\overline{\text{Trop}(\mathcal{P})} \subset \mathbb{T}P^N$ defined in Section 2.1. This definition is equivalent to the one given in [Sha], but has the advantage of rendering the necessary lemmas more transparent. Recall that given a primitive simplex Δ containing $\Delta(\mathcal{P})$, we denote by $u_0, \dots, u_N$ the outward primitive integer normal vectors to the faces of Δ.

Definition 3.1. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex containing $\Delta(\mathcal{P})$. Given two fan tropical curves $C_1, C_2 \subset Trop(\mathcal{P})$, let $\overline{C}_i$ denote their compactifications in $\overline{Trop(\mathcal{P})} \subset \mathbb{T}P^N$. Let $p_I \in \overline{Trop(\mathcal{P})}$ be a corner point and suppose that $\overline{C}_1$ and $\overline{C}_2$ have each exactly one ray passing through p_I. Then define the intersection multiplicity of $\overline{C}_1$ and $\overline{C}_2$ at the corner p_I as follows:*

1. *If $I = \{i, j\}$ choose an affine chart U_l for $l \notin I$ and let $\pi_{i,j} : U_l \longrightarrow \mathbb{T}^2$ be the projection from Section 2.1. Suppose the ray of $\pi_{i,j}(\overline{C}_1 \cap U_l) \subset \mathbb{T}^2$ has weight w_1 and primitive integer direction (p_1, q_1), and similarly the ray of $\pi_{i,j}(\overline{C}_2 \cap U_l) \subset \mathbb{T}^2$ has weight w_2 and primitive integer direction (p_2, q_2) then,*
$$(\overline{C}_1.\overline{C}_2)_{p_I} = w_1 w_2 \min\{p_1 q_2, q_1 p_2\}.$$
2. *If $|I| > 2$ choose an affine chart, $U_l \ni p_I$ for $l \notin I$, and a projection $\pi_{i,j} : U_l \longrightarrow \mathbb{T}^2$ where $i, j \in I$ such that the rays of $\overline{C}_1$ and $\overline{C}_2$ are contained in the union of the closed faces generated by u_i, u_j, see Figure 7. Then*
$$(\overline{C}_1.\overline{C}_2)_{p_I} = (\pi_{i,j}(\overline{C}_1 \cap U_l).\pi_{i,j}(\overline{C}_2 \cap U_l))_{(-\infty,-\infty)}.$$

We extend this intersection multiplicity by distributivity in the case when $\overline{C}_1$ and $\overline{C}_2$ have several rays passing through p_I.

In part (2) of the above definition, if the two rays of C_1 and C_2 are contained in the same open face generated by u_i and u_I then we are free to choose j as we wish. In this case even the result of the projection $\pi_{i,j}$ does not depend on the choice of j.

Next we define the degree of a tropical curve $C \subset \text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$. Suppose that a vector $v \in \mathbb{Z}^N$ is contained in $\text{Trop}(\mathcal{P})$. By the construction of $\text{Trop}(\mathcal{P})$ described in Section 2.1, v is contained in a cone generated by u_i, u_I for some $I \in \mathbf{p}(\mathcal{A})$ and $i \in I$. Therefore, there is a unique expression, $v = \rho_i(v)u_i + \rho_I(v)u_I$ where $\rho_i(v), \rho_I(v)$ are non-negative integers. Set $r_i(v) = \rho_i(v) + \rho_I(v)$ if the direction v is contained in a cone generated by u_i and u_I for some $I \ni i$, and $r_i(v) = 0$ otherwise. Given an edge $e \in \text{Edge}(C)$, we denote by v_e the primitive integer vector of e pointing outward from the vertex of C.

Definition 3.2. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex containing $\Delta(\mathcal{P})$. Let $C \subset Trop(\mathcal{P})$ be a fan tropical curve and $i \in \{0, \dots, N\}$. We define the degree of C with respect to Δ as*

$$\deg_\Delta(C) = \sum_{e \in Edge(C)} w_e r_i(v_e).$$

It follows from the balancing condition that the above definition is independent of the choice of u_i. In fact as we will see in the proof of Lemma 3.4, the degree of a curve $C \subset \mathbb{R}^N$ with respect to Δ is the multiplicity of the tropical stable intersection in $\mathbb{R}^n$ of the curve C and the tropical variety dual to Δ. This next example demonstrates the dependence of $\deg_\Delta(C)$ on the choice of Δ in the case when $\Delta(\mathcal{P}) \subsetneq \Delta$.

Example 3.3. Consider a plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ given by the zero locus of the equation $z_2 + z_3 + 1 = 0$. The corresponding line arrangement is drawn on the right of Figure 3 and $\text{Trop}(\mathcal{P}) \subset \mathbb{R}^3$ contains the affine line in direction $(1, 0, 0)$. The support of $\mathcal{P}$ is

$$\Delta(\mathcal{P}) = Conv\{(0,0,0), (0,1,0), (0,0,1)\} \subset \mathbb{R}^3.$$

Consider the two simplicies containing $\Delta(\mathcal{P})$,

$$\Delta = Conv\{(0,0,0), (0,1,0), (0,0,1), (1,0,0)\},$$

and

$$\Delta' = Conv\{(0,0,0), (0,1,0), (0,0,1), (-1,0,0)\}.$$

Now consider the 4-valent fan tropical curve $C \subset \text{Trop}(\mathcal{P})$ with edges of weight 1 in directions:

$$(1,1,1), \quad (-1,0,0), \quad (0,-1,0) \quad \text{and} \quad (0,0,-1).$$

Calculating the degree using Definition 3.2 we obtain, $\deg_\Delta(C) = 1$ and $\deg_{\Delta'}(C) = 2$.

Lemma 3.4. *Let $\mathcal{C} \subset (\mathbb{C}^*)^N$ be a complex algebraic curve, and $\Delta \subset \mathbb{R}^N$ a primitive N-simplex. Then $\deg(\overline{\mathcal{C}}) = \deg_\Delta(Trop(\mathcal{C}))$, where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in the toric compactification of $(\mathbb{C}^*)^N$ to $\mathbb{C}P^N$ given by Δ.*

Proof. Let H be the tropicalisation of a hypersurface $\mathcal{H}$ of $(\mathbb{C}^*)^N$ with Newton polygon Δ. Then let $\overline{\mathcal{H}} \subset \mathbb{C}P^N$ be the closure of $\mathcal{H}$ in the toric compactification given by Δ. Then $\overline{\mathcal{H}}$ is a hyperplane, thus of degree 1. Given $i \in \{0, \dots, N\}$, there is a translation $H' = H + v_i$ where $v_i \in \mathbb{R}^N$, such that H' intersects the tropical curve C in a finite number of points and only in the face of H' orthogonal to u_i. Then each such edge e intersecting H' does so with tropical multiplicity $w_e r_i(v_e)$, in the notation of Definition 3.2. Therefore by Definition 3.2 $\deg_\Delta(C) = \deg(H'.C) = \deg(H.C)$, where $H'.C$ is the stable intersection from [RGST05].

The translation H' is approximated by the family $\mathcal{H}_t = \{(t^{v_1} z_1, \dots t^{v_N} z_N \mid (z_1, \dots, z_N) \in \mathcal{H}\}$ in the sense that $\lim_{t\to\infty} \text{Log}_t(\mathcal{H}_t) = H'$. The intersection of H' and C is proper, so by Theorem 8.8 of [Kat09], for t large enough, $\deg(H'.C)$ is the intersection number of $\mathcal{H}_t$ and $\mathcal{C}$. Compactifying $\mathbb{R}^N$ to $\mathbb{T}P^N$ by way of Δ, the closures $\overline{H'}$ and $\overline{C}$ do not intersect at the boundary for a generic translation. Therefore $\deg_\Delta(C) = \deg(H'.C) = \deg(\overline{\mathcal{H}}.\overline{\mathcal{C}}) = \deg(\overline{\mathcal{C}})$, and the lemma is proved. □

The next definition is an equivalent definition of the tropical intersection multiplicity of two tropical curves C_1 and C_2 contained in a fan tropical plane. This definition is equivalent to the one from [Sha], and so it is independent of the choice of Δ.

Definition 3.5. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex containing $\Delta(\mathcal{P})$. Given two fan tropical curves C_1 and C_2 in $Trop(\mathcal{P})$, we define their tropical intersection*

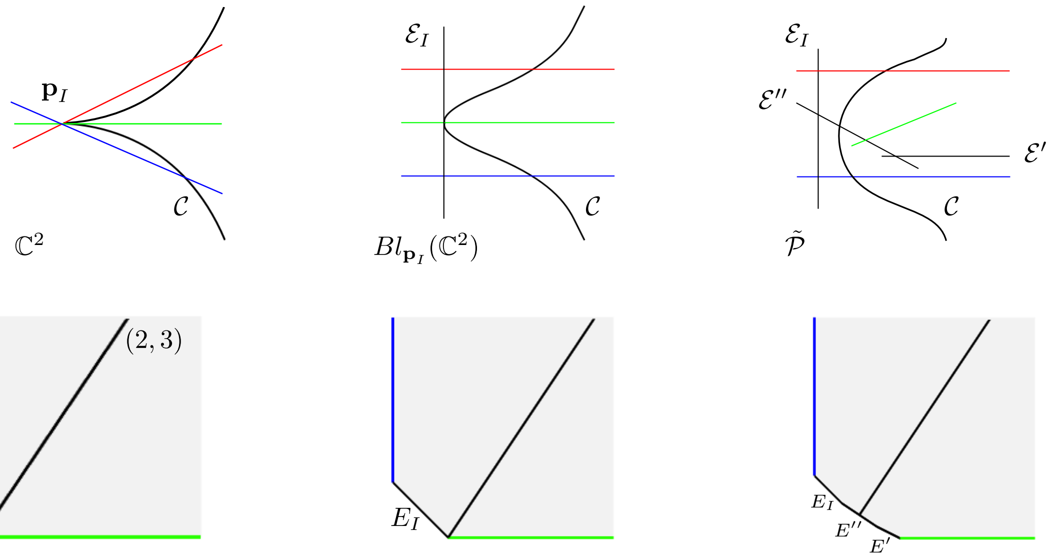

FIGURE 11. An illustration of the compactification $\tilde{\mathcal{P}}$ at a single corner point $\mathbf{p}_{i,j,k}$, along with the tropicalisation of a projection $\pi_{i,j}$, and the toric blow ups.

number in Trop($\mathcal{P}$) as

$$C_1.C_2 = \deg_\Delta(C_1).\deg_\Delta(C_1) - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (\overline{C}_1.\overline{C}_2)_{p_I}.$$

In order to interpret the above tropical intersection number in the case of two approximable tropical curves, we must first describe an appropriate compactification of the open space $\mathcal{P}$, in general distinct from $\overline{\mathcal{P}} = \mathbb{C}P^2$. Recall that in Section 2.2 a compactification $\mathcal{X}$ of $(\mathbb{C}^*)^N$ is called **compatible** with a complex curve $\mathcal{C}$ if $\mathcal{X}$ is a toric compactification of $(\mathbb{C}^*)^N$ such that for any two irreducible boundary divisors $\mathcal{D}_i$ and $\mathcal{D}_j$ of $\mathcal{X}$, the intersection $\mathcal{D}_i \cap \mathcal{D}_j \cap \mathcal{C}$ is empty. For two complex algebraic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$, we call a compactification $\mathcal{X}$ of $(\mathbb{C}^*)^N$ **compatible** with $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{P}$ if it is compatible with both curves and the compactification $\tilde{\mathcal{P}} \subset \mathcal{X}$ of $\mathcal{P}$ is a non-singular surface. We refer to [Tev07] for more details about compactification of subvarieties of $(\mathbb{C}^*)^N$.

Example 3.6. Here we describe a compactification of $(\mathbb{C}^*)^N$ compatible with two given complex curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ that we will use in the rest of the article. First let $\Sigma \subset \mathbb{R}^N$ be the complete unimodular fan dual to a primitive N-simplex Δ containing $\Delta(\mathcal{P})$. Let $\tilde{\Sigma}$ be a unimodular completion of $\Sigma \cup \text{Trop}(\mathcal{C}_1) \cup \text{Trop}(\mathcal{C}_2) \subset \mathbb{R}^N$, and $\tilde{\mathcal{X}}(\Delta)$ denote the corresponding compactification of $(\mathbb{C}^*)^N$. Then $\tilde{\mathcal{X}}(\Delta)$ is compatible with $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{P}$. This compactification $\tilde{\mathcal{P}}$ of $\mathcal{P}$ is the minimal compactification of $(\mathbb{C}^*)^N$ compatible with $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{P}$. Moreover, it is obtained from $\mathbb{C}P^2$ by blowing up points in $\mathbf{p}(\mathcal{A})$ and points above them which are intersection of boundary divisors (see Figure 11).

The following theorem states the correspondence between the complex and tropical intersection numbers for curves.

Theorem 3.7. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two algebraic curves in $\mathcal{P}$, and let $C_i = \text{Trop}(\mathcal{C}_i)$. Then*

$$\tilde{\mathcal{C}}_1.\tilde{\mathcal{C}}_2 = C_1.C_2$$

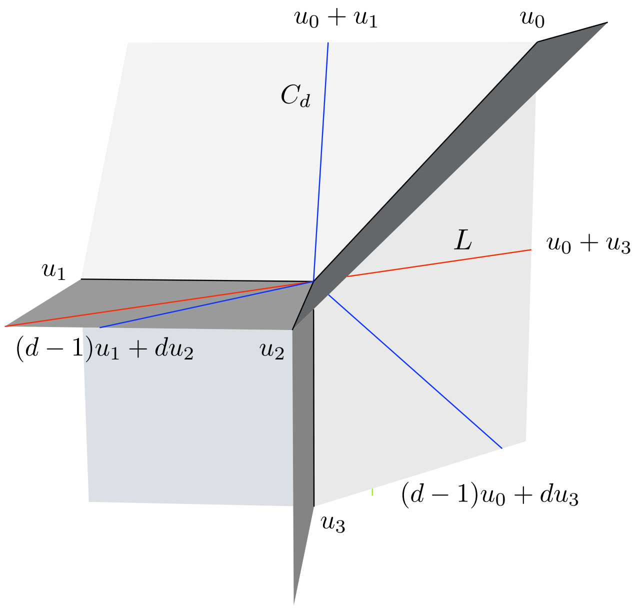

FIGURE 12. The curve C_d and the line L from Example 3.9.

where $\tilde{\mathcal{C}}_i$ is the compactification of $\mathcal{C}_i$ in the compatible toric compactification $\tilde{\mathcal{P}} \subset \tilde{\mathcal{X}}(\Delta)$ from Example 3.6.

Note that the above theorem does not depend on the initial choice of simplex Δ used to construct the compatible compactification in Example 3.6. We postpone the proof of Theorem 3.7 to present the following corollaries and examples.

Example 3.8. Let P be the tropicalisation of a uniform hyperplane $\mathcal{P} \subset (\mathbb{C}^*)^3$ and let $L \subset P$ be the affine line in the direction $(1, 1, 0)$. It is the red curve depicted in Figure 1. Since $\mathcal{P}$ is uniform $\Delta(\mathcal{P})$ is a 3 dimensional simplex. Using Definition 3.5 we compute $L.L = -1$. This tropical line is approximable by a line $\mathcal{L} \subset \mathcal{P}$. Viewing $\mathcal{P}$ as a complement of four generic lines $\mathcal{L}_1, \dots, \mathcal{L}_4$ in $\mathbb{C}P^2$, then for some labeling of these lines, $\mathcal{L}$ is the line passing through the two points $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_3 \cap \mathcal{L}_4$. Again see Figure 1 for a real drawing of the five lines. The blow up of $\mathbb{C}P^2$ at the two points $\mathcal{L}_1 \cap \mathcal{L}_2$ and $\mathcal{L}_3 \cap \mathcal{L}_4$ gives the desired compactification $\tilde{\mathcal{P}}$. The proper transform of $\mathcal{L}$ in $\tilde{\mathcal{P}}$ is indeed a curve of self intersection -1.

Example 3.9. We recall the example presented in Section 4 of [Sha]. Let us consider P and L as in Example 3.8, and let C_d be the trivalent tropical curve of degree d centered at the vertex of P with weight one rays in the directions

$$u_0 + u_1, \quad (d-1)u_0 + du_3, \quad \text{and} \quad (d-1)u_1 + du_2,$$

see Figure 12. These curves intersect at two corner points of $\overline{P} \subset \mathbb{T}P^3$. Locally at these two corners the curves appear as rays passing through the corner of $\mathbb{T}^2$, L in the direction $(-1, -1)$ and C_d in the direction $(-d, 1-d)$. Using Definition 3.5 we have

$$C_d.L = d \cdot 1 - 2(d-1) = -d + 2.$$

Corollary 3.10. *Let $\mathcal{P}$ be a non-degenerate plane and $C \subset Trop(\mathcal{P})$ be an irreducible approximable fan tropical curve. If $D \subset Trop(\mathcal{P})$ is also an irreducible fan tropical curve such that $D \neq C$ and $C.D < 0$ then D is not approximable.*

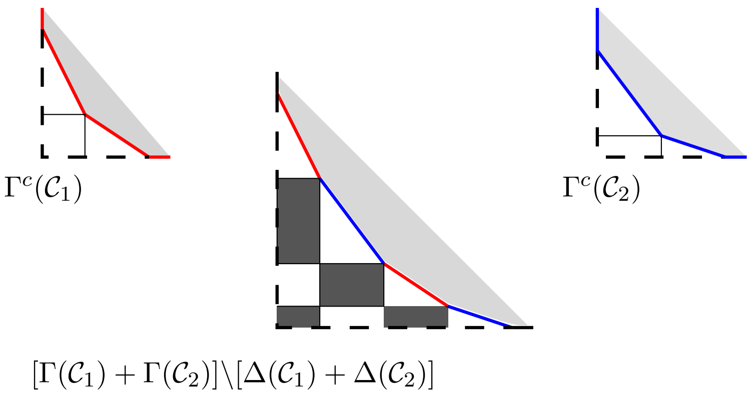

FIGURE 13. The polygons of the proof of Lemma 3.12.

It follows from this corollary and Example 3.8 that the curve C_d from Example 3.9 is not approximable for $d \geq 3$, this was already shown in [Sha].

Corollary 3.11. *Let $\mathcal{P}$ be a non-degenerate plane and $C \subset Trop(\mathcal{P})$ be an irreducible approximable fan tropical curve. If $C.C < 0$ then C is finely approximated by a unique complex curve $\mathcal{C} \subset \mathcal{P}$.*

Before returning to the proof of Theorem 3.7 we prove the next lemma. Recall in Section 2.3 for a Newton polytope of a curve $\Delta(\mathcal{C})$, we defined $\Gamma(\mathcal{C}) = Conv\{\Delta(\mathcal{C}) \cup (0,0)\}$, and $\Gamma^c(\mathcal{C}) = \Gamma(\mathcal{C})\backslash\Delta(\mathcal{C})$.

Lemma 3.12. *Let $\Delta(\mathcal{C}_1), \Delta(\mathcal{C}_2)$ be the Newton polygons of two affine algebraic curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{C}^2$ with respect to a fixed coordinate system and $Trop(\mathcal{C}_i) = C_i$ for $i = 1, 2$. Then,*

$$(C_1.C_2)_{(-\infty,-\infty)} = MV(\Gamma(\mathcal{C}_1), \Gamma(\mathcal{C}_2)) - MV(\Delta(\mathcal{C}_1), \Delta(\mathcal{C}_2)).$$

Where MV denotes the mixed volume of the two Newton polytopes.

Proof. To shorten notation we will denote $\Delta(\mathcal{C}_i)$ by Δ_i and analogously for Γ_i and Γ_i^c. When $\Delta_i = \Gamma_i$ for $i = 1$ or 2 we have $(C_1.C_2)_{(-\infty,-\infty)} = 0$ and also $MV(\Delta_1, \Delta_2) = MV(\Gamma_1, \Gamma_2)$. Otherwise, $\Gamma_i^c = \Gamma_i\backslash\Delta_i \neq \emptyset$ for both $i = 1, 2$. Figure 13 shows examples of the non-convex polygons,

$$[\Gamma_1 + \Gamma_2]\backslash[\Delta_1 + \Delta_2], \quad \Gamma_1^c \quad \text{and} \quad \Gamma_2^c.$$

Observe that

$$MV(\Gamma_1, \Gamma_2) - MV(\Delta_1, \Delta_2) = A([\Gamma_1 + \Gamma_2]\backslash[\Delta_1 + \Delta_2]) - A(\Gamma_1^c) - A(\Gamma_2^c).$$

The intersection $\Delta_1 \cap \Gamma_1^c$ consists of a collection of edges which will be called outward edges of Δ_1 and we will denote by σ_i. Similarly the edges of $\Delta_2 \cap \Gamma_2^c$ will be called the outward edges of Δ_2 and denoted τ_j.

Subdividing the polygons $[\Gamma_1 + \Gamma_2]\backslash[\Delta_1 + \Delta_2]$, Γ_1^c and Γ_2^c as in Figure 13, we see that the above difference in areas is the sum of the areas of all the shaded rectangles in $[\Gamma_1+\Gamma_2]\backslash[\Delta_1+\Delta_2]$ in Figure 13. Each such shaded rectangle is formed from a pair of outward edges $\sigma_1 \subset \Delta_1 \cap \Gamma_1^c$, $\sigma_2 \subset \Delta_2 \cap \Gamma_2^c$. Suppose the primitive outward vectors of σ_1, σ_2 have directions (p_1, q_1), (p_2, q_2) respectively, and also that σ_1 and σ_2 have integer lengths w_1 and w_2 respectively. Then the area of such a rectangle is given by $w_1 w_2 \min\{p_1 q_2, q_1 p_2\}$.

By duality, a ray of the tropical curve C_1 passing through $(-\infty, -\infty)$ with direction (p_1, q_1) corresponds to the outward edge of Δ_1 with the same normal direction (p_1, q_1). The weight on the edge of the tropical curve corresponds to the integer length of the corresponding edge of Δ, again denote this w_1. The same is true of a ray of the curve C_2 of direction (p_2, q_2), with weight w_2 and the polytope Δ_2. By Definition 3.1 these rays contribute exactly $w_1 w_2 \min\{p_1 q_2, q_1 p_2\}$ to the tropical intersection multiplicity at the corner $(-\infty,-\infty)$. The difference in the mixed volumes $MV(\Gamma_1,\Gamma_2) - MV(\Delta_1,\Delta_2)$ is distributive amongst the outward edges of Δ_1 and Δ_2 and so is the tropical intersection multiplicity at the corner, thus the lemma is proved. □

Corollary 3.13. *Let $C \subset \mathbb{T}^2$ be a tropical curve, then*

$$(C^2)_{(-\infty,-\infty)} = A(\Gamma^c(C)).$$

Together with the next corollary, the above lemma relates the intersection product of two curves after blowing up the necessary points above a single $\mathbf{p}_I$. Given two algebraic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in a plane $\mathcal{P}$ and a primitive N-simplex Δ containing $\Delta(\mathcal{P})$, recall the compactification $\tilde{\mathcal{P}}$ of $\mathcal{P}$ constructed in Example 3.6. This compactification is obtained by performing a sequence of blowups of $\overline{\mathcal{P}} = \mathbb{C}P^2$ starting with the points $\mathbf{p}_I \in \mathbf{p}(\mathcal{A})$ and then continuing at points infinitely close to $\mathbf{p}_I$ which are intersection of the boundary divisors. Let $\tilde{\mathcal{P}}_I$ be the surface obtained from $\overline{\mathcal{P}} = \mathbb{C}P^2$ by making all necessary blowups only at and above the point $\mathbf{p}_I$. Applying the relation between mixed volumes and intersection numbers from toric geometry when $I = \{i,j\}$ (see Section 5.4 of [Ful93]) we obtain the following corollary to Lemma 3.12.

Corollary 3.14. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane with corresponding line arrangement $\mathcal{A}$ and let Δ be a primitive N-simplex containing $\Delta(\mathcal{P})$. Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{P}$ be two complex curves and C_1, C_2 their respective tropicalisations. If $\boldsymbol{p}_{i,j} \in \boldsymbol{p}(\mathcal{A})$ then,*

$$\tilde{\mathcal{C}}_1.\tilde{\mathcal{C}}_2 = \deg_\Delta(C_1)\deg_\Delta(C_2) - (\overline{C}_1.\overline{C}_2)_{p_{i,j}},$$

where $\tilde{\mathcal{C}}_k$ is the closure of $\mathcal{C}_k$ in $\tilde{\mathcal{P}}_{i,j}$, and $\overline{C}_k = Trop(\overline{\mathcal{C}}_k)$, where $\overline{\mathcal{C}}_k$ is the closure of $\mathcal{C}_k$ in the toric compactification of $(\mathbb{C}^*)^N$ given by Δ.

Proof of Theorem 3.7. Again let $\overline{\mathcal{P}} = \mathbb{C}P^2$ denote the closure of $\mathcal{P}$ in the toric compactification of $(\mathbb{C}^*)^N$ given by Δ. Then by Bézout's Theorem and Lemma 3.4,

$$\overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = \deg_\Delta(C_1)\deg_\Delta(C_2).$$

We claim that after the sequence of blowups starting at $\mathbf{p}_I$, the degree of the intersections of the curves after the blow up decreases by the tropical multiplicity $(C_1.C_2)_{p_I}$ at the corresponding point.

When $I = \{i,j\}$, the claim follows directly from Corollary 3.14. When $|I| = m > 2$, suppose the tropicalisations $\overline{C}_1, \overline{C}_2$ each have a single ray passing through the point $p_I \in \overline{P}$ and that the ray of $\overline{C}_1$ is contained in the face generated by u_i and u_I and the ray of $\overline{C}_2$ is contained in the face generated by u_j and u_I. We denote by $\mathcal{E}_I$ the proper transform in $\tilde{\mathcal{P}}_I$ of the exceptional divisor of the blowup of $\mathbb{C}P^2$ at the point $\mathbf{p}_I$ (see Figure 11).

If $i \neq j$ then after blowing up at $\mathbf{p}_I$ the proper transforms of $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ do not intersect at any points $\mathcal{E}_I \cap \mathcal{L}_{i'}$ for $i' \in I$, and further blow ups do not affect the intersection number of the curves. In a chart given by the projection $\pi_{i,j}$ the blowup at $\mathbf{p}_I$ is toric, therefore after the blowup the intersection of the curves decreases by $(\pi_{i,j}(C_1).\pi_{i,j}(C_2))_{(-\infty,-\infty)}$, which by Definition 3.1 is $(C_1.C_2)_{p_I}$. See Figure 7.

If $i = j$ then after blowing up at $\mathbf{p}_I$ the proper transforms $\mathcal{C}_1', \mathcal{C}_2'$ can contain $\mathcal{E}_I \cap \mathcal{L}_{i'}$ if and only if $i' = i$. Therefore, in a chart $\pi_{i,i'}$ for any $i' \in I$ all further blowups at points above $\mathbf{p}_I$ are toric and by applying Corollary 3.14 the claim is proved.

The claim holds in the case when several rays of the tropical curves C_1, C_2 pass through p_I by distributivity. Continuing the process at each point $\mathbf{p}_I \in \mathbf{p}(\mathcal{A})$ we obtain the theorem. □

4. Obstructions coming from the adjunction formula

The adjunction formula for a non-singular curve $\mathcal{C}$ in a non-singular compact complex surface $\mathcal{X}$ reduces to (see [Sha94])

$$g(\mathcal{C}) = \frac{K_{\mathcal{X}}.\mathcal{C} + \mathcal{C}^2 + 2}{2}$$

where $g(\mathcal{C})$ is the genus of $\mathcal{C}$, and $K_{\mathcal{X}}$ is the canonical class of $\mathcal{X}$. If $\mathcal{C}$ is singular but reduced, the right hand side of the above formula defines the arithmetic genus of the curve and we denote it by $g_a(\mathcal{C})$. As before, denote by $g(\mathcal{C})$ the geometric genus of the curve $\mathcal{C}$, i.e. the genus of its normalisation. If $\mathcal{C}$ is an irreducible curve, we have $0 \leq g(\mathcal{C}) \leq g_a(\mathcal{C})$. We interpret this bound on the level of the tropical curve in order to prove Theorem 4.1 which appeared in a simplified version in Theorem 1.3.

Beforehand, we need to introduce some more notation. Given a non-degenerate plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ choose a primitive N-simplex Δ containing $\Delta(\mathcal{P})$. Recall that if the arrangement $\mathcal{A}$ corresponding to $\mathcal{P} \subset (\mathbb{C}^*)^N$ contains a point $\mathbf{p}_I$, then the fan tropical plane $P = \text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ contains a ray in the corresponding direction $u_I = \sum_{i \in I} u_i$, where $u_0, \dots, u_N$ are the outgoing primitive normal vectors to the simplex Δ. Given a fan tropical curve $C \subset P$, let w_I denote the weight of the edge of C in the direction u_I, with the convention that $w_I = 0$ if C does not contain a ray in this direction. Note that this definition depends on the directions of u_I, therefore it depends on the choice of Δ when $\dim(\Delta(\mathcal{P})) = N - 1$.

Theorem 4.1. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and $C \subset Trop(\mathcal{P})$ a fan tropical curve. If C is finely approximable by a complex curve $\mathcal{C} \subset \mathcal{P}$, then for a primitive N-simplex $\Delta \subset \mathbb{R}^N$ containing $\Delta(\mathcal{P})$ we have*

$$2g(\mathcal{C}) \leq C^2 + (N-2)\deg_\Delta(C) - \sum_{e_i \in Edge(C)} w_{e_i} - \sum_{\boldsymbol{p}_I \in \boldsymbol{p}(\mathcal{A})} (|I| - 2) w_I + 2,$$

with equality if and only if $\mathcal{C}$ is non-singular.

For a uniform plane, all of the points of the corresponding arrangement are pairs i, j, and $\Delta(\mathcal{P})$ is N-dimensional, therefore Δ is unique. This accounts for the simplified version of the above theorem stated as Theorem 1.3.

Proof. Let $\tilde{\mathcal{P}}, \tilde{\mathcal{C}}_1$, and $\tilde{\mathcal{C}}_2$ be the closures of $\mathcal{P}, \mathcal{C}_1$, and $\mathcal{C}_2$ in the corresponding compatible compactification $\tilde{\mathcal{X}}(\Delta)$ of $(\mathbb{C}^*)^N$ described in Example 3.6. Let $\pi : \tilde{\mathcal{P}} \longrightarrow \mathbb{C}P^2$ denote the contraction map. The boundary $\partial\tilde{\mathcal{P}} = \tilde{\mathcal{P}} \backslash \mathcal{P}$ is a collection of non-singular divisors consisting of the proper transforms of the $N+1$ lines in $\overline{\mathcal{P}} \setminus \mathcal{P}$ along with all exceptional divisors. Given $\mathbf{p}_I \in \mathbf{p}(\mathcal{A})$, we denote by $\mathcal{E}_I$ the proper transform in $\tilde{\mathcal{P}}$ of the exceptional divisor of the blowup of $\mathbb{C}P^2$ at the point $\mathbf{p}_I$, by $\partial\tilde{P}$ the sum of all divisors in $\tilde{\mathcal{P}} \backslash \mathcal{P}$, and by $\mathcal{L}$ the divisor class of a line in $\mathbb{C}P^2$. Note that the divisors $\mathcal{E}_I$ are contained in the support of $\partial\tilde{P}$. We will prove in Lemma 4.3 that the canonical class of $\tilde{\mathcal{P}}$ can be written in the following way:

$$K_{\tilde{\mathcal{P}}} = (N-2)\pi^*\mathcal{L} - \partial\tilde{\mathcal{P}} - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (|I| - 2)\mathcal{E}_I.$$

With $K_{\tilde{\mathcal{P}}}$ written this way, we may calculate $K_{\tilde{\mathcal{P}}}.\mathcal{C}$ using just the tropical curve $C = \text{Trop}(\mathcal{C})$. Firstly, $\pi^*\mathcal{L}.\mathcal{C} = \deg_\Delta(C)$. By definition of the weights of the edges of $\text{Trop}(C)$ we have

$$\mathcal{E}_I.\tilde{\mathcal{C}} = w_I \quad \text{and} \quad \partial\mathcal{P}.\tilde{\mathcal{C}} = \sum_{e \in \text{Edge}(C)} w_e.$$

Therefore,

$$K_{\tilde{\mathcal{P}}}.\mathcal{C} = (N-2)\deg_\Delta(C) - \sum_{e\in \mathrm{Edge}(C)} w_e - \sum_{\mathbf{p}_I\in \mathbf{p}(\mathcal{A})} (|I|-2)w_I.$$

By Theorem 3.7 we have $\tilde{\mathcal{C}}^2 = C^2$. Applying the adjunction formula for $\tilde{\mathcal{C}} \subset \tilde{\mathcal{P}}$ we obtain the claimed inequality. □

Corollary 4.2. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and $C \subset Trop(\mathcal{P})$ a fan tropical curve. If C is finely approximable by a complex curve $\mathcal{C} \subset \mathcal{P}$, then for a primitive N-simplex $\Delta \subset \mathbb{R}^N$ containing $\Delta(\mathcal{P})$ we have*

$$C^2 + (N-2)\deg_\Delta(C) - \sum_{e\in Edge(C)} w_e - \sum_{\boldsymbol{p}_I\in \boldsymbol{p}(\mathcal{A})} (|I|-2)w_I + 2 \geq 0.$$

The following lemma completes the proof of Theorem 4.1.

Lemma 4.3. *Using the same notations as in the proof of Theorem 4.1, we have*

$$K_{\tilde{\mathcal{P}}} = (N-2)\pi^*\mathcal{L} - \partial\tilde{\mathcal{P}} - \sum_{\boldsymbol{p}_I\in \boldsymbol{p}(\mathcal{A})} (|I|-2)\mathcal{E}_I.$$

Proof. To see that the canonical class can be expressed as claimed we first start with

$$K_{\mathbb{C}P^2} = -3\mathcal{L} = -\sum_{i=0}^{N} \mathcal{L}_i + (N-2)\mathcal{L},$$

where the $\mathcal{L}_i$'s are the lines in $\overline{\mathcal{P}} \setminus \mathcal{P}$. If $\pi' : \mathcal{P}' \longrightarrow \mathbb{C}P^2$ is the blowup of $\mathbb{C}P^2$ at the point $\mathbf{p}_I$, the canonical classes are related as follows,

$$K_{\mathcal{P}'} = \pi'^* K_{\mathbb{C}P^2} + \mathcal{E}_I.$$

Then,

$$K_{\mathcal{P}'} = (N-2)\pi'^*\mathcal{L} - \pi'^*(\sum_{i=0}^{N}\mathcal{L}_i) + \mathcal{E}_I = (N-2)\pi'^*\mathcal{L} - \sum_{i=0}^{N}\tilde{\mathcal{L}}_i - |I|\mathcal{E}_I + \mathcal{E}_I,$$

where $\tilde{\mathcal{L}}_i$ is the proper transform of $\mathcal{L}_i$. Moreover $\partial\mathcal{P}' = \sum_{i=0}^N \tilde{\mathcal{L}}_i + \mathcal{E}_I$, so

$$K_{\mathcal{P}'} = (N-2)\pi'^*\mathcal{L} - \partial\mathcal{P}' - (|I|-2)\mathcal{E}_I.$$

Blowing up further at points above $\mathbf{p}_I$ that are the intersection of two boundary divisors, the exceptional divisor is again a boundary divisor of the new surface. Continuing the process at each $\mathbf{p}_I$ to obtain $\tilde{\mathcal{P}}$ and we have:

$$K_{\tilde{\mathcal{P}}} = (N-2)\pi^*\mathcal{L} - \partial\tilde{\mathcal{P}} - \sum_{\mathbf{p}_I\in \mathbf{p}(\mathcal{A})} (|I|-2)\mathcal{E}_I,$$

which completes the proof. □

5. Obstruction coming from intersection with Hessian

Consider a plane algebraic curve $\mathcal{C}$ in $\mathbb{C}P^2$ given by the homogeneous equation $F(z,w,u) = 0$. The Hessian of the polynomial $F(z,w,u)$, denoted by $Hess_F(z,w,u)$, is the homogeneous polynomial defined as

$$Hess_F(z,w,u) = \det\begin{pmatrix} \frac{\partial^2 F}{\partial^2 z} & \frac{\partial^2 F}{\partial z\partial w} & \frac{\partial^2 F}{\partial z\partial u} \\ \frac{\partial^2 F}{\partial z\partial w} & \frac{\partial^2 F}{\partial^2 w} & \frac{\partial^2 F}{\partial w\partial u} \\ \frac{\partial^2 F}{\partial z\partial u} & \frac{\partial^2 F}{\partial w\partial u} & \frac{\partial^2 F}{\partial^2 u} \end{pmatrix}.$$

If $Hess_F$ is not the null polynomial, it defines a curve $Hess_{\mathcal{C}}$ called the *Hessian* of $\mathcal{C}$. The polynomial $Hess_{\mathcal{C}}$ of course depends on the chosen coordinate system on $\mathbb{C}P^2$, however the curve $Hess_{\mathcal{C}}$ does not, i.e. the curve $Hess_{\mathcal{C}}$ is invariant under projective change of coordinates in $\mathbb{C}P^2$. Note that if $\mathcal{C}$ has degree d, then $Hess_{\mathcal{C}}$ has degree $3(d-2)$ and intersects $\mathcal{C}$ in finitely many points if $\mathcal{C}$ is reduced and does not contain a line as a component. Intersecting a curve with its Hessian detects singularities and inflection points of the curve. In particular, if $\mathcal{L}$ is a line and $p \in \mathcal{C}$ are such that $(\mathcal{C}.\mathcal{L})_p = m$, then $(\mathcal{C}.Hess_{\mathcal{C}})_p \geq m-2$.

Recall that A stands for the lattice area, and that we have defined in Section 2.3,

$$\Gamma(\mathcal{C}) = Conv(\Delta(\mathcal{C}) \cup \{(0,0\}), \quad \text{and} \quad \Gamma^c(\mathcal{C}) = \Gamma(\mathcal{C}) \setminus \Delta(\mathcal{C}).$$

Lemma 5.1. *Let $\mathcal{C}$ be an algebraic curve in $\mathbb{C}P^2$, and let us fix a coordinate system on $\mathbb{C}P^2$. If $\mathcal{C}$ is reduced and does not contain any line as a component, then*

$$(\mathcal{C}.Hess_{\mathcal{C}})_{[0:0:1]} \geq 3A(\Gamma^c(\mathcal{C})) + r_0(\mathcal{C}) - 2v_0(\mathcal{C}) - 2h_0(\mathcal{C})$$

where

- $r_0(\mathcal{C}) = Card(e \cap \mathbb{Z}^2) - 1$ *if there exists an edge e of $\Gamma^c(\mathcal{C})$ of slope -1, and* $r_0(\mathcal{C}) = 0$ *otherwise;*
- $v_0(\mathcal{C}) = Card(\Gamma^c(\mathcal{C}) \cap (\{0\} \times \mathbb{Z})) - 1$*;*
- $h_0(\mathcal{C}) = Card(\Gamma^c(\mathcal{C}) \cap (\mathbb{Z} \times \{0\})) - 1$*.*

Proof. The intersection multiplicity at $[0:0:1]$ of the curve $\mathcal{C}$ and its Hessian is bigger than the number of inflection points in $\mathbb{C}P^2$ of a curve with Newton polygon $\Gamma(\mathcal{C})$ minus the number of inflection points in $\mathbb{C}P^2 \setminus \{[0:0:1]\}\}$ of a curve with Newton polygon $\Delta(\mathcal{C})$. Hence the result follows from [BLdM, Proposition 6.1]. □

Example 5.2. We will use Lemma 5.1 in the two following simple situations:

- if p is a non-degenerate node of a complex curve $\mathcal{C}$, then $(\mathcal{C}.Hess_{\mathcal{C}})_p \geq 6$;
- if the curve $\mathcal{C}$ has a unique branch at a point p, then $(\mathcal{C}.Hess_{\mathcal{C}})_p \geq 3M_pm_p - 2M_p - 2m_p$ where m_p is the multiplicity of C at p, and M is the maximal order of contact of a line with $\mathcal{C}$ at p (note that $M_p > m_p$ and that there exists a unique line $\mathcal{L}$ such that $(\mathcal{C}.\mathcal{L})_p > m_p$).

Consider a plane $\mathcal{P} \subset (\mathbb{C}^*)^N$ and a tropical morphism $f : C' \to \text{Trop}(\mathcal{P})$, denote the image by $C = f(C')$. As usual, given a primitive N-simplex Δ containing $\Delta(\mathcal{P})$, denote by $u_0, \ldots, u_N$ the outward primitive integer vectors normal to the faces of Δ. We also denote by $\mathcal{A}$ the line arrangement $\overline{\mathcal{P}} \setminus \mathcal{P}$. We define the three following subsets of Edge(C'):

$Bis_I(C') = \{e \in \text{Edge}(C') \mid u_{f,e} = u_I\}$,
$Bis(C') = \bigcup_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} Bis_I(C')$,
$K_w(C') = \{e \in \text{Edge}(C') \mid \exists i, \ u_{f,e} = u_i, \ \text{and } w_{f,e} > 1\}$,
$K_1(C') = \{e \in \text{Edge}(C') \mid \exists i, \ u_{f,e} = u_i, \ \text{and } w_{f,e} = 1\}$.

Note that if $\dim(\Delta(\mathcal{P})) = N-1$ all of the the above defined sets are dependent on the choice of N-simplex Δ, since they depend on the vectors u_i. Finally, we denote by $m_I(C)$ the multiplicity of the curve C at the point p_I, i.e. its intersection multiplicity at p_I with the ray u_I. For a simple expression of this multiplicity, let $Edge_I(C') = \{e \in \text{Edge}(C') \mid p_I \in \overline{f(e)}\}$. For an edge $e \in \text{Edge}_I(C')$, the vector $u_{f,e}$ has a unique expression $p_e u_I + q_e u_k$ for some $k \in I$. Then the multiplicity at point p_I is given by,

$$m_I(C) = \sum_{e \in Edge_I(C')} w_{f,e} p_e.$$

Again, the multiplicity $m_I(C)$ is dependent on the choice of N-simplex Δ, when a choice exists.

We can now state and prove the main result of this section.

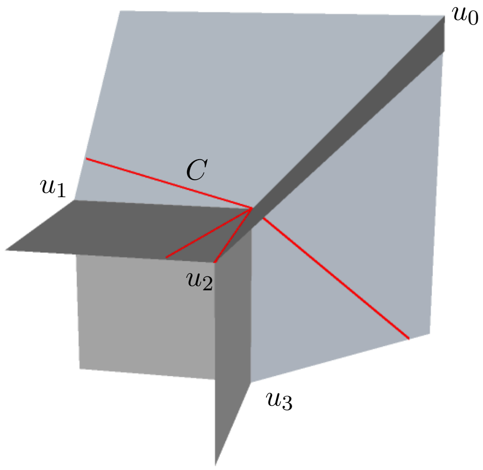

FIGURE 14. The 4-valent curve from Corollary 5.4.

Theorem 5.3. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and Δ a primitive N-simplex containing $\Delta(\mathcal{P})$. Let $f : C' \to Trop(\mathcal{P})$ be a tropical morphism such that $C = f(C')$ is an irreducible tropical curve and $\deg_\Delta(C) > 1$. If the morphism f is finely approximable in $\mathcal{P}$, then*

$$3C^2 + 2(N-2)\deg_\Delta(C) - \sum_{\boldsymbol{p}_I \in \boldsymbol{p}(\mathcal{A})} 2(|I|-2)m_I(C) - \sum_{e \in Edge(C')} (3w_{e,f} - 2) - |K_1(C')| \geq 0.$$

Proof. Suppose that $f : C' \to \text{Trop}(\mathcal{P})$ is finely approximable by $\mathcal{F} : \mathcal{C} \to \mathcal{P}$. Since $\mathcal{F}(\mathcal{C})$ is irreducible, we will identify $\mathcal{C}$ and $\mathcal{F}(\mathcal{C})$ in $\mathcal{P}$. Consider $\overline{\mathcal{C}}$ the closure of $\mathcal{C}$ in $\overline{\mathcal{P}} = \mathbb{C}P^2$, and define $q_1, \ldots, q_s$ the points in $\overline{\mathcal{C}} \cap (\mathcal{A} \setminus \mathbf{p}(\mathcal{A}))$ for which the mulitplicity of intersection is at least 2. We denote by m_j this intersection multiplicity at the point q_j.

Since the number of intersection points of $\overline{\mathcal{C}}$ with its Hessian in $\overline{\mathcal{P}} \setminus \{q_1, \ldots, q_s, \mathbf{p}_I \in \mathbf{p}(\mathcal{A})\}$ is non-negative and the Hessian is of degree $3(d-2)$, we have

$$(1) \qquad 3d(d-2) - \sum_{\mathbf{p}_I \in \mathbf{p}(\mathcal{A})} (\overline{\mathcal{C}}.Hess_{\overline{\mathcal{C}}})_{\mathbf{p}_I} - \sum_{j=1}^{s} (m_j - 2) \geq 0.$$

It is immediate from the definition of the weights of the tropicalisation of a morphism from Definition 2.8 that

$$\sum_{j=1}^{s} (m_j - 2) = \sum_{e \in K_w(C')} (w_{f,e} - 2).$$

It remains to estimate the quantities $(\overline{\mathcal{C}}.Hess_{\overline{\mathcal{C}}})_{\mathbf{p}_I}$. As before, we denote $\partial\overline{\mathcal{P}} = \overline{\mathcal{P}} \setminus \mathcal{P}$, and we claim that

$$(2) \quad (\overline{\mathcal{C}}.Hess_{\overline{\mathcal{C}}})_{\mathbf{p}_I} \geq 3(\overline{\mathcal{C}}^2)_{p_I} - 2(\overline{\mathcal{C}}.\partial\overline{\mathcal{P}})_{\mathbf{p}_I} - 2|Bis_I(C')| + 3 \sum_{e \in Bis_I(C')} w_{f,e} + 2(|I|-2)m_I(C).$$

To prove inequality (2), we have to estimate the number of inflection points that are contained in the Milnor fiber F_I of $\overline{\mathcal{C}}$ at $\mathbf{p}_I$. Let us denote by $b_1, \ldots, b_k$ the local branches of $\overline{\mathcal{C}}$ at p_I. Note that these branches are in one to one correspondence with the edges of $\text{Edge}_I(C')$. A small perturbation of F_I can be constructed as follows: in a small Milnor ball centered at $\mathbf{p}_I$, we translate each branch b_i such that they intersect transversally; then we replace each singular point by its Milnor fiber to obtain a surface Γ. Let us denote by m_i the multiplicity of b_i at $\mathbf{p}_I$, and by M_i the maximal order of contact of a line with b_i at $\mathbf{p}_I$. Note that if $e_i \in Bis_I(C')$, then $(b_i.\partial\overline{\mathcal{P}}) = |I|m_i$, and that

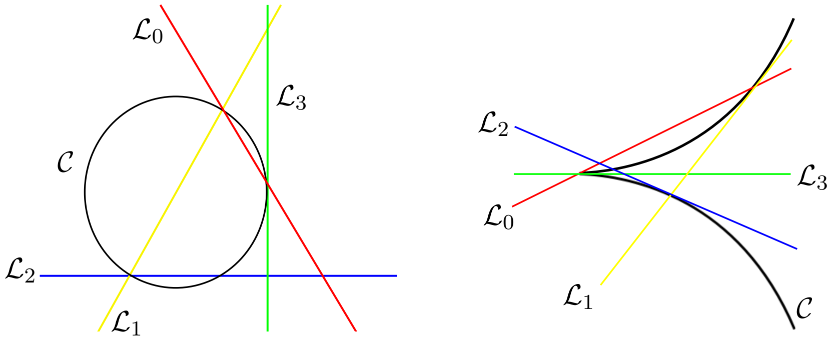

FIGURE 15. The conic and the cuspidal cubic from Corollary 5.4 with their positions with respect to the 4 lines.

if $e_i \notin Bis_I(C')$, then $(b_i.\partial\overline{\mathcal{P}}) = (|I|-1)m_i + M_i$. In addition, note that if $e_i \in Bis_I(C')$, then $m_i = w_{f,e_i}$ and $M_i \geq m_i + 1$. Combining both the second and first part of Example 5.2, the number of inflection points contained in Γ is at least,

$$6\sum_{i\neq j}(b_i.b_j)_{\mathbf{p}_I} + \sum_{e_i\notin Bis_I(C')}(3M_im_i - 2M_i - 2m_i) + \sum_{e_i\in Bis_I(C')}(3(m_i+1)m_i - 4m_i - 2).$$

By Definition 3.5, we have that

$$\sum_{e_i\notin Bis_I(C')} M_im_i + \sum_{e_i\in Bis_I(C')} m_i^2 + 2\sum_{i\neq j}(b_i.b_j)_{\mathbf{p}_I} = (\overline{C}^2)_{p_I}.$$

Combining the above two lines we obtain Inequality (2).

Hence we get from Inequality (1) and (2) that

$$\begin{aligned}(3)\qquad & 3d(d-2) - \textstyle\sum_{\mathbf{p}_I\in\mathbf{p}(\mathcal{A})}\left(3(\overline{C}^2)_{p_I} - 2(\overline{\mathcal{C}}.\partial\overline{\mathcal{P}})_{\mathbf{p}_I} + 2(|I|-2)m_I(C)\right) \\ & \qquad +2|Bis(C')| - 3\textstyle\sum_{e\in Bis(C')} w_{f,e} - \sum_{e\in K_w(C')}(w_{f(e)}-2) \geq 0\end{aligned}$$

By Definition 3.5, we have

$$\sum_{0\leq i<j\leq N}(\overline{C}^2)_{p_{i,j}} = d^2 - C^2.$$

Summing up all intersection multiplicities of $\overline{\mathcal{C}}$ with $\overline{\mathcal{P}}\setminus\mathcal{P}$, we get

$$\mathcal{C}.\partial\overline{\mathcal{P}} = (N+1)d = \sum_{\mathbf{p}_I\in\mathbf{p}(\mathcal{A})}(\overline{\mathcal{C}}.\partial\overline{\mathcal{P}})_{\mathbf{p}_I} + \sum_{e\in K_w(C')} w_{f(e)} + |K_1(C')|.$$

Plugging all of the latter equalities into Inequality (3) we obtain the desired inequality. $\square$

As an application, we prove the following corollary.

Corollary 5.4. *Let $\mathcal{P}$ be a uniform plane in $(\mathbb{C}^*)^3$, and denote by u_0, u_1, u_2, and u_3 the primitive integer directions of its four edges. Let C be a fan tropical curve in Trop($\mathcal{P}$) with four rays of weight 1 in the primitive integer directions*

$$u_1 + (d-1)u_2, \quad (d-1)u_1 + u_0, \quad du_3 + (d-1)u_0, \quad \textit{and} \quad u_2,$$

(see Figure 14). Then C is approximable by a complex curve $\mathcal{C}\subset\mathcal{P}$ if and only if $d = 1, 2$ or 3.

Proof. The tropical cycle induced by C is irreducible, hence if an approximation $\mathcal{C}$ exists it must be an irreducible and reduced curve. Also, as $\mathcal{P}$ is uniform we have $\Delta(\mathcal{P}) = \Delta$, and $\deg_\Delta(C) = d$.

If $d = 1$ the tropical curve C is the skeleton of the plane Trop$(\mathcal{P})$ and it is approximated by a generic line $\mathcal{L} \subset \mathcal{P}$. For $d > 1$ the closure of the tropical curve $\overline{C} \subset \overline{\text{Trop}(\mathcal{P})} \subset \mathbb{T}P^3$ contains the corner points $p_{i,j} \subset \overline{\text{Trop}(\mathcal{P})}$ for $\{i,j\} = \{0,1\}, \{0,3\}, \{1,2\}$. At these corner points we have the polygons

$$\Gamma^c_{i,j} = Conv\{(0,0),(0,1),(d-1,0)\}, \text{ for } \{i,j\} = \{0,1\} \text{ and } \{1,2\}$$

and

$$\Gamma^c_{0,3} = Conv\{(0,0),(0,d),(d-1,0)\}.$$

It is immediate that $\mathcal{C}$ exists if $d = 2$ or 3; the curves relative to the line arrangements are drawn in Figure 15. For $d = 3$ the curve has a cusp at $\mathbf{p}_{0,3}$ and simple tangencies to $\mathcal{L}_1$ and $\mathcal{L}_0$ at the points $\mathbf{p}_{1,2}$ and $\mathbf{p}_{0,1}$. For $d = 2$, the curve is a conic passing through the point $\mathbf{p}_{0,3}$ with a tangency to the line $\mathcal{L}_3$, and also passing through the points $\mathbf{p}_{0,1}$ and $\mathbf{p}_{1,2}$ having intersection multiplicity one with the respective lines at these points.

Let us now prohibit the remaining situations. An easy computation yields $C^2 = 2 - d$, then according to Theorem 5.3, if $\mathcal{C}$ exists then

$$3(2-d) + 2d - 2 \geq 0$$

which reduces to $d \leq 4$. We are left to study by hand the case $d = 4$. Choose coordinates on $\overline{\mathcal{P}} = \mathbb{C}P^2$ such that $\mathcal{L}_1, \mathcal{L}_0, \mathcal{L}_3$ are coordinates axes. In this coordinate system, $\Delta(\mathcal{C}) = Conv\{(0,4),(0,3),(1,0)\}$. It is not hard to verify (see for example [BLdM, Lemma 6.4]) that such a curve cannot have an inflection point on $\mathcal{L}_0 \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$ □

These curves will be revisited in Section 7 as Vigeland lines in a tropical surface.

6. Trivalent fan tropical curves in planes.

In this section we classify all 3-valent tropical curves $C \subset \text{Trop}(\mathcal{P}) \subset \mathbb{R}^N$ which are finely approximable in a given non-degenerate plane $\mathcal{P} \subset (\mathbb{C}^*)^N$. To this end we first focus on the case when $\mathcal{P}$ is a uniform hyperplane in $(\mathbb{C}^*)^3$.

6.1. Tropical curves in $\mathbb{R}^3$ with $\text{aff}_C \leq 2$.

Given a fan tropical curve $C \subset \mathbb{R}^N$, we denote by Aff(C) its affine span in $\mathbb{R}^N$, and by aff_C the dimension of Aff(C). The space Aff(C) has a natural tropical structure since it is the tropicalisation of a binomial surface in $(\mathbb{C}^*)^N$.

In [BK], T. Bogart and E. Katz gave some combinatorial obstructions to the approximation of a fan tropical curve C in a uniform tropical plane in $\mathbb{R}^3$ satisfying $\text{aff}_C \leq 2$. Combining their results with our own we obtain the complete classification of such approximable tropical curves.

Theorem 6.1. *Let $\mathcal{P}$ be a uniform plane in $(\mathbb{C}^*)^3$, and let us denote by u_0, u_1, u_2, and u_3 the four rays of Trop$(\mathcal{P})$. Let $C \subset$ Trop$(\mathcal{P})$ be a reduced fan tropical curve in $\mathbb{R}^3$ with $\text{aff}_{f(C)} \leq 2$. Then the curve C is finely approximable in $\mathcal{P}$ if and only if one of the two following conditions hold*

1. *C is equal to the tropical stable intersection of Aff$(f(C))$ and Trop$(\mathcal{P})$ (see* [RGST05]*);*
2. *C has three edges e_1, e_2, and e_3 satisfying*

$$w_{e_1}u_{e_1} = u_i + du_k, \quad w_{e_2}u_{e_2} = u_j + du_l, \quad w_{e_3}u_{e_3} = (d-1)(u_i + u_j)$$

with $\{i,j,k,l\} = \{0,1,2,3\}$. In particular, if $d = 1$ the fan tropical curve C is 2-valent with directions $u_i + u_j$, and $u_k + u_l$.

Moreover in this case $C = f(C_0)$ where C_0 is a fan tropical curve with $d+1$ edges, and $f : C_0 \to$ Trop$(\mathcal{P})$ is a tropical morphism finely approximable in $\mathcal{P}$ by a rational curve with $d+1$ punctures. In particular, f maps all edges of C_0 to Trop$(\mathcal{P})$ with weight 1, and $(d-1)$ edges of C_0 are mapped to e_3 by f.

The proof of Theorem 6.1 decomposes into several steps. Let us first recall two lemmas from [BK].

Lemma 6.2 (Bogart-Katz, [BK])**.** *Let C be a tropical curve in $\mathbb{R}^3$ such that $\mathrm{aff}_C \leq 2$ and which is approximable by a reduced and irreducible complex algebraic curve $\mathcal{C} \subset (\mathbb{C}^*)^3$. Then there exists a reduced and irreducible binomial algebraic surface $\mathcal{H} \subset (\mathbb{C}^*)^3$ such that $Trop(\mathcal{H}) = Aff(C)$ and $\mathcal{C} \subset \mathcal{H}$.*

Lemma 6.3 (Bogart-Katz, [BK])**.** *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and let $\mathcal{H} \subset (\mathbb{C}^*)^3$ be a reduced and irreducible binomial surface. Then either the curve $\mathcal{P} \cap \mathcal{H}$ is non-singular or it has a unique singular point which is a node in $(\mathbb{C}^*)^3$. Moreover, if $\mathcal{P} \cap \mathcal{H}$ has two irreducible components $\mathcal{C}_1$ and $\mathcal{C}_2$, then the embedded tropical curve $Trop(\mathcal{C}_1) \cup Trop(\mathcal{C}_2)$ is 4-valent, the two tropical curves $Trop(\mathcal{C}_1)$ and $Trop(\mathcal{C}_2)$ are at most 3-valent, and at least one of them is 2-valent.*

Lemma 6.3 implies immediately that a fan tropical curve $C \subset \mathbb{R}^3$ with $\mathrm{aff}_C \leq 2$ which is finely approximable in a uniform plane $\mathcal{P} \subset (\mathbb{C}^*)^3$ and not equal to the tropical stable intersection of $\mathrm{Trop}(\mathcal{P})$ and $\mathrm{Aff}(C)$ must be either 2 or 3-valent.

Lemma 6.4. *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and let $\mathcal{H} \subset (\mathbb{C}^*)^3$ be a reduced and irreducible binomial surface. We denote by $\Delta_{\mathcal{P},\mathcal{H}}$ the Newton polytope of the tropical surface $Trop(\mathcal{P}) \cup Trop(\mathcal{H})$, and by $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}})$ the toric variety defined by $\Delta_{\mathcal{P},\mathcal{H}}$. Let $\overline{\mathcal{P}}$ and $\overline{\mathcal{H}}$ be respectively the closure of $\mathcal{P}$ and $\mathcal{H}$ in $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}})$, and let $\overline{\mathcal{C}} = \overline{\mathcal{P}} \cap \overline{\mathcal{H}}$. Then the curve $\overline{\mathcal{C}}$ is reduced and $\overline{\mathcal{C}}^2 = 0$ in $\overline{\mathcal{P}}$. Moreover, if $\overline{\mathcal{C}}$ is reducible, then $\overline{\mathcal{C}}$ has exactly two irreducible components $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$, and $\overline{\mathcal{C}}_1^2 = \overline{\mathcal{C}}_2^2 = -1$ in $\overline{\mathcal{P}}$.*

In particular, $Trop(\mathcal{C}_1)^2 = Trop(\mathcal{C}_2)^2 = -1$ in $Trop(\mathcal{P})$.

Proof. We define $\mathcal{C} = \mathcal{P} \cap \mathcal{H} = \overline{\mathcal{C}} \cap (\mathbb{C}^*)^3$. According to Lemma 6.3, the curve $\mathcal{C}$ has at most one singular point, so it has to be reduced and cannot have more than two irreducible components. Hence the same is true for $\overline{\mathcal{C}}$. Since $\overline{\mathcal{H}}^2 = 0$, we also have $\overline{\mathcal{C}} = 0$ in $\overline{\mathcal{P}}$. Suppose that $\overline{\mathcal{C}}$ has two irreducible components $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$. Since $\overline{\mathcal{H}}^2 = 0$ we have

$$\overline{\mathcal{C}}_1.\overline{\mathcal{C}} = \overline{\mathcal{C}}_2.\overline{\mathcal{C}} = 0$$

which implies that

$$\overline{\mathcal{C}}_1^2 + \overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = \overline{\mathcal{C}}_2^2 + \overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = 0.$$

So we are left to show that $\overline{\mathcal{C}}_1.\overline{\mathcal{C}}_2 = 1$. Since the curve $\overline{\mathcal{C}}$ is reducible, it follows from Lemma 6.3 that $\mathcal{C}$ has a unique singular point, which is a node, in $(\mathbb{C}^*)^3$. Hence the result will follow from the fact that $\overline{\mathcal{C}}$ intersects the boundary $\mathcal{X}(\Delta_{\mathcal{P},\mathcal{H}}) \setminus (\mathbb{C}^*)^3$ transversally at non-singular points of $\overline{\mathcal{C}}$.

To prove this last claim, we may assume that $\mathcal{H}$ is a subtorus of $(\mathbb{C}^*)^3$. In this case, there is a surjection $\phi : \mathrm{Hom}((\mathbb{C}^*)^3, \mathbb{C}^*) \otimes \mathbb{R} \to \mathrm{Hom}(\mathcal{H}, \mathbb{C}^*) \otimes \mathbb{R}$. Moreover, if $Q \in Hom((\mathbb{C}^*)^3, \mathbb{C}^*)$ is an equation of $\mathcal{P}$ in $(\mathbb{C}^*)^3$, then $\phi(Q)$ is an equation of $\mathcal{C}$ in $\mathcal{H}$. In particular, the Newton polygon of $\phi(Q)$ is dual to the tropical curve $\mathrm{Trop}(\mathcal{C})$, seen as a tropical curve in $\mathrm{Trop}(\mathcal{H})$. According to Lemma 6.3, the tropical curve $\mathrm{Trop}(\mathcal{C})$ is 4-valent, so the Newton polygon of $\phi(Q)$ is a quadrangle. The polynomial Q has exactly 4 monomials and $\phi(Q)$ has no few monomials than Q, so we get that $\phi(Q)$ also has exactly 4 monomials. In particular, the only non-zero coefficients of $\phi(Q)$ are the vertices of its Newton polygon. This implies that $\overline{\mathcal{C}}$ intersects the boundary $\overline{\mathcal{H}} \setminus \mathcal{H}$ transversally at non-singular points of $\overline{\mathcal{C}}$. □

To prove Theorem 6.1, we need to list all possible 3-valent fan tropical plane curves C with $\mathrm{aff}_C \leq 2$ and to compute C^2 for each of them. This is the content of the next lemma.

Lemma 6.5. *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and let $C \subset Trop(\mathcal{P})$ be an irreducible 3-valent fan tropical curve with $\mathrm{aff}_C \leq 2$. Let $u_0, u_1, u_2,$ and u_3 be the primitive integer directions of the edges*

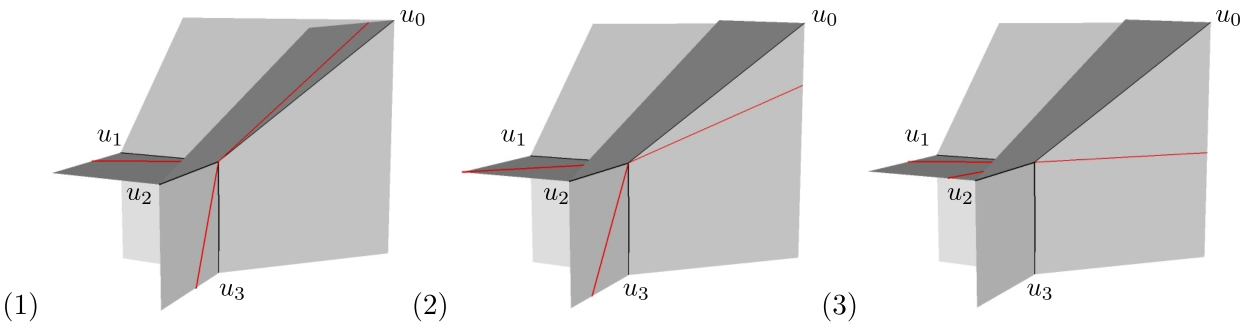

FIGURE 16. The three types of curves from Lemma 6.5.

of Trop($\mathcal{P}$), and let e_1, e_2, *and* e_3 *denote the edges of* C. *Then, up to the action of* $\mathcal{S}_4$ *on the rays of Trop($\mathcal{P}$) and re-ordering of the edges of* C, *the curve* C *is one of the following types:*

(1) *There exists* $0 \leq \alpha, \beta$ *with* $\gcd(d, \alpha, \beta) = 1$ *and* $\alpha + \beta \leq d$, *such that*

$$w_{e_1} u_{e_1} = du_1 + \alpha u_2, \quad w_{e_2} u_{e_2} = \beta u_2 + du_3, \quad \textit{and} \quad w_{e_3} u_{e_3} = (d - \alpha - \beta) u_2 + du_4,$$

see Figure 16 (1). In this case the curve C *is the tropical intersection of Trop($\mathcal{P}$) and Aff(C), and* $C^2 = 0$;

(2) *There exists* $0 \leq \alpha, \beta \leq d$ *with* $\gcd(d, \alpha, \beta) = 1$ *such that*

$$w_{e_1} u_{e_1} = du_1 + \alpha u_2, \quad w_{e_2} u_{e_2} = (d - \alpha) u_2 + (d - \beta) u_3, \quad \textit{and} \quad w_{e_3} u_{e_3} = \beta u_3 + du_4,$$

see Figure 16 (2). In this case, $C^2 = -\alpha\beta$;

(3) *There exists* $0 \leq \alpha < \beta \leq d$ *with* $\gcd(d, \alpha, \beta) = 1$ *such that*

$$w_{e_1} u_{e_1} = \alpha u_1 + \beta u_2, \quad w_{e_2} u_{e_2} = (d - \alpha) u_1 + (d - \beta) u_2, \quad \textit{and} \quad w_{e_3} u_{e_3} = du_3 + du_4,$$

see Figure 16 (3). In this case, $C^2 = -d^2 + \beta d - \alpha\beta$.

Note that cases (1) and (2) when $\alpha = \beta = 0$ (and consequently $d = 1$) coincide with the case (3) when $\alpha = 0$ and $\beta = d = 1$. In addition, notice that when $d > 1$ and $\alpha, \beta > 0$, an approximation of a curve from case (1) must intersect the line arrangement $\overline{\mathcal{P}} \setminus \mathcal{P}$ in only three collinear points $\mathbf{p}_{1,2}, \mathbf{p}_{2,3}, \mathbf{p}_{2,4}$. Under the same conditions on d, α and β, an approximation of a curve from case (2) must intersect the arrangement in only the three points $\mathbf{p}_{1,2}, \mathbf{p}_{2,3}, \mathbf{p}_{3,4}$. Finally an approximation of a curve from case (3) may only intersect the arrangement in two points $\mathbf{p}_{1,2}, \mathbf{p}_{3,4}$, if we assume again the same restrictions on d, α and β.

Proof. The intersection numbers follow from a direct computation. In case (1), we have to prove in addition that the curve C is the tropical intersection of Trop($\mathcal{P}$) and Aff(C), which is non-trivial only for $\alpha \neq 0$ and $\beta \neq 0$. If C' denotes this tropical intersection, it is clear that C and C' have the same underlying sets. Since C is irreducible, it remains to prove that C' is also irreducible.

Without loss of generality, we can assume that

$$u_0 = (1, 1, 1), \quad u_1 = (-1, 0, 0), \quad u_2 = (0, -1, 0), \quad \text{and} \quad u_3 = (0, 0, -1).$$

The surface Aff(C) is given by a classical linear equation of the form

$$ax + by + cz = 0 \quad \text{with} \quad \gcd(a, b, c) = 1.$$

Let us denote by $w_{1,2}$ (resp. $w_{2,3}$) the weight of the edge of C' lying in the convex cone spanned by u_1 and u_2 (resp. u_3 and u_2). A computation gives $w_1 = \gcd(a, b)$, and $w_2 = \gcd(b, c)$. Hence w_1 and w_2 are relatively prime and C' is irreducible, which implies $C' = C$. This completes the proof. $\square$

Remark 6.6. Note that the same proof gives that the tropical stable intersection of *any* tropical surface of degree 1 in $\mathbb{R}^3$ made of an edge and 3 faces with *any* non-singular binomial tropical surface in $\mathbb{R}^3$ is *always* irreducible.

End of the proof of Theorem 6.1: It follows from Lemma 6.4 that if the tropical curve C is finely approximable in $\mathcal{P}$, then $C^2 = 0$ or -1 in $\text{Trop}(\mathcal{P})$. Now, it follows from Lemma 6.5 that

- the case when $C^2 = 0$ corresponds to the case (1) of Theorem 6.1. The approximation of such a tropical curve follows from the fact that tropical stable intersections are always approximable (see [OP]).
- the case when $C^2 = -1$ corresponds to the case (2) of Theorem 6.1. Up to automorphism of $\mathbb{C}P^2$ we are free to fix the 4 generic lines of the arrangement corresponding to $\mathcal{P}$ as we wish. Let $\mathcal{L}_2$ be the line at infinity and

$$\mathcal{L}_1 = \{y = 0\}, \quad \mathcal{L}_3 = \{x = 0\}, \quad \text{and} \quad \mathcal{L}_4 = \{x + y - 1 = 0\}.$$

The existence of an approximation in $\mathcal{P}$ of the tropical morphism $f : C_0 \to \text{Trop}(\mathcal{P})$ is equivalent to the existence of an irreducible complex algebraic curve $\mathcal{C}$ in $\mathbb{C}^2$ defined by the equation $yP(x) - 1 = 0$ where $P(x)$ is a complex polynomial of degree $d-1$ with no multiple root, and such that $\mathcal{C}$ has order of contact d at $(0, 1)$ with the line $\mathcal{L}_4$. It is not hard to check that one has to have $P(x) = 1 + x + \ldots + x^{d-1}$. The curve $\mathcal{C}$ has the correct order of contact with each of the lines $\mathcal{L}_i$ of $\overline{\mathcal{P}} \backslash \mathcal{P}$ in order to tropicalise to $C \subset P$, and it is immediate that $\mathcal{C}$ is a rational curve with $d + 1$ punctures.

This completes the proof of the theorem. □

Notice that by Corollary 3.11, the above constructed complex curve $\mathcal{C}$ is the unique curve which approximates a tropical curve C from Case (2) of Theorem 6.1, since $C^2 = -1$.

6.2. **Classification of 3-valent fan tropical curves in a tropical plane.** The complete classification of fan tropical curves $C \subset \mathbb{R}^3$ with $\text{aff}_C \leq 2$ which are finely approximable in a non-degenerate plane in $(\mathbb{C}^*)^3$ can be extended to classify all 3-valent fan tropical curves finely approximable in a given plane of any codimension. To this aim the next lemma determines whether a tropical curve $L \subset \text{Trop}(\mathcal{P})$ with $\deg_\Delta(L) = 1$ and any valency is approximable.

Lemma 6.7. *Let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane and Δ a primitive N-simplex containing $\Delta(\mathcal{P})$. Let $L \subset Trop(\mathcal{P}) \subset \mathbb{R}^N$ be a k-valent fan tropical curve with $\deg_\Delta(L) = 1$. Then L is approximated by some line $\mathcal{L} \subset \mathcal{P}$ if and only if for some $0 \leq m \leq k$ the arrangement determined by $\mathcal{P}$ contains m collinear points $\boldsymbol{p}_{I_1}, \ldots, \boldsymbol{p}_{I_m}$ such that the sets $I_1, \ldots, I_m$ are disjoint and $I_1 \cup \cdots \cup I_m = \{0, \ldots, N\} \backslash J$ where $|J| = k - m$; moreover, the k rays of L are in the directions*

$$u_{I_1}, \ldots u_{I_m}, \quad \textit{and} \quad \{u_i \mid i \in J\},$$

and $\mathcal{L}$ is the line passing through the m collinear points $\boldsymbol{p}_{I_i}$.

Proof. Suppose a curve $\mathcal{L}$ approximates L. Since $\deg_\Delta(L) = 1$, the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ must be a line in the compactification of $\mathcal{P}$ to $\overline{\mathcal{P}} = \mathbb{C}P^2$ given by Δ. Therefore $\overline{\mathcal{L}}$ intersects each line in the arrangement determined by $\mathcal{P}$ exactly once. Since L is k-valent and each edge is of weight one, the line $\overline{\mathcal{L}}$ may intersect the arrangement in only k points. These k intersection points induce a partition of the set of lines in the arrangement into k subsets. The subsets of size greater than one correspond to points $\mathbf{p}_{I_i}$ of the arrangement through which the line $\mathcal{L}$ passes. □

Before considering the general case we remark that not every tropical plane $P \subset \mathbb{R}^N$ contains trivalent fan tropical curves. In fact, in order for P to contain a trivalent curve there must exist

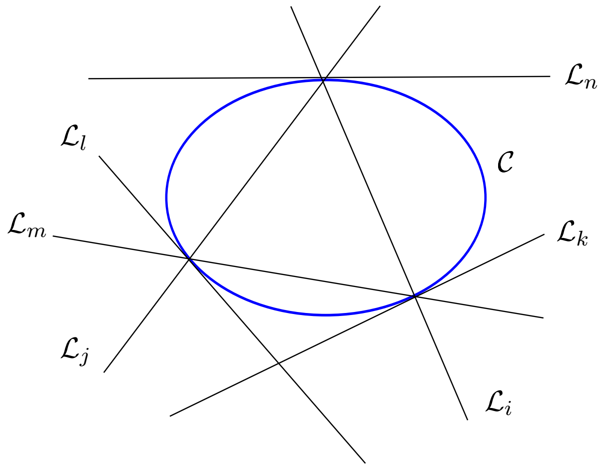

FIGURE 17. The curve $\mathcal{C}$ from Lemma 6.8 with respect to the lines indexed by i, j, k, l, m, n.

three sets I_1, I_2, I_3 satisfying: $I_1 \cup I_2 \cup I_3 = \{0, \dots, N\}$ and if $|I_i| > 1$ then p_{I_i} is a point of the corresponding line arrangement.

Given two line arrangements $\mathcal{A} \subset \mathcal{A}'$ in $\mathbb{C}P^2$, there is a natural inclusion of their respective planes $i : \mathcal{P}' \hookrightarrow \mathcal{P}$.

Lemma 6.8. *Let $\mathcal{P} \subset (\mathbb{C}^*)^3$ be a uniform plane, and denote the lines of the associated arrangement by $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$, and $\mathcal{L}_l$. In addition, let $\mathcal{C}_2 \subset \mathcal{P}$ be the degree 2 curve from part (2) of Theorem 6.1. Then the further fan tropical curves are finely approximated in the plane $\mathcal{P}'$ in following cases (see Figure 17):*

(1) *the plane $\mathcal{P}' \subset (\mathbb{C}^*)^4$ corresponds to the arrangement of 5 lines obtained by adding to the arrangement of $\mathcal{P}$ the unique line $\mathcal{L}_m$ which passes through the two points $\boldsymbol{p}_{i,k}$, $\boldsymbol{p}_{j,l}$; the three rays of $C \subset Trop(\mathcal{P}')$ are of weight one with primitive integer directions*
$$u_i + u_j, \quad u_i + 2u_k + u_m, \quad \text{and} \quad u_j + 2u_l + u_m.$$
(2) *the plane $\mathcal{P}' \subset (\mathbb{C}^*)^4$ corresponds to the arrangement of 5 lines obtained by adding to the arrangement of $\mathcal{P}$ the unique line $\mathcal{L}_n$ which is tangent to $\mathcal{C}_2$ at the point $\boldsymbol{p}_{i,j}$; the three rays of $C_2 \subset Trop(\mathcal{P}')$ are of weight one with primitive integer directions*
$$u_i + u_j + 2u_n, \quad u_i + 2u_k, \text{ and } \quad u_j + 2u_l.$$
(3) *the plane $\mathcal{P}' \subset (\mathbb{C}^*)^5$ corresponds to the arrangement of 6 lines obtained from the arrangement of $\mathcal{P}$ by adding the lines $\mathcal{L}_m$ and $\mathcal{L}_n$ from parts (1) and (2); the three rays of $C \subset Trop(\mathcal{P}')$ are of weight one and with primitive integer directions*
$$u_i + u_j + 2u_m, \quad u_i + 2u_k + u_n, \quad \text{and} \quad u_j + 2u_l + u_n.$$

Proof. In each of the three above cases the tropical curves are approximated by the curve $\mathcal{C} = \mathcal{C}_2 \cap \mathcal{P}'$. □

Theorem 6.9. *Let $N \geq 3$, let $\mathcal{P} \subset (\mathbb{C}^*)^N$ be a non-degenerate plane, and let $C \subset Trop(\mathcal{P})$ be an irreducible 2 or 3-valent fan tropical curve. Then the curve C is finely approximable in $\mathcal{P}$ if and only if we are in one of the following cases:*

(1) *there exists a primitive N-simplex Δ containing $\Delta(\mathcal{P})$ such that $\deg(C)_\Delta = 1$ and C and $\mathcal{P}$ satisfy Lemma 6.7;*
(2) *C and $\mathcal{P}$ satisfy one of the three situations described in Lemma 6.8;*

(3) *the plane* $\mathcal{P} \subset (\mathbb{C}^*)^3$ *is non-uniform and* C *is any irreducible trivalent fan tropical curve;*
(4) *the plane* $\mathcal{P} \subset (\mathbb{C}^*)^3$ *is uniform and* C *is a trivalent curve from part* (2) *of Theorem 6.1 or part* (1) *of Lemma 6.5.*

As a remark we mention that the trivalent lines with $N \geq 6$ in case (1) of Theorem 6.9 and the curves of case (2) and (3) of Lemma 6.8 are **exceptional**, in the sense that for a generic choice of plane $\mathcal{P}$ which tropicalises to the fans in each of these cases (i.e. whose line arrangement has the right intersection lattice), the corresponding tropical curve will not be approximable.

Proof. All of the above tropical curves were shown to be approximable in the corresponding plane in Lemmas 6.7, 6.8, Remark 6.6, and Theorem 6.1.

Let C be an irreducible 2 or 3-valent fan tropical curve which is finely approximable by a curve $\mathcal{C}$ in some plane $\mathcal{P}$. It remains to show that the pair $(\mathcal{P}, \mathcal{C})$ is one of those described in the theorem. According to Lemma 6.7, this is true if $\deg_\Delta(C) = 1$, for some primitive N-simplex Δ containing $\Delta(\mathcal{P})$ so let us suppose that $\deg_\Delta(C) \geq 2$ for all such Δ.

Suppose first that the arrangement determined by $\mathcal{P}$ contains a uniform subarrangement $\mathcal{A}_0$ of 4 lines $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$, and $\mathcal{L}_l$ yielding a plane $\mathcal{P}_0$. In this case $\Delta(\mathcal{P})$ must be an N-simplex thus the degree of a tropical curve $C \subset \text{Trop}(\mathcal{P})$ is determined by $\Delta(\mathcal{P})$ itself so we denote it simply by $\deg(C)$. Then there is a natural inclusion $\mathcal{P} \hookrightarrow \mathcal{P}_0$. If $\mathcal{C} \subset \mathcal{P}$ approximates C, let $\mathcal{C}_0 \subset \mathcal{P}_0$ be the closure of $\mathcal{C}$ in $\mathcal{P}_0$ and $C_0 \subset \text{Trop}(\mathcal{P}_0)$ its tropicalisation. Then $\deg(C_0) = \deg(C) \geq 2$, which implies by Theorem 6.1 that the curve C_0 is trivalent. Therefore C_0 is either of type (1) from Lemma 6.5 or from case (2) of Theorem 6.1.

Suppose it is the former. Since $\deg(C_0) \geq 2$, up to relabeling the four lines in $\mathcal{A}_0$, it follows from Lemma 6.5 that $\mathcal{C}_0 \subset \mathcal{P}_0$ intersects this uniform arrangement in the three points $\mathbf{p}_{i,j}, \mathbf{p}_{i,k}, \mathbf{p}_{i,l}$. Let $\mathcal{L}$ be another line of the arrangement $\mathcal{A}$ determined by $\mathcal{P}$. Since C is trivalent, the line $\mathcal{L}$ must intersect $\mathcal{C}$ with multiplicity $\deg(C)$ at one of these three points, which is impossible according to Lemma 6.5.

If $C_0 \subset \text{Trop}(\mathcal{P}_0)$ satisfies part (2) of Theorem 6.1, then $\mathcal{C}_0$ is the unique curve given in the proof of Theorem 6.1 (up to relabelling the four lines in $\mathcal{A}_0$). This curve intersects the arrangement $\mathcal{A}_0$ in the points $\mathbf{p}_{i,j}, \mathbf{p}_{i,k}$, and $\mathbf{p}_{j,l}$. Note that the only singular point of $\overline{\mathcal{C}} \subset \overline{\mathcal{P}}$ may be at the point $\mathbf{p}_{i,j}$ and that the tangent line to $\overline{\mathcal{C}}_0$ at the points $\mathbf{p}_{i,k}$ and $\mathbf{p}_{j,l}$ is already contained in the arrangement $\mathcal{A}_0$. Therefore, any other line passing through $\mathbf{p}_{i,k}$ or $\mathbf{p}_{j,l}$ has intersection multiplicity 1 with $\overline{\mathcal{C}}_0$ at this point. Let $\mathcal{L}$ be a line in $\mathcal{A} \setminus \mathcal{A}_0$. Since the tropical curve C is trivalent, we are in one of the following two situations:

(1) the line $\mathcal{L}$ passes through the points $\mathbf{p}_{i,k}, \mathbf{p}_{j,l}$, and the sum of the intersection multiplicities of $\overline{\mathcal{C}}$ and $\mathcal{L}$ at these two points is equal $\deg(C)$; since the intersection multiplicity of $\overline{\mathcal{C}}$ and $\mathcal{L}$ is 1 at these points, this is possible only if $d = 2$;
(2) the line $\mathcal{L}$ passes through the point $\mathbf{p}_{i,j}$, $\mathbf{p}_{i,k}$, or $\mathbf{p}_{j,l}$, and intersects $\overline{\mathcal{C}}$ with multiplicity $\deg(C)$ at this point; since the intersection multiplicity of $\overline{\mathcal{C}}$ and $\mathcal{L}$ is 1 at $\mathbf{p}_{i,k}$ and $\mathbf{p}_{j,l}$, the line $\mathcal{L}$ necessarily passes through $\mathbf{p}_{i,j}$, which is an ordinary point of multiplicity $d-1$ of $\overline{\mathcal{C}}$; since C is 3-valent, the line $\mathcal{L}$ must have the same intersection multiplicity with all local branches of $\overline{\mathcal{C}}$ at $\mathbf{p}_{i,j}$, which is possible only if $d = 2$.

Hence if we are not in cases (1) or (4) from the statement of the theorem, we are necessarily in case (2).

If the arrangement $\mathcal{A}$ does not contain a uniform subarrangement of 4 lines then according to Lemma 6.10 below, all but one line of $\mathcal{A}$ must belong to the same pencil. Then, $\Delta(\mathcal{P})$ is $N-1$ dimensional and we must choose an N-simplex Δ containing $\Delta(\mathcal{P})$. The arrangement $\mathcal{A}$ contains a subarrangement $\mathcal{A}'_0$ of 4 lines, 3 of which belong to the same pencil, and defining a plane $\mathcal{P}'_0$. As previously, if $\mathcal{C} \subset \mathcal{P}$ approximates C, let $\mathcal{C}'_0 \subset \mathcal{P}'_0$ be the closure of $\mathcal{C}$ in $\mathcal{P}'_0$ and $C'_0 \subset \text{Trop}(\mathcal{P}'_0)$ its tropicalisation. Since $\deg_\Delta(C'_0) \geq 2$, the curve C'_0 is trivalent. Hence according to Remark 6.6, C'_0 is the tropical stable intersection of $\text{Trop}(\mathcal{P}'_0)$ and $\text{Aff}(C)$. If $\mathcal{C}'_0$ does not pass through the triple point of $\mathcal{A}'_0$, then since $\text{Trop}(\mathcal{C})$ is trivalent we must have $\mathcal{A} = \mathcal{A}'_0$. If $\mathcal{C}'_0$ passes through the triple point of $\mathcal{A}'_0$, then there exist $|I| - 2$ lines of $\mathcal{A}$ such that each branch of $\mathcal{C}$ at $\mathbf{p}_I$ has order of contact $\deg_\Delta(C)$ with these lines. This implies that $|I| = 3$, which completes the proof. □

Lemma 6.10. *If $\mathcal{A}$ is a line arrangement not containing a uniform subarrangement of 4 lines then all but one of the lines in $\mathcal{A}$ are contained in the same pencil.*

Proof. By assumption not all lines of $\mathcal{A}$ belong to the same pencil, so there is a subarrangement of 3 lines $\mathcal{L}_i, \mathcal{L}_j$, and $\mathcal{L}_k$, that is uniform. Every other line in $\mathcal{A}$ must belong to the pencil determined by a pair of lines in this subarrangement, otherwise there would be 4 lines forming a uniform subarrangement. If two of the additional lines indexed by l, m belong to different pencils given by say $\mathbf{p}_{i,j}$ and $\mathbf{p}_{i,k}$ then the subarrangement given by j, k, l, m is uniform, and we obtain a contradiction. □

7. Application to tropical lines in tropical surfaces

In the space of tropical surfaces of degree d in $\mathbb{T}P^3$, there exists an open subset of surfaces which contain (potentially infinitely many) tropical lines. In this section we prove that a generic non-singular tropical surface S in $\mathbb{R}^3$ of degree 3 contains finitely many tropical lines L such that the pair (S, L) is approximable. In addition, a generic non-singular tropical surface S of degree greater than 3 contains no tropical lines L such that the pair (S, L) is approximable. Therefore, if we restrict to approximable lines the tropical situation is analogous to the classical algebro-geometric one.

Examples of generic non-singular tropical surfaces in $\mathbb{R}^3$ of any degree containing infinitely many lines were first constructed by Vigeland in [Vig09]. In [Vig], Vigeland later classified by combinatorial type all tropical lines in generic non-singular tropical surfaces. In this entire section, we denote by Δ_d the simplex

$$\Delta_d = Conv\{(0,0,0), (d,0,0), (0,d,0), (0,0,d)\},$$

and by $F_1, \ldots, F_4$ its facets.

7.1. 1-parametric families of lines.

A simplex $\Delta \subset \Delta_d$ with vertices in $\mathbb{Z}^3$ is said to be **d-pathological** if Δ is primitive (i.e. of lattice area 1) and if Δ has one edge in $F_i \cap F_j$, one edge in F_k, and one edge in F_l for $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Theorem 7.1 (Vigeland, [Vig])**.** *Let S be a generic non-singular tropical surface of degree $d \geq 3$ in $\mathbb{R}^3$ with Newton polytope Δ_d. If S contains a 1-parametric family of tropical lines, then there exists a vertex p of S, dual to a d-pathological simplex, such that any line in the family is contained in the fan $p + Star_p(S)$.*

Conversely, any compact tropical surface S in $\mathbb{R}^3$ containing a d-pathological simplex in its dual subdivision contains infinitely many tropical lines.

Examples of tropical surfaces with infinitely many lines from [Vig09] were constructed using the d-pathological simplex with vertices $(0,0,0), (1,0,0), (0,1,d-1)$, and $(d-1,0,1)$. It is easy to see that any d-pathological simplex defines a tropical fan containing a 1-parametric family of tropical lines. This family consists of a unique 4-valent line and the remaining lines have two 3-valent vertices. Here we prove that among all of these families, there is only a single line which is approximable. Before giving the rigourous statement, lets us first describe in details tropical fans with Newton polygon a d-pathological simplex.

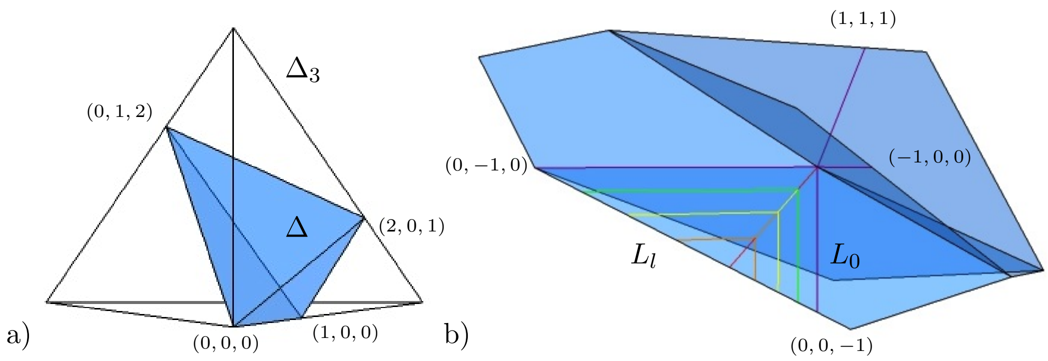

FIGURE 18. a) A pathological 3-simplex Δ from Theorem 7.2 drawn inside Δ_3. b) The vertex of the surface dual to Δ and the infinite family of tropical lines. By Theorem 7.2 only L_0 is approximable in S.

Let Δ be such a d-pathological simplex with $d \geq 3$. Without loss of generality, we can suppose that Δ has one edge in $F_1 \cap F_2$. Hence either Δ has one edge in $F_3 \cap F_4$, or Δ has one edge in F_3 and the last one in F_4. The first case is impossible since Δ would not be primitive. Hence, up to permutation of the coordinates, the vertices of Δ are

$$(0,0,0), \quad (1,0,0), \quad (0,d-\alpha,\alpha), \quad (d-\beta-\gamma,\beta,\gamma)$$

with the conditions that

$$0<\alpha<d, \quad 0<\beta+\gamma<d, \quad \begin{vmatrix} 1 & 0 & d-\beta-\gamma \\ 0 & d-\alpha & \beta \\ 0 & \alpha & \gamma \end{vmatrix} = \gamma(d-\alpha)-\alpha\beta = 1,$$

to ensure that Δ is primitive. See Figure 18 for an example.

Let S be a tropical surface in $\mathbb{R}^3$ with Newton polygon Δ. Without loss of generality, we may assume that the vertex of S is the origin. By a computation we get that S has 4 rays with the 4 following primitive outgoing directions

$$u_0 = (0,\alpha,\alpha-d), \quad u_1 = (0,-\gamma,\beta), \quad u_2 = (-1,-\alpha(d-\beta-\gamma),(d-\alpha)(d-\beta-\gamma)),$$
$$u_3 = (1,\gamma+\alpha(d-\beta-\gamma-1),-\beta-(d-\alpha)(d-\beta-\gamma-1)).$$

The tropical surface S contains the following one parameter family of tropical lines $(L_l)_{l\in\mathbb{R}_{\geq 0}}$: the tropical line L_l has one vertex V_1 at $(0,0,0)$ adjacent to 3 rays with outgoing directions

$$U_1 = (0,-1,-1), \quad U_2 = (-1,0,0), \quad \text{and} \quad U_3 = (1,1,1),$$

and another vertex V_2 at $(0,-l,-l)$ adjacent to 3 rays with outgoing directions

$$-U_1, \quad U_4 = (0,-1,0), \quad \text{and} \quad U_5 = (0,0,-1).$$

If $l=0$, then L_0 is a tropical line with one 4-valent vertex. The tropical line L_l is indeed in S since we have (see Figure 18)

$$U_1 = (\beta+\gamma)u_0 + du_1, \quad U_2 = (d-\beta-\gamma)u_0 + u_2,$$
$$U_3 = (d-1)u_2 + du_3, \quad U_4 = \beta u_0 + (d-\alpha)u_1, \quad U_5 = \gamma u_0 + \alpha u_1.$$

Theorem 7.2. *Let $\mathcal{S} \subset (\mathbb{C}^*)^3$ be an algebraic surface with Newton polytope a pathological d-simplex Δ. The tropical line $Star_{(0,0,0)}(L_l) \subset S$ is approximable by a complex algebraic line $\mathcal{L}_l \subset \mathcal{S}$ if and only if $l=0$ and S has Newton polytope $Conv\{(0,0,0),(1,0,0),(0,1,2),(2,0,1)\}$.*

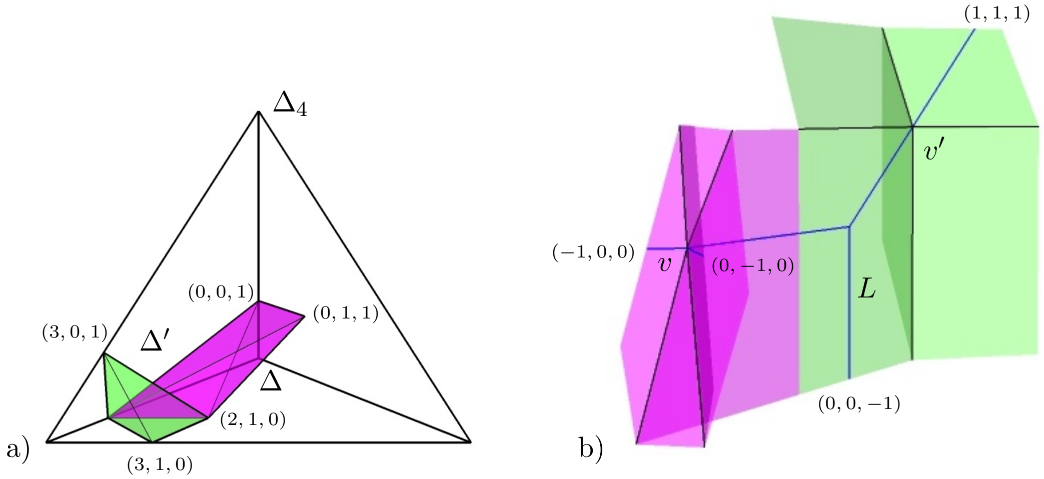

FIGURE 19. a) A pair (Δ, Δ') of type I in Δ_4. b) The corresponding vertices v, v' along with the isolated line L.

In the case of 3-valent tropical lines two instances of Theorem 7.2 were already known, namely the cases $(\alpha, \beta, \gamma) = (d-1, 0, 1)$ ([BK]) and $(\alpha, \beta, \gamma) = (1, d-2, 1)$ ([Sha], see also Example 3.9).

Proof. The case $l > 0$ follows from initial degeneration and the classification given in Theorem 6.1. Suppose now that $l = 0$. We have that $C^2 = -(\beta + \gamma)(d-1) + 1$ which implies that

$$d + C^2 - \sum_{e \in \mathrm{Edge}(L_l)} w_e + 2 = -(d-1)(\beta + \gamma - 1).$$

Hence, if $\beta + \gamma > 1$, then the result follows from Theorem 1.3. If $\beta + \gamma = 1$, then since $\gamma(d-\alpha) - \alpha\beta = 1$ we deduce that $\beta = 0$ and $\gamma = 1$, and so $\alpha = d-1$, and these are the Vigeland lines. The result now follows from Corollary 5.4. □

7.2. **Isolated lines.** A pair (Δ, Δ') of simplices Δ and Δ' contained in Δ_d and with vertices in $\mathbb{Z}^3$ is said to be ***d*-pathological** if

(1) Δ and Δ' are primitive and intersect along a common edge e;
(2) Δ has 2 edges distinct from e and contained in the faces F_i and F_j of Δ_d;
(3) one of the two situations occurs:
  (a) the edge e is contained in F_k, and the opposite edge of Δ' is contained in F_l; in this case we say that the pair (Δ, Δ') is of type I (see Figure 19);
  (b) the polytope Δ' has a face F containing e and intersecting F_k and F_l; in this case we say that the pair (Δ, Δ') is of type II (see Figure 20);
(4) the set $\{i, j, k, l\}$ is equal to the set $\{1, 2, 3, 4\}$.

Isolated lines on a generic non-singular tropical surface of degree $d \geq 4$ in $\mathbb{R}^3$ have also been classified by combinatorial type by Vigeland.

Theorem 7.3 (Vigeland, [Vig])**.** *Let S be a generic non-singular tropical surface of degree $d \geq 4$ in $\mathbb{R}^3$ with Newton polytope Δ_d. If S contains an isolated tropical line L, then S contains two vertices v and v', respectively dual to the simplices Δ and Δ', such that the pair (Δ, Δ') is d-pathological, and such that L has one vertex at v and*

(a) *passes through v' if (Δ, Δ') is of type I;*
(b) *has another vertex on the edge of S dual to F if (Δ, Δ') is of type II.*

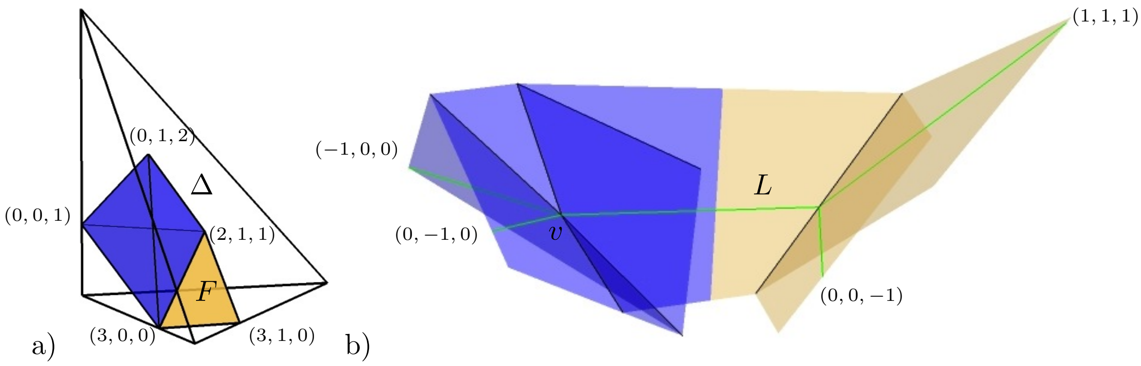

FIGURE 20. a) A pathological pair (Δ, Δ') of type II in Δ_4. b) The corresponding vertex and edge of the dual surface along with the isolated line L.

According to next theorem, none of the tropical lines of Theorem 7.3 are approximable in S.

Theorem 7.4. *Let S be a generic non-singular tropical surface of degree $d \geq 4$ in $\mathbb{R}^3$ with Newton polytope Δ_d, and let $(\mathcal{S}_t)$ be a 1-parametric family of algebraic surfaces in $(\mathbb{C}^*)^3$ with Newton polytope Δ_d such that*

$$\lim_{t\to+\infty} Log_t(\mathcal{S}_t) = S.$$

Suppose that the tropical surface S contains a tropical line L. Then there does not exist a 1-parametric family of lines $\mathcal{L}_t \subset \mathcal{S}_t$ such that

$$\lim_{t\to+\infty} Log_t(\mathcal{L}_t) = L.$$

Proof. Suppose that such a 1-parametric family of lines $(\mathcal{L}_t)$ exists. It follows from Theorem 7.2 that the tropical line L is isolated. Hence the surface S contains two vertices v and v', respectively dual to the simplices Δ and Δ', such the pair (Δ, Δ') is d-pathological, and the line L is as described in Theorem 7.3. By initial degeneration, the family $(\mathcal{S}_t, \mathcal{L}_t)$ produces an approximation of the pair $(Star_v(S), Star_v(L))$ by a constant family. Let us denote by u_0, u_1, u_2, and u_3 the primitive integer directions of the rays of $Star_v(S)$. Since L is isolated the fan tropical curve $Star_v(L)$ cannot be equal to the tropical stable intersection of $\text{Aff}(Star_v(L))$ and $Star_v(S)$. Moreover L only has edges of weight 1, so according to Theorem 6.1 the curve $Star_v(L)$ has degree 2 in $Star_v(S)$. Without loss of generality, we may assume that the 4 rays of $Star_v(S)$ satisfy one of the two cases:

Case 1: $2u_0 + u_3 = (-1, 0, 0)$, $2u_1 + u_2 = (0, -1, 0)$, and $u_2 + u_3 = (1, 1, 0)$,

Case 2: $2u_0 + u_1 = (-1, 0, 0)$, $u_1 + u_3 = (0, -1, 0)$, and $2u_2 + u_3 = (1, 1, 0)$.

See Figure 21 for the dual polytope Δ in each case.

Case 1 corresponds to the situation when the two edges of Δ contained in the faces F_i, F_j are opposite edges of Δ and Case 2 is when these two edges are adjacent, see Figure 21.

In Case 1 we may set

$$u_0 = (a, b, c), \quad u_1 = (-a-1, -b-1, -c), \quad u_2 = (2a+2, 2b+1, 2c), \quad \text{and} \quad u_3 = (-2a-1, -2b, -2c).$$

Since Δ is primitive, any three vectors among u_0, u_1, u_2, and u_3 form a basis for the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$, hence

$$\pm 1 = \begin{vmatrix} a & -a-1 & -2a-1 \\ b & -b-1 & -2b \\ c & -c & -2c \end{vmatrix}$$

which gives $c = \pm 1$.

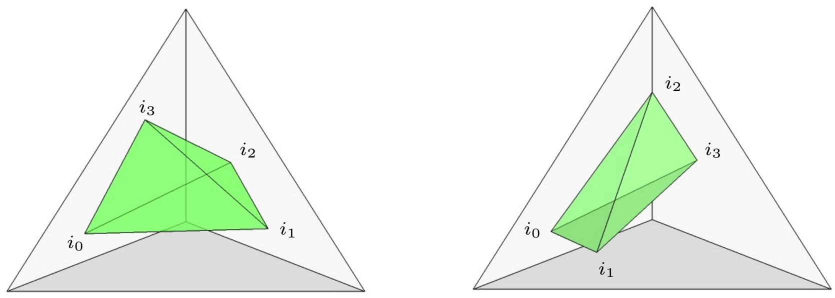

FIGURE 21. The two types of polytopes $\Delta \subset \Delta_d$ dual to the vertex v yielding the lines $Star_v(L)$. Case 1 is on the left and Case 2 on the right.

Denote by i_0, i_1, i_2, and i_3 the vertices of Δ in such a way that i_j is dual to the region of $\mathbb{R}^3 \setminus Star_v(S)$ which contains the vector $-u_j$. The edge e of Δ is dual to the face of $Star_v(S)$ generated by u_2 and u_3, so that $e = [i_0; i_1]$. Then we have

$$i_0 = (\alpha_0, 0, \gamma_0), \quad i_1 = (0, \beta_1, \gamma_1), \quad i_2 = (0, \beta_2, \gamma_2), \quad \text{and} \quad i_3 = (\alpha_3, 0, \gamma_3).$$

The edge e is orthogonal to both u_2 and u_3, therefore it is in direction $(-2c, 2c, 2a - 2b + 1)$, since $c = \pm 1$ the direction of e is $(-2, 2, \pm(2a - 2b + 1))$. From this we deduced that

$$i_0 = (2, 0, \gamma_0) \quad \text{and} \quad i_1 = (0, 2, \gamma_0 \pm (2a - 2b + 1)).$$

Now we have to distinguish the two cases depending on the type of the pair (Δ, Δ').

If (Δ, Δ') is of type I, then the edge e is contained in a face of Δ_d which is neither $\{x = 0\}$ nor $\{y = 0\}$. Moreover, a, b are integers so $2b - 2a - 1 \neq 0$ for any choice of a, b. Therefore e cannot be contained in the face $\{z = 0\}$, and neither can it be contained in $\{x + y + z = d\}$ by symmetry.

If (Δ, Δ') is of type II, then up to a change of coordinates we have

$$i_0 = (2, 0, 0), \quad i_1 = (0, 2, d - 2), \quad i_2 = (0, \beta_2, \gamma_2), \quad \text{and} \quad i_3 = (\alpha_3, 0, \gamma_3).$$

Moreover, the third vertex of the face F of Δ' has coordinates $(\alpha, d - \alpha, 0)$. Let us denote by (β, γ, δ) the coordinates of the fourth vertex of Δ'. Since the polytope Δ' is primitive, we must have

$$\pm 1 = \begin{vmatrix} -2 & \alpha - 2 & \beta - 2 \\ 2 & d - \alpha & \gamma \\ d - 2 & 0 & \delta \end{vmatrix} = (d - 2)C,$$

where $C \in \mathbb{Z}$ is some constant. Therefore, we obtain $d = 3$, so no isolated lines of this type exist when $d \geq 4$.

Now consider Case 2, here we may set:

$$u_0 = (a, b, c), \quad u_1 = (-2a - 1, -2b, -2c), \quad u_2 = (-a, -b + 1, -c), \quad \text{and} \quad u_3 = (2a + 1, 2b - 1, 2c).$$

By a calculation of the determinant of the vectors u_0, u_1, u_2 we find again that $c = \pm 1$. Again the edge e is orthogonal to both u_2, u_3 so that it is in the direction $(c, -c, b - a - 1)$, since $c = \pm 1$ this becomes, $(1, -1, \pm(b - a - 1))$.

In this case the vertices of the polytope Δ have coordinates:

$$i_0 = (\alpha_0, 0, \gamma_1), \quad i_1 = (\alpha_1, \beta_1, \gamma_1), \quad i_2 = (0, 0, \gamma_2), \quad \text{and} \quad i_3 = (0, \beta_3, \gamma_3).$$

The edge $[i_0; i_2]$ is orthogonal to both u_1, u_3 by tropical duality. Hence this edge has direction $(\pm 2, 0, 1 + 2a)$, so that $\alpha_0 = 2$.

Suppose the pair (Δ, Δ') is of type I, by a change of coordinates we may assume the edge e lies in the face $\{z = 0\}$. Hence $\gamma_0 = \gamma_1 = 0$ and the edge e must have direction $(1, -1, 0)$. Therefore,

$i_0 = (2,0,0)$ and $i_1 = (1,1,0)$. Now, Δ' has vertices i_0, i_1, $(\alpha, \beta, d-\alpha-\beta)$, and $(\gamma, \delta, d-\gamma-\delta)$. The polytope Δ' is primitive so,

$$\pm 1 = \begin{vmatrix} -1 & \alpha-2 & \gamma-2 \\ 1 & \beta & \delta \\ 0 & d-\alpha-\beta & d-\gamma-\delta \end{vmatrix} = (d-2)C,$$

where $C \in \mathbb{Z}$ is some constant. This implies $d = 3$, so again there are no isolated lines of this type for $d \geq 4$.

Finally, if (Δ, Δ') is of type II, then i_0 is in $\{y = z = 0\}$ and i_1 is in $\{x + y + z = d\}$, so we have $i_0 = (2,0,0)$ and $i_1 = (1,1,d-2)$. Now Δ' has two other vertices $(\alpha, d-\alpha, 0)$ and (β, γ, δ), and since Δ' is primitive we have

$$\pm 1 = \begin{vmatrix} -1 & \alpha-2 & \beta-2 \\ 1 & d-\alpha & \gamma \\ d-2 & 0 & \delta \end{vmatrix} = (d-2)C,$$

where $C \in \mathbb{Z}$ is some constant. Once again we obtain $d = 3$ and there are no isolated lines of this type for $d \geq 4$. This completes the proof. □

7.3. **Singular tropical lines.** To conclude this paper, let us point out a strange phenomenon in tropical geometry that we were not aware of before starting this investigation. Let S be a generic non-singular tropical surface of degree $d \geq 3$ in $\mathbb{T}P^3$ which has a vertex dual to a d-pathological simplex with $(\alpha, \beta, \gamma) \neq (d-1, 0, 1)$ and $(1,1,0)$. Then any tropical line L in the corresponding 1-parameter family is singular when considered as a tropical curve in S. Indeed the self-intersection of such a line is $-(\beta+\gamma)(d-1)+1$ in S, which is far from being equal to $2-d$, which is the self-intersection of a complex algebraic line in a complex algebraic surface of degree d in $\mathbb{C}P^3$. In particular, this means that none of these tropical lines satisfy the adjunction formula in S, i.e. they are singular.

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