$L^2$-Betti numbers of groups

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Les Diablerets
Kervaire seminar
10 March 2020
1 Motivation for $L^2$-cohomology

Let $X$ be a finite complex. The Betti numbers

$$b_i(X) = \dim H^i(X, \mathbb{R})$$

are homotopy invariants of $X$. The Euler characteristic is

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Let $\hat{X}$ be a $d$-sheeted covering of $X$. Then

$$\chi(\hat{X}) = d \cdot \chi(X).$$

However: in general $b_i(\hat{X}) \neq d \cdot b_i(X)$.

**Example 1.** Take $X = \hat{X} = S^1$, the unit circle in $\mathbb{C}$. The map $\hat{X} \to X : z \mapsto z^d$ is a $d$-sheeted covering. But $b_i(X) = b_i(\hat{X}) = 1$ for $i = 0, 1$. 
Goal: By means of the universal cover $\tilde{X}$ of $X$, construct $L^2$-Betti numbers $b_i^{(2)}(X) \geq 0$ such that:

B1: $b_i^{(2)}(X)$ is a homotopy invariant of $X$.

B2: $\chi(X) = \sum_i (-1)^i b_i^{(2)}(X)$.

B3: If $\hat{X}$ is a $d$-sheeted cover of $X$, then $b_i^{(2)}(\hat{X}) = d \cdot b_i^{(2)}(X)$.

B4: (Lück approximation) Suppose $G = \pi_1(X)$, and if $(G_j)_{j \geq 1}$ is a family of finite index normal subgroups that decreases to $\{1\}$ for $j \to \infty$, then with $X_j = \tilde{X}/G_j$ we have:

$$b_i^{(2)}(X) = \lim_{j \to \infty} \frac{b_i(X_j)}{[G : G_j]}.$$ 

THIS CAN BE DONE!
2 A few successes

2.1 The Hopf conjecture

Conjecture 1. (Hopf 1931, Chern 1955) Let $M^{2n}$ be a closed manifold carrying a Riemannian metric of negative sectional curvature. Then $(-1)^n \chi(M) > 0$.

The conjecture is known in dimensions 2 (Gauss-Bonnet formula) and 4 (Milnor 1955); it is open in general. Using $L^2$-Betti numbers, Gromov proved in 1991:

**Theorem 2.1.** The Conjecture holds for Kähler manifolds.

2.2 Deficiency of groups

For $G$ a finitely presented group, the deficiency of $G$ is:

$$def(G) = \max \{ g - r : G \text{ admits a presentation on } g \text{ generators and } r \text{ relations} \}.$$ 

It is a measure of the “complexity” of $G$. 
Let $BG$ be the classifying space of $G$: a complex such that $\pi(BG) = G$ and $\tilde{BG}$ is contractible ($BG$ is unique up to homotopy). Set $b^{(2)}(G) = b^{(2)}(BG)$.

**Theorem 2.2.** (B. Eckmann) $\text{def}(G) \leq 1 + b^{(2)}_1(G)$. Moreover, if $\text{def}(G) = 1$ and $b^{(2)}_1(G) = 0$, then $G$ admits a 2-dimensional model for $BG$. \(\square\)

### 2.3 Structure of groups

In 1993 Gromov conjectured that, if

$$1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$$

is a short exact sequence of infinite groups which are fundamental groups of finite aspherical complexes, then $b^{(2)}_1(G) = 0$. Actually a much stronger statement holds!
**Theorem 2.3.** (Gaboriau 2000; Lück 1995 under extra assumption $\mathbb{Z} \subset G/N$). Let

$$1 \to N \to G \to G/N \to 1$$

be a short exact sequence of infinite groups. If $N$ is finitely generated (as a group), then $b_1^{(2)}(G) = 0$.

**Corollary 2.4.** (Schreier 1930) Any (non-trivial) finitely generated normal subgroup of the free group $F_n$, has finite index.

**Proof:** Assume $N \triangleleft F_n$, with $n \geq 2$, and $N$ finitely generated. Since $b_1^{(2)}(F_n) > 0$, we either have $N$ finite (hence trivial) or $G/N$ finite. ■
2.4 Cost and measurable group theory (Gaboriau 2002)

Theorem 2.5. $\mathcal{C}(G) - 1 \geq b_1^{(2)}(G) - b_0^{(2)}(G)$, where $\mathcal{C}$ denotes the cost.

Theorem 2.6. Assume $G$ admits a free, treeable action. Then $b_k^{(2)}(G) = 0$ for $k > 1$. Moreover $b_1^{(2)}(G) = 0$ if and only if $G$ is amenable.
3 G-dimension

For $G$ a countable group, let $L(G)$ be the group von Neumann algebra, i.e. the commutant of the left regular representation $\lambda$ of $G$ on $\ell^2(G)$. The map

$$L(G) \to \ell^2(G) : S \mapsto S(\delta_1)$$

is an embedding with dense image. The functional

$$\tau(S) = \langle S(\delta_1) | \delta_1 \rangle$$

defines a trace on $L(G)$ (i.e. $\tau(ST) = \tau(TS)$), which is positive ($\tau(S^*S) \geq 0$) and faithful ($\tau(S^*S) = 0 \iff S = 0$).

Let $\mathcal{H}$ be a Hilbert space. We extend $\tau$ to a densely defined trace on $L(G) \otimes B(\mathcal{H})$ by:

$$\tau(S \otimes T) =: \tau(S) \cdot Tr(T).$$

Let $G$ act on $\ell^2(G) \otimes \mathcal{H}$ by $\lambda \otimes 1$. 
If $V$ is a closed $G$-invariant subspace of $\ell^2(G) \otimes \mathcal{H}$, and $P$ is the orthogonal projection onto $V$, we define the $G$-dimension of $V$ as:

$$\dim G V = \tau(P) \in [0, +\infty].$$

**Example 2.** $\dim G \ell^2(G) = \tau(1) = 1$.

**Properties:**

D1: $\dim G V = 0 \Leftrightarrow V = 0$.

D2: If $V$ is $G$-isomorphic to a dense subspace of $W$, then $\dim G V = \dim G W$.

D3: (additivity) $\dim G (V \oplus W) = \dim G V + \dim G W$.

D4: (continuity) If $(V_j)_{j>0}$ is a decreasing sequence of $G$-invariant subspaces, then

$$\dim G (\bigcap_{j>0} V_j) = \lim_{j \to \infty} \dim G V_j.$$  

D5: (finite index subgroups) If $[G : H] = d$, and $V$ is $G$-invariant: $\dim H(V) = d \cdot \dim G V$. 
Example 3. 1. If $G$ finite: $\dim_G V = \frac{\dim V}{|G|}$.

2. For $G = \mathbb{Z}$, by Fourier series for $n \in \mathbb{Z}$ the shift operator $\lambda(n)$ becomes multiplication by $e^{2\pi in\theta}$ on $L^2(S^1, \mu)$ ($\mu$ the normalized Lebesgue measure). Invariant subspaces of this action are of the form

$$\mathcal{H}_B = \{ f \in L^2(S^1) : f \equiv 0 \text{ a.e. on } S^1 \setminus B \},$$

with $B$ a Borel subset of $S^1$. The corresponding projection is multiplication by the characteristic function $\chi_B$ of $B$. Hence

$$\dim_{\mathbb{Z}} \mathcal{H}_B = \int_{S^1} \chi_B \, d\mu = \mu(B);$$

this takes all values in $[0, 1]$. 
4 $L^2$-cohomology

Let $X$ be a finite complex, $\tilde{X}$ its universal cover. Let $\tilde{X}^k$ denote the set of $k$-cells of $\tilde{X}$. Consider the complex of $\ell^2$-cochains:

$$\ell^2(\tilde{X}^0) \xrightarrow{d_0} \ell^2(\tilde{X}^1) \xrightarrow{d_1} \ell^2(\tilde{X}^2) \rightarrow \ldots$$

where $(d_k f)(\sigma) = \sum_{\tau \subset \sigma} (-1)^{e(\tau,\sigma)} f(\tau)$, for $\sigma \in \tilde{X}^{k+1}$.

**Example 4.** $(d_0 f)(e) = f(e^+) - f(e^-)$ for $e \in \tilde{X}^1$.

**Definition 4.1.** The $L^2$-cohomology of $\tilde{X}$ is $H^k_{(2)}(\tilde{X}) = \ker d_k / \text{Im } d_{k-1}$.

We can realize the quotient as a subspace because we use Hilbert spaces. Let $\mathcal{H}^k_{(2)}(X) = \ker d_k \cap (\text{Im } d_{k-1})^\perp$ be the space of harmonic $k$-cycles: then we have a canonical identification $\mathcal{H}^k_{(2)}(\tilde{X}) = H^k_{(2)}(\tilde{X})$. 
Example 5. \((d_0^* f)(x) = \sum_{e: e^+ = x} f(e) - \sum_{e: e^- = x} f(e)\) for \(x \in \tilde{X}^0\).
For $G = \pi_1(X)$, all $\ell^2(\tilde{X}^k)$’s become $G$-invariant subspaces of $\ell^2(G) \otimes \ell^2(N)$, so we may define:

**Definition 4.2.** The $k$-th $L^2$-Betti number of $X$ is $b_{k}^{(2)}(X) = \dim_G H_{k}^{(2)}(\tilde{X})$.

**Proposition 4.3.** (*Atiyah 1976*) $\chi(X) = \sum_i (-1)^i b_i^{(2)}(X)$.

**Proof:** Set $\ell_{even/odd}^2(\tilde{X}) =: \oplus_{k \ even/odd} \ell^2(\tilde{X}^k)$. Consider

$$S = d + d^* : \ell_{even}^2(\tilde{X}) \to \ell_{odd}^2(\tilde{X}).$$

Then

$$\sum_k (-1)^k b_k^{(2)}(X) = \dim_G \ker S - \dim_G \ker S^*$$

$$= (\dim_G \ker S + \dim_G \overline{Im S}) - (\dim_G \overline{Im S} + \dim_G \ker S^*)$$

$$= \dim_G \ell_{even}^2(\tilde{X}) - \dim_G \ell_{odd}^2(\tilde{X}).$$
Choosing representatives for orbits of the free $G$-action on $\tilde{X}^k$, identify $\ell^2(\tilde{X}^k)$ with $\ell^2(G)|X^k|$ in a $G$-equivariant way. So $\dim_G \ell^2(\tilde{X}^k) = |X^k|$ and $\sum_k (-1)^k b_k^{(2)}(X) = \sum_k (-1)^k |X^k| = \chi(X)$. ■

**Theorem 4.4.** (Dodziuk 1977) $H_k^{(2)}(\tilde{X})$ is a homotopy invariant of $X$. □

Also enjoyable:

**Proposition 4.5.** (Poincaré duality) If $X$ is a triangulation of a closed orientable manifold $M^n$, then $b_k^{(2)}(X) = b_{n-k}^{(2)}(X)$.

5 Invariants of discrete groups

Suppose the countable group $G$ has a finite classifying space $BG$ (=Eilenberg-McLane space $K(G,1)$). Define:

$$b_k^{(2)}(G) = b_k^{(2)}(BG).$$

(Well-defined by Dodziuk’s theorem).
Example 6. 1. Construct $BG$ from the presentation 2-complex of $G$. So the 1-skeleton of $\tilde{X}$ is a Cayley graph of $G$, hence $\tilde{X}^0 = G$ and $H_0^{(2)}(\tilde{X})$ is the space of square-integrable constant functions on $G$. So

$$b_0^{(2)}(G) = \begin{cases} 0 & \text{if } G \text{ infinite} \\ 1/|G| & \text{if } G \text{ finite} \end{cases}$$

Moreover $b_1^{(2)}(G) \leq \dim_G \ell^2(\tilde{X}^1)$, hence $b_1^{(2)}(G) \leq d(G)$, the minimal number of generators of $G$.

2. $G = \mathbb{Z}^n$. Then $BG = \mathbb{T}^n$, which is a double cover of itself. So $b_k^{(2)}(\mathbb{Z}^n) = b_k^{(2)}(\mathbb{T}^n) = 2b_k^{(2)}(\mathbb{T}^n)$, hence $b_k^{(2)}(\mathbb{Z}^n) = 0$.

3. $G = \mathbb{F}_n$. Then we may take for $X = BG$ a bouquet of $n$ circles, so

$$\chi(X) = 1 - n = b_0^{(2)}(\mathbb{F}_n) - b_1^{(2)}(\mathbb{F}_n) = -b_1^{(2)}(\mathbb{F}_n)$$

hence $b_1^{(2)}(\mathbb{F}_n) = n - 1$. 
4. $G = \pi_1(\Sigma_g)$, with $\Sigma_g$ a closed Riemann surface of genus $g > 0$. Take $BG = \Sigma_g$, so by Poincaré duality:

$$\chi(\Sigma_g) = 2 - 2g = b_0^{(2)}(G) - b_1^{(2)}(G) + b_2^{(2)}(G) = -b_1^{(2)}(G)$$

hence $b_1^{(2)}(G) = 2g - 2$.

**Theorem 5.1.** The property $b_k^{(2)}(G) = 0$ is invariant under:

- quasi-isometry (Soardi for $k = 1$, Pansu for $k$ arbitrary);
- measure equivalence (Gaboriau)

6 Methods of computation

**Theorem 6.1.** (*Cheeger-Gromov 1986*) There is a Küneth formula:

$$b_k^{(2)}(G_1 \times G_2) = \sum_{i=0}^{k} b_i^{(2)}(G_1)b_{k-i}^{(2)}(G_2).$$
Theorem 6.2. (Cheeger-Gromov 1986; Paschke 1992) For $G = A \ast_C B$:

$$b_1^{(2)}(G) - b_0^{(2)}(G) = (b_1^{(2)}(A) - b_0^{(2)}(A)) + (b_1^{(2)}(B) - b_0^{(2)}(B)) - (b_1^{(2)}(C) - b_0^{(2)}(C)).$$

Example 7. For $G = SL_2(\mathbb{Z}) = (\mathbb{Z}/6) \ast_{\mathbb{Z}/2} (\mathbb{Z}/4)$:

$$b_1^{(2)}(G) = (0 - \frac{1}{6}) + (0 - \frac{1}{4}) - (0 - \frac{1}{2}) = \frac{1}{12}.$$
<table>
<thead>
<tr>
<th>Infinite $G$</th>
<th>$b^{(2)}_1(G)$</th>
<th>$b^{(2)}_k(G)$ ($k \geq 2$)</th>
<th>Who?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_n$</td>
<td>$n - 1$</td>
<td>0</td>
<td>Cheeger – Gromov 1986</td>
</tr>
<tr>
<td>$SL_2(\mathbb{Z})$</td>
<td>$\frac{1}{12}$</td>
<td>0</td>
<td>Lueck 1994</td>
</tr>
<tr>
<td>$\pi_1(\Sigma_g)$</td>
<td>$2g - 2$</td>
<td>0</td>
<td>Bekka – V. 1996</td>
</tr>
<tr>
<td>Amenable</td>
<td>0</td>
<td>0</td>
<td>Borel 1985</td>
</tr>
<tr>
<td>Thompson's $F$</td>
<td>0</td>
<td>0</td>
<td>Borel 1985</td>
</tr>
<tr>
<td>Prop. (T)</td>
<td>0</td>
<td>*</td>
<td>Martin – V. 2003</td>
</tr>
<tr>
<td>Lattice in $SL_n(\mathbb{R})$, $n \geq 3$</td>
<td>0</td>
<td>0</td>
<td>Borel 1985</td>
</tr>
<tr>
<td>Lattice in $SO(2n + 1, 1)$</td>
<td>0</td>
<td>0</td>
<td>Lueck 1995</td>
</tr>
<tr>
<td>$G \rtimes \mathbb{Z}$ ($G$ fin. gen.)</td>
<td>0</td>
<td>0</td>
<td>Dicks – Linnell 2006</td>
</tr>
<tr>
<td>$H \wr G$</td>
<td>0</td>
<td>*</td>
<td>Fernos – V. 2016</td>
</tr>
<tr>
<td>$1$ – relator on $g$ gen.</td>
<td>$\max{g - 2, 0}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Higman's group $H_4$</td>
<td>0</td>
<td>$b^{(2)}_2 = 1$</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>$b^{(2)}_k = 0$ ($k &gt; 2$)</td>
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</tbody>
</table>
**Theorem 6.3.** (Borel 1985) Let $G$ be a lattice in a rank 1 simple Lie group $S$:

<table>
<thead>
<tr>
<th>$S$</th>
<th>$b_n^{(2)}(G) &gt; 0$</th>
<th>Others : 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(2n, 1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$SU(n, 1)$</td>
<td></td>
<td></td>
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<tr>
<td>$Sp(n, 1)$</td>
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</tbody>
</table>

THANK YOU FOR YOUR ATTENTION!

Following a question: D. Osin (2008) constructed finitely generated torsion groups with $b_{1}^{(2)}(G) > 0$, where non-amenability follows from non-vanishing of $b_{1}^{(2)}$. 