Quasi-isometric diversity of marked groups

A. Minasyan, D. Osin, S. Witzel

March 10, 2020
Quasi-isometry between metric spaces

Let \((S, d_S), (T, d_T)\) be metric spaces.

**Definition (Gromov)**

A map \(f: S \to T\) is a **quasi-isometry** if \(\exists \lambda, c, \varepsilon\) such that

(a) \(\frac{1}{\lambda} d_T(f(x), f(y)) - c \leq d_S(x, y) \leq \lambda d_T(f(x), f(y)) + c\) \(\forall x, y \in S;\)

(b) \(f(S)\) is \(\varepsilon\)-dense in \(T\).
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\(S\) and \(T\) are **quasi-isometric**, written \(S \sim_{q.i.} T\), if there \(\exists\) a q.i. \(S \to T\).
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\(\sim_{q.i.}\) is an equivalence relation on the class of metric spaces.
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For \(G = \langle X \rangle\), \(d_X\) denotes the word metric.
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**Example.**

1. If \(G\) is a finitely generated group and \(X, Y\) are finite generating sets of \(G\), then \((G, d_X) \sim_{q.i.} (G, d_Y)\).
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**Definition (Gromov)**

A map \(f : S \rightarrow T\) is a **quasi-isometry** if \(\exists \lambda, c, \varepsilon\) such that

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2. \((G, d_X) \sim_{q.i.} \text{Cay}(G, X)\).
Theorem (Svarc-Milnor Lemma)

*If* $S$ *is a geodesic metric space and* $G \bowtie S$ *isometrically, properly, and cocompactly, then* $G$ *is finitely generated and* $G \sim_{q.i.} S$. 
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Cay$(\mathbb{Z} \oplus \mathbb{Z}, X) \subset \mathbb{R}^2 = \tilde{T}^2$

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Examples.

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2. If $G$ is finitely generated and $H \leq G$ is of finite index, then $G \sim_{q.i.} H$. (Hint: consider $H \curvearrowright \text{Cay}(G, X)$).
How many quasi-isometry classes are there?

**Theorem (Grigorchuk, 1984)**  
There exist \(2^{\aleph_0}\) quasi-isometry classes of finitely generated groups.

**Invariant:** growth functions.

**Theorem (Bowditch, 1998)**  
There exist \(2^{\aleph_0}\) quasi-isometry classes of finitely generated \(C'(1/6)\) groups.  
**Invariant:** growth of tout loops.

**Theorem (Cornulier-Tessera, 2013)**  
There exist \(2^{\aleph_0}\) quasi-isometry classes of finitely generated solvable groups.  
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— \((G, A)\) and \((H, B)\) are \(k\)-similar, denoted \((G, A) \approx_k (H, B)\), if there is an isomorphism between balls of radius \(k\) in \(\text{Cay}(G, A)\) and \(\text{Cay}(H, B)\) mapping edges labelled by \(a_i\) to edges labelled by \(b_i\), \(i = 1, \ldots, n\).
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\mathcal{G}_n = \{ (G, (a_1, \ldots, a_n)) \mid G = \langle a_1, \ldots, a_n \rangle \}/ \approx
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\textbf{Theorem}

\textit{For all \(n\), \(G_n\) is a totally disconnected, compact, metrizable space.}
Let $F_n = F(x_1, \ldots, x_n)$ be the free group of rank $n$. Endow $2^{F_n}$ with the product topology.

Proposition
The map $(G, A) \mapsto \ker \varepsilon(G, A)$ defines a homeomorphism $G_n \to \mathcal{N}(F_n)$.

Examples.
1. $\lim_{i \to \infty} (\mathbb{Z}/m\mathbb{Z}, \{1\}) = (\mathbb{Z}, \{1\})$.
2. If $\mathcal{N} = \bigcup_{i=1}^{\infty} \mathcal{N}_i$ or $\mathcal{N} = \bigcap_{i=1}^{\infty} \mathcal{N}_i$, then $\lim_{i \to \infty} \mathcal{N}_i = \mathcal{N}$ in $\mathcal{N}(F_n)$.
In particular, the set of finitely presented groups is dense in $G_n$ and every residually finite group is a limit of finite groups.
Let $F_n = F(x_1, \ldots, x_n)$ be the free group of rank $n$. Endow $2^{F_n}$ with the product topology.

$$\mathcal{N}(F_n) = \{ N \triangleleft F_n \} \subseteq 2^{F_n}.$$
Equivalent definition (the Chabauty space)

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A subset of a topological space is **meager** (respectively, **comeager**) if it is a union of countably many nowhere dense sets (respectively, an intersection of countably many sets with dense interiors).

**Theorem (Minasyan–O.–Witzel)**

Let \( n \in \mathbb{N} \) and let \( S \) be a non-empty closed subset of \( G \). Suppose that every non-empty open subset of \( S \) contains at least two non-quasi-isometric groups. Then \( S \) is quasi-isometrically diverse.

A subset of a topological space is **perfect** if it is closed and has no isolated points.

**Corollary**

Suppose that \( S \subseteq G \) is perfect and contains a dense subset of finitely presented groups. Then \( S \) is quasi-isometrically diverse.
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**Definition**

A subset $S \subseteq G_n$ is **quasi-isometrically diverse** if every comeagre subset of $S$ has $2^{\aleph_0}$ quasi-isometry classes.

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The main theorem

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Suppose that $S \subseteq G_n$ is perfect and contains a dense subset of finitely presented groups. Then $S$ is quasi-isometrically diverse.
A diverse zoo of monsters

Corollary (Minasyan–O.–Witzel)

There are \(2^{\aleph_0}\) quasi-isometry classes of finitely generated simple (torsion, divisible, property (T), ...) groups.

Idea of the proof.

Let \(H_n\) be the set of all \((G, A) \in G_n\) such that \(G\) is non-elementary, hyperbolic, and has trivial finite radical. Generic groups in \(H\) are simple. — (Olshanskii + \(\varepsilon\)) Simple groups are dense in \(H_n\).

— Simplicity is definable by a \(\Pi_2\)-sentence in \(L_{\omega_1, \omega}\):

\[
\forall a \forall b (b \neq 1 \rightarrow (\bigvee_{\ell=1}^{\infty} \exists t_1, \ldots, \exists t_{\ell} a = t_{\ell} - t_{\ell-1} b \pm t_{\ell} b \pm 1 t_{\ell} \cdots t_{\ell} b \pm 1 t_{\ell} \cdots t_{\ell} b)).
\]

It follows that simple groups form a \(G_\delta\) set in \(G_n\).

Corollary (Minasyan–O.–Witzel)

There are \(2^{\aleph_0}\) quasi-isometry classes of torsion free Tarski Monsters.
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Idea of the proof. Let $\mathcal{H}_n$ be the set of all $(G, A) \in G_n$ such that $G$ is non-elementary, hyperbolic, and has trivial finite radical. Generic groups in $\overline{\mathcal{H}}$ are simple.

— (Olshanskii + $\varepsilon$) Simple groups are dense in $\overline{\mathcal{H}_n}$.
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It follows that simple groups form a $G_\delta$ set in $G_n$. 

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Another application

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\{1\} \to \mathbb{Z}_2^\infty \to G \to \mathbb{Z}_2 \text{ wr } \mathbb{Z} \to \{1\},
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where $\mathbb{Z}_2^\infty$ is central in $G$. In particular,

(a) there exist continuously many quasi-isometry classes of finitely generated groups of asymptotic dimension 1;

(b) there exist continuously many quasi-isometry classes of finitely generated center-by-metabelian groups (i.e., groups $G$ satisfying the identity $[G'', G] = 1$).

Remarks.

1. Every finitely presented group of asymptotic dimension 1 is virtually free (Fujiwara-Whyte, 2007).
2. Cornulier-Tessera groups are (nilpotent of class 3)-by-abelian, while our groups are (nilpotent of class 2)-by-abelian.
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$G$ and $H$ are **elementarily equivalent** if they satisfy the same sentences in the first-order language of groups.

Elementary equivalence preserves many geometric properties of f.g. groups: polynomial growth (Gromov); hyperbolicity (Sela, André); quasi-isometry type of polycyclic groups (Sabbagh-Wilson, Raphael).

Problem
To what extent does the first order theory of a finitely generated group determine its quasi-isometry type?

Let $H_{tf}$ denote the set of all non-cyclic torsion-free hyperbolic groups in $G$.

**Theorem (O., 2020)**
Generic groups in $H_{tf}$ are elementarily equivalent.

**Corollary**
There exist $2^{\aleph_0}$ pairwise non-quasi-isometric finitely generated groups having the same elementary theory.
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Let \((G, X), (H, Y) \in \mathcal{G}_n\). A map \(\phi: G \to H\) is a **pointed \(N\)-quasi-isometry** if

1. \(\phi(1) = 1\) and
2. \(\phi\) is a quasi-isometry between \((G, d_X)\) and \((H, d_Y)\) with parameters \(\lambda = c = \varepsilon = N\).

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*For any \(n, N \in \mathbb{N}\), the set*

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Theorem

Let \( n \in \mathbb{N} \) and let \( S \) be a non-empty closed subset of \( G_n \). Suppose that every non-empty open subset of \( S \) contains at least two non-quasi-isometric groups. Then \( S \) is quasi-isometrically diverse.
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Theorem (The Big Gun of DST a.k.a. Silver Dichotomy)

A Borel equivalence relation on a Polish space has either at most countably many or $2^{\aleph_0}$ equivalence classes.
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