Quasi-isometric diversity of marked groups

A. Minasyan, D. Osin, S. Witzel

March 10, 2020

Quasi-isometry between metric spaces

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Definition (Gromov)

A map $f\colon S \to T$ is a quasi-isometry if $\exists\, \lambda, c, \varepsilon$ such that

(a) $\frac{1}{\lambda}\mathrm{d}_T(f(x), f(y)) - c \leq \mathrm{d}_S(x, y) \leq \lambda \mathrm{d}_T(f(x), f(y)) + c \quad \forall\, x, y \in S$;

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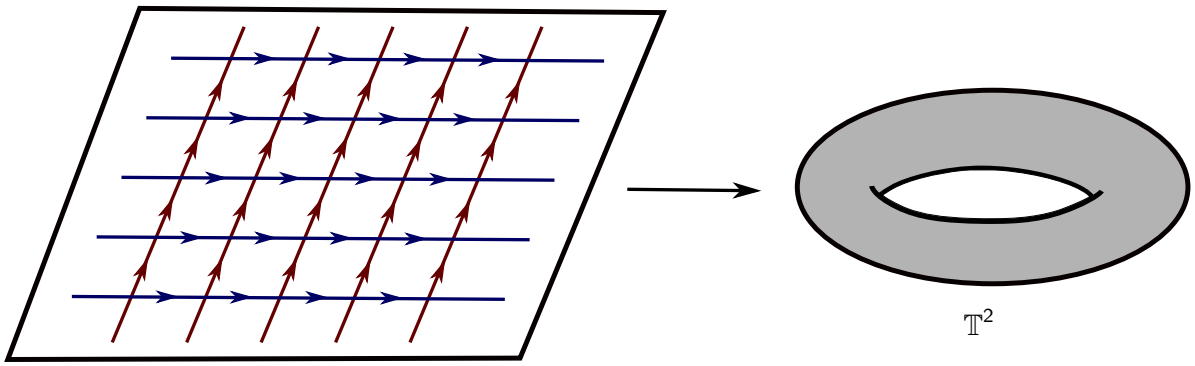

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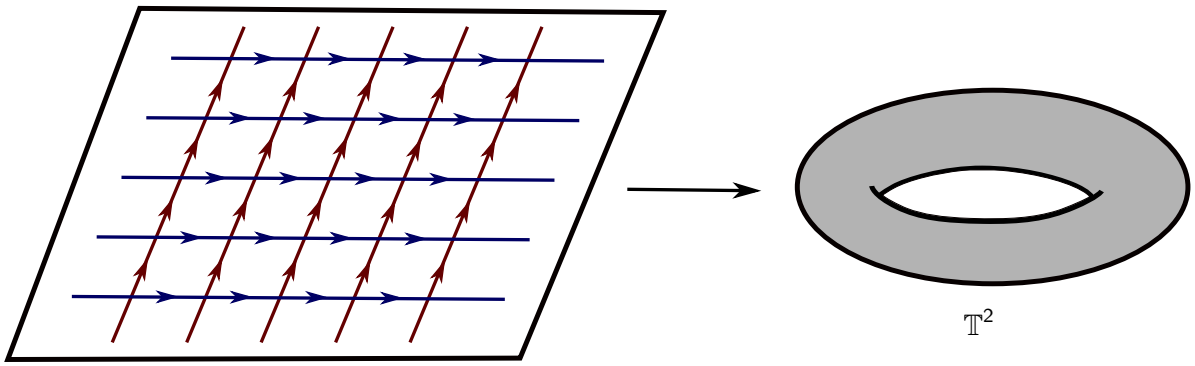

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2. If G is finitely generated and $H \leq G$ is of finite index, then $G \sim_{q.i.} H$. (Hint: consider $H \curvearrowright Cay(G, X)$).

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Invariant: The set of ultrafilters corresponding to simply connected asymptotic cones.

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— (G, A) and (H, B) are k-similar, denoted $(G, A) \approx_k (H, B)$, if there is an isomorphism between balls of radius k in $Cay(G, A)$ and $Cay(H, B)$ mapping edges labelled by a_i to edges labelled by b_i, $i = 1, \ldots, n$.

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Theorem

For all n, $\mathcal{G}_n$ is a totally disconnected, compact, metrizable space.

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In particular, the set of finitely presented groups is dense in $\mathcal{G}_n$ and every residually finite group is a limit of finite groups.

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Let $n \in \mathbb{N}$ and let $\mathcal{S}$ be a non-empty closed subset of $\mathcal{G}_n$. Suppose that every non-empty open subset of $\mathcal{S}$ contains at least two non-quasi-isometric groups. Then $\mathcal{S}$ is quasi-isometrically diverse.

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Corollary

Suppose that $\mathcal{S} \subseteq \mathcal{G}_n$ is perfect and contains a dense subset of finitely presented groups. Then $\mathcal{S}$ is quasi-isometrically diverse.

A diverse zoo of monsters

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There are $2^{\aleph_0}$ quasi-isometry classes of torsion free Tarski Monsters.

Another application

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There exist $2^{\aleph_0}$ pairwise non-quasi-isometric finitely generated groups G splitting as

$$\{1\} \to {\mathbb{Z}_2}^{\infty} \to G \to \mathbb{Z}_2 \operatorname{wr} \mathbb{Z} \to \{1\}, \tag{1}$$

where ${\mathbb{Z}_2}^{\infty}$ is central in G. In particular,

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2. Cornulier-Tessera groups are (nilpotent of class 3)-by-abelian, while our groups are (nilpotent of class 2)-by-abelian.

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Corollary

There exist $2^{\aleph_0}$ pairwise non-quasi-isometric finitely generated groups having the same elementary theory.

Pointed quasi-isometries

Let $(G, X), (H, Y) \in \mathcal{G}_n$. A map $\phi\colon G \to H$ is a pointed N-quasi-isometry if

- $\phi(1) = 1$ and
- ϕ is a quasi-isometry between (G, d_X) and (H, d_Y) with parameters $\lambda = c = \varepsilon = N$.

We write $(G, X) \rightsquigarrow_{N-q.i.} (H, Y)$ when such a map exists.

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For any $n, N \in \mathbb{N}$, the set

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Every pointed N-q.i. maps elements of length ℓ to elements of length at most $N(\ell + N)$. This allows us to build an N-q.i. $\phi\colon (G, X) \to (H, Y)$ as a limit of N-quasi-isometries $\phi_i\colon (G_i, X_i) \to (H_i, Y_i)$.

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Theorem

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A Borel equivalence relation on a Polish space has either at most countably many or $2^{\aleph_0}$ equivalence classes.

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