Université de Genève
Section de Mathématiques

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Lecture notes of the mini-course
given by N. Varopoulos on the

Discrete and continuous Dirichlet problem in Lipschitz domains

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Contents

Lecture I

Introduction to the discrete and continuous Dirichlet problem

Contents of the lecture

The aim of this course is to investigate the links between the solutions of the classical and discrete Dirichlet problem in a Lipschitz domain of $\mathbb{R}^d$. This series of lectures will be based on parts of the recent paper [Var09]. The latter being rather long and technical, we will try to restructure these lectures in such a way that it will *facilitate* its reading. We also hope that the present lecture notes[1] will help the reader in understanding the various mathematical tools used in [Var09] and the articulation of the proofs given there.

[1]Comments and remarks on these lectures notes are of course very welcome !

1 The Dirichlet problem

The scope of these lectures is the Dirichlet problem which we briefly recall in the continuous and discrete settings in the next two paragraphs.

1.1 The classical Dirichlet problem

Consider $\Omega \subset \mathbb{R}^d$ a bounded connected domain with a reasonable boundary, say a C^1-boundary, and f a continuous function on Ω^c, the complement of Ω. The classical Dirichlet problem consists in finding a function u on $\mathbb{R}^d$ that is continuous and is such that $u_{|\Omega}$ is harmonic and $u_{|\Omega^c} = f$. Recall that a function u is said to be *harmonic* in a domain $\mathcal{O}$ if $\Delta u = 0$ in $\mathcal{O}$, where $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the usual euclidian Laplacian.

Example I.1 — If $\Omega = (0,1) \subset \mathbb{R}$, $f \equiv 0$ on $(-\infty, 0]$ and $f \equiv 1$ on $[1, +\infty)$, then the Dirichlet problem admits a unique solution $u(x) = x$ in Ω.

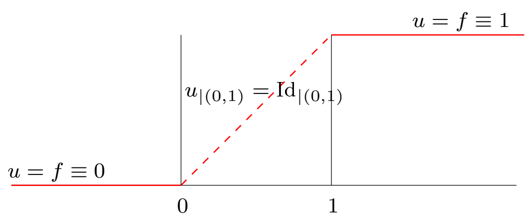

Figure 1: Example of Dirichlet problem on the real line.

Example I.2 — If $\Omega = \{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 < 1\}$ and $f(x,y) = x^2 + y^2$ on Ω^c, then the Dirichlet problem admits a unique solution $u \equiv 1$ in Ω.

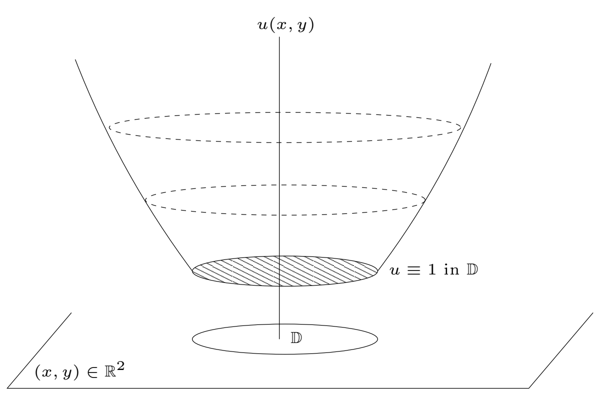

Figure 2: Example of Dirichlet problem on the real plane.

1.2 The discrete version

Let us now describe the Dirichlet problem in a discrete setting. Consider $\Omega \subset \mathbb{R}^d$ as in the first paragraph and $\mathbb{Z}^d \subset \mathbb{R}^d$ the lattice points. The discrete Laplacian Δ_d is an operator acting on the functions on $\mathbb{Z}^d$. For such a function ϕ, $\Delta_d\phi$ is defined as

$$\Delta_d\phi(x) := \phi(x) - \frac{1}{2d}\sum_{i=1}^{d} \phi(x \pm e_i), \quad \forall\, x \in \mathbb{Z}^d,$$

where the e_i's are the basis vectors of the lattice. A function ϕ is said harmonic for the discrete Laplacian in Ω if $\Delta_d\phi \equiv 0$ in Ω, that is to say if it satisfies the average property .

Proposition I.1 — *There exists a unique function u_d on $\mathbb{Z}^d$ such that $u_{d|\Omega\cap\mathbb{Z}^d}$ is harmonic and $u_{d|\Omega^c} = f$.*

Example I.3 — Here is an example of dicrete Dirichlet problem in the open disk of radius $\sqrt{2}$ which contains five points of the lattice $\mathbb{Z}^2$.

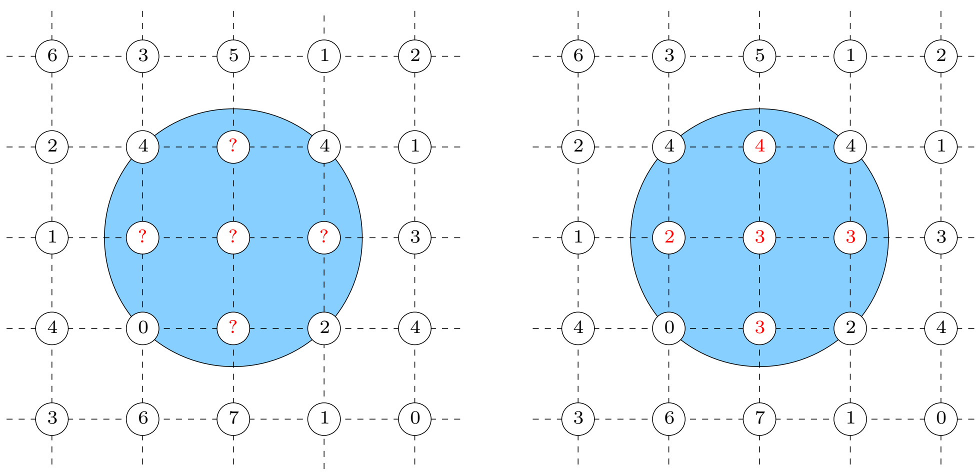

Figure 3: Example of discrete Dirichlet problem in the disk of radius $\sqrt{2}$.

Proof. exercise ! See [BJS79] or [Law91] p. 24-27 for a probabilistic proof. Hint : observe that it amounts to showing the existence and uniqueness of solution of a system of linear equations, one for each $x \in \mathbb{Z}^d$. In fact, only finitely many equations need to be considered, namely the equations comming from $x \in \mathbb{Z}^d$ with $dist(x, \Omega) \le C$ for C large enough. For $x \in \Omega$, these equations are homogeneous, for $x \notin \Omega$, the equation is naturally $u_d(x) = f(x)$. To end the proof, remark that the corresponding homogeneous system, *i.e.* $f \equiv 0$, has only $u_d \equiv 0$ for solution. This can be proved by "the maximum principle" that asserts that when ϕ is real and harmonic in Ω, no local maximum exists in $\Omega \cap \mathbb{Z}^d$ unless ϕ is constant. □

In broad terms, the issue that will be adressed in these lectures is to find (best possible) estimates for $||u - u_d||_\infty$, the uniform distance between the solutions of the continuous and discrete Dirichlet problem.

1.3 A generalization of the discrete case

One important way one can generalize the discrete Laplacian Δ_d is by setting

$$\Delta_\mu f(x) := f \star (\delta - \mu)(x) = f(x) - \int f(x-y) d\mu(y),$$

where $\mu \in \mathbb{P}(\mathbb{R}^d)$ is a compactly supported and centered probability measure with covariance Id :

$$\int x d\mu = 0, \qquad \int x_i x_j \, d\mu = \delta_{ij}.$$

Of course, this includes the case of discrete measures $\mu \in \mathbb{P}(\mathbb{Z}^d)$ (centered with covariance Id), such that there exists $C_0 > 0$ with $\mu(\pm e_i) \geq C_0$ and $diam(\text{supp}\mu) \leq C_0$. We can then generalize our previous problem by generalizing the notion of discrete harmonic function in Ω to $\Delta_\mu f = 0$. The importance of this generalization lies mostly in the fact that we can take for μ a smooth measure $d\mu(x) = \phi(x)dx$ for say $\phi \in C_0^\infty(\mathbb{R}^d)$ and then when f is continuous in $\mathbb{R}^d$, we say that f is harmonic Ω with respect to μ if $f \star (\delta - \mu) \equiv 0$ in Ω.

2 The classical inhomogeneous Dirichlet problem and the Green function

Let Ω be as in paragraph 1 and Φ some continuous function on Ω. Then the inhomogeneous Dirichlet problem consists in finding a continuous function U defined on Ω such that $\Delta U = \Phi$ on Ω and $U_{|\partial\Omega} = 0$. If the original $f \in C(\Omega^c)$ admits an extension $\widetilde{f}$ in Ω such that $\Delta \widetilde{f} = \Phi$ then the function $u = \widetilde{f} - U$ is harmonic in Ω and coincides with f on $\partial\Omega$ *i.e.* we are back to the classical homogeneous Dirichlet problem.

2.1 The Green function

In this setting, the Green function $G(x, y)$, $x, y \in \Omega$, is defined as the solution of the generalized inhomogeneous Dirichlet problem with the following conditions : $\forall\, y \in \Omega$

$$\begin{cases} \Delta_x G(x, y) = \delta_y(x) & \text{the Dirac mass in the distribution sense,} \\ \\ G(., y)_{|\partial\Omega} \equiv 0. \end{cases}$$

Remark I.1 — In our definition $\Delta = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ *i.e.* the positive self adjoint operator.

The solution of the general inhomogeneous Dirichlet problem can then be expressed in terms of G as

$$U(x) = \langle G(x, .),\ \Phi(.)\rangle = \int G(x, y)\Phi(y)dy.$$

In fact, one then has

$$\Delta_x U(x) = \int \Delta_x G(x, y)\Phi(y)dy = \int \delta_y(x)\Phi(y)dy = \Phi(x).$$

Several lecture courses at all level can be given on the classical Green function. We do not intend to do that of course !

2.2 Brownian motion and the heat diffusion kernel

For the people familiar with Brownian motion, one way to capture the Green function is to start with $(B_t)_{t\geq 0}$ the standard Brownian motion in $\mathbb{R}^d$ starting at $x \in \Omega$ and define the transition kernel in Ω :

$$p_t(x, y) := \mathbb{P}_x\left(B_t \in dy,\ B_s \in \Omega,\ \forall\, 0 < s < t\right).$$

Then it not difficult to verify that the Green function in Ω is given by

$$G(x, y) = \int_0^\infty p_t(x, y)dt.$$

Formally, p_t is the kernel of the diffusion semigroup $T_t := e^{-t\Delta}$ and since T_t satifies the heat equation

$$\Delta T_t = -\frac{\partial}{\partial t} T_t,$$

we can integrate and get $\Delta G = T_0 = \delta$ as required.

3 The discrete Green function and random walks

3.1 Discrete Green function via random walks

The only reason we brought Brownian motion in the picture in the previous section is that the definition of the Green function given there generalizes naturally for random walks $Z_n \in \mathbb{Z}^d$ where

$$\mathbb{P}\left(Z_{n+1} = x \,|\, Z_n = y\right) = \mu(y - x),$$

and where $\mu \in \mathbb{P}(\mathbb{Z}^d)$. Here the simple random walks are the ones that give $\Delta_\mu = \Delta_d$ in paragraphs 1.2 and 1.3 and $\mu(\pm e_i) = 1/2d$. These are the nearest neighbour Bernoulli random walks. We can then define for $t = 0, 1, 2, \ldots$

$$p_t(x, y) := \mathbb{P}_x\left(Z(t) = y,\ Z(s) \in \Omega,\ s = 0, 1, 2, \ldots, t\right)$$

and we can use this to define

$$G_d(x, y) = G_\mu(x, y) := \sum_{n \geq 0} p_n(x, y).$$

The convergence is of course an issue but we have by definition

$$\left(p_n(x, .) \star \mu(.)\right)(y) = p_{n+1}(x, y), \quad x, y \in \Omega.$$

So formally, for $x, y \in \Omega$

$$\left(G(x, .) \star \mu(.)\right)(y) = G(x, y) - \delta_x(y),$$

or

$$\left(G(x, .) \star \Delta_\mu(.)\right)(y) = \delta_x(y)$$

and we have thus the defining properties of the Green function in the discrete setting. The above definition of the discrete Green function extends in the obvious way to smooth measures of the form $d\mu = \phi(x)dx$, $\phi \in C_0^\infty$.

3.2 Remarks related to the symmetry of μ

When μ is symmetric $\mu(x) = \mu(-x)$ the Green function is also symmetric

$$G(x, y) = G(y, x),$$

but not otherwise. In the symmetric case, we can afford to be much less vigilent with the notations for example, if $f_x(.) := G(., x)$ then we have also $f_x(.) = G(x, .)$. So an expression such as $G(x) \star f$ makes unambiguous since it equals $G(x, .) \star f$ and $G(., x) \star f$. So while nowhere in these lectures will it be necessary to assume that the measures involved are symmetric, in writing formulas down, we will do as it is...

4 Dilatations and Lipschitz boundaries

4.1 Green function and dilatations

It is well known that Brownian motion is scale invariant in the sense that when starting from zero, the two processes $(B_t)_{t\geq 0}$ and $(\lambda^{-1}B_{\lambda^2 t})_{t\geq 0}$ have the same law for all $\lambda > 0$. So is the Green function in so far that if we dilate Ω to $\lambda\Omega \subset \mathbb{R}^d$, the corresponding Green function G_λ is $\lambda^{2-d}G$. It is therefore vital that the conditions we impose on Ω are dilation invariant. Assume that $\partial\Omega = \{x \in \mathbb{R}^d,\ f(x) = 0\}$ for some function f. The regularity condition of the boundary of Ω, C^1-regularity or even C^α with $\alpha > 1$, is not dilation invariant (because the continuity of ∇f imposes a quantitative condition on $\omega(\delta)$ the module of continuity of ∇f). The correct condition must therefore be $\nabla f \in \mathbb{L}^\infty$, *i.e.* Ω is a Lipschitz domain.

4.2 Lipschitz domains

To fix the ideas, we will stand from the canonical model (or building block) of Lipschitz domains : the half space type

$$\Omega = \{x = (x_1, \dots, x_d) = (\mathbf{x}, x_d),\ x_d > \phi(\mathbf{x})\}\,,$$

for some Lipschitz function ϕ : $|\phi(\mathbf{x}) - \phi(\mathbf{x}')| \leq A|\mathbf{x} - \mathbf{x}'|$, $\forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^{d-1}$.

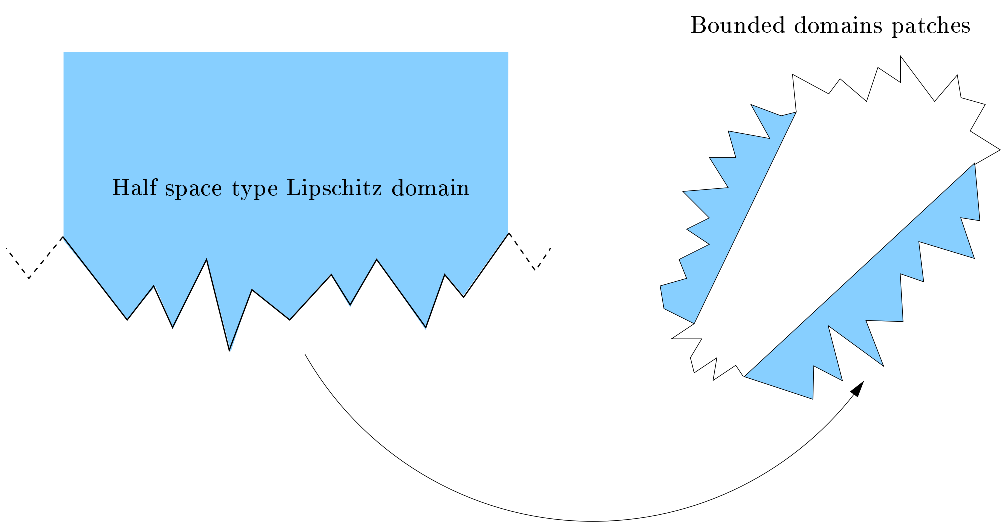

Figure 4: Patches of half space type Lipschitz domains.

The bounded Lipschitz domains we will consider in the sequel are build up from these by finitely many patches as above on their boundary (see figure 4).

The Lipschitz regularity constant $\mathrm{Lip}\,\Omega = \mathrm{Lip}\,\phi$ of such a domain is defined as the optimal A such that the above inequality holds. For any domain and $x \in \Omega$ we denote throughout

$$\delta(x) := dist(x, \partial\Omega).$$

Remark I.2 — One important issue concerning the dilatation is that in what follows Brownian motion (alternatively the euclidian Laplacian) must always be scaled so as to be consistant with the simple random walk *i.e.* we assume

$$\mathbb{E}\left[B_i(1)B_j(1)\right] = \delta_{ij}.$$

5 Statement of the main theorem

5.1 Main result on Green functions

Let us now state the main result of [Var09]. Here Ω is a half space Lipschitz domain as above and G, G_d are the euclidian and discrete Green function defined above. The operator G_d could be instead the more general G_μ's for measures as above but to fix ideas we shall consider the simple random walk.

Theorem I.1 — *There exists a constant C only depending on $Lip\,\Omega$ such that for all $x, y \in \Omega \cap \mathbb{Z}^d$, with $\delta(x), \delta(y), |x-y| > C$:*

$$|G_d(x,y) - G(x,y)| \leq CG(x,y)\left(\delta(x)^{-1} + \delta(y)^{-1} + |x-y|^{-1}\right).$$

The scope of these lectures is to explain the proof of this theorem. It is rather difficult and technical. In the next lecture, we will sketch the proof of a weaker version of theorem I.1, where the -1 on the right are replaced by $-\varepsilon$ with $0 < \varepsilon < 1$. The main difficulty in theorem I.1 is precisely to get the same estimate with $\varepsilon = 1$.

5.2 Letting the mesh goes to zero

Naturally, the above result can be scaled so that it applies for a finer and finer mesh *i.e.* $\mathbb{Z}^d$ is replaced by $\varepsilon\mathbb{Z}^d$ for some small ε. The theorem is then adapted to bounded Lipschitz domains as ε goes to zero, when the interior points look very far from the boundary in the scale ε. Here is a typical easy corollary of our main theorem that illustrates the issue.

Corollary I.1 — *Let Ω be some convex domain and let F be some smooth function in Ω. Let u^ε be the finite difference discrete solution of the inhomogeneous Dirichlet problem :*

$$\Delta_\varepsilon u^\varepsilon = F, \quad u^\varepsilon_{|\Omega^c} = 0,$$

and let u^0 be the euclidian solution of

$$\Delta u^0 = F, \quad u^0_{|\Omega^c} = 0.$$

Then we have

$$|u^\varepsilon(x) - u^0(x)| \leq C\varepsilon \left(||F||_\infty + ||\nabla F||_\infty\right), \quad x \in \Omega \cap \varepsilon\mathbb{Z}^d, \quad \delta(x) \geq C\varepsilon.$$

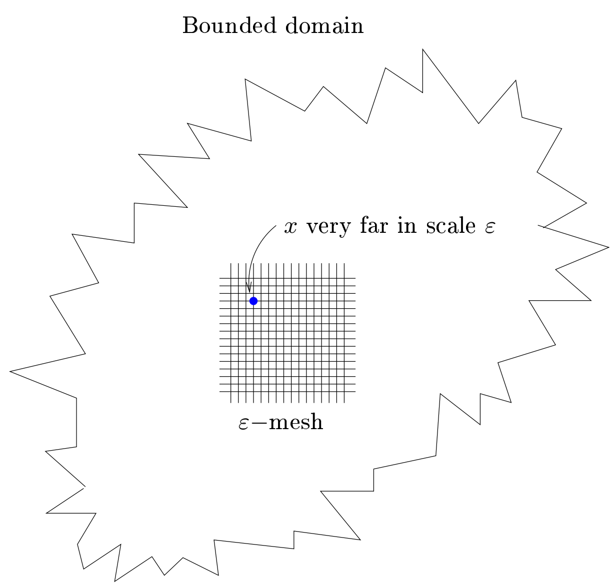

Figure 5: Letting the mesh goes to zero.

Remark I.3 — The constant here only depends on the eccentricity of Ω (no smoothness is assumed on $\partial\Omega$, see examples below). Smoothness of F in one form or another is on the other hand essential because for general $F \in \mathbb{L}^\infty(\Omega)$, we cannot define u^ε.

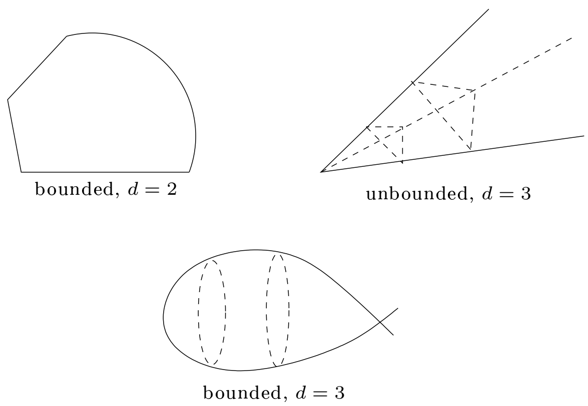

FIGURE 6: Examples of convex domains where corollary I.1 applies.

Lecture II

Towards a weaker version of the main theorem

Contents of the lecture

As announced at the end of the first lecture, we give here the proof, more precisely we sketch the proof of a weaker version of theorem I.1. The notes are organized as follows. In the first section, we state the weaker version of theorem I.1, the "$\varepsilon-$approximation" of the theorem. We claim that this weaker result is essentially equivalent to an estimate, the so called $\varepsilon-$estimate, involving convolution of Green functions. As a motivation, we then explain in a very heuristic way how the $\varepsilon-$approximation of the theorem can be derived from this $\varepsilon-$estimate. In the second section, we recall some tools/results of potential theory in Lipschitz domains needed for the proof of the $\varepsilon-$estimate. We first state estimates in the interior of the domain, and then near the boundary. In the third section, we finally bring these tools together to get closed to the $\varepsilon-$estimate.

1 A weaker version of the main theorem

1.1 The $\varepsilon-$approximation the main theorem

Let Ω be a half space type Lipschitz domain introduced in the last lecture and let G, G_μ be the euclidian and discrete associated Green functions. In this second lecture, we want to establish the following weaker version of our main theorem, the so called $\varepsilon-$approximation of theorem I.1 :

Theorem II.1 ($\varepsilon-$approximation of theorem I.1) — *For all $0 < \varepsilon < 1$ there exists a constant C that depends only on $Lip\,\Omega$ and ε such that for $\delta(x)$, $\delta(y), |x-y| > C$:*

$$|G_\mu(x,y) - G(x,y)| \leq CG(x,y)R_\varepsilon(x,y),$$

where

$$R_\varepsilon(x,y) := \left(\delta(x)^{-\varepsilon} + \delta(y)^{-\varepsilon} + |x-y|^{-\varepsilon}\right).$$

By "establish", we do not mean here that we will give a complete and detailed proof of this result but rather give an idea of the various tools involved in the proof and explain how these tools interact.

Remark II.1 — Naturally, theorem I.1 corresponds to the limit case when $\varepsilon = 1$. In fact the main difficulty of theorem I.1 is precisely to go from $\varepsilon < 1$ to $\varepsilon = 1$: this requires much more work than the one needed to establish the weaker version of the theorem.

In the next paragraph, we will explain that, after several non trivial steps, the proof theorem II.1 above reduces to showing an estimate of the following type :

$$\int_\Omega G(x_1,x)G(x,x_2)\delta(x)^{-2-\varepsilon}dx \leq CG(x_1,x_2)R_\varepsilon(x_1,x_2). \qquad (\varepsilon\text{-estimate})$$

What would really be needed to prove theorem I.1 is the same estimate but with $\varepsilon = 1$, that is

$$\int_\Omega G(x_1,x)G(x,x_2)\delta(x)^{-3}dx \leq CG(x_1,x_2)\left(\delta(x_1)^{-1} + \delta(x_2)^{-1} + |x_1-x_2|^{-1}\right).$$

Unfortunately, this is clearly false as the simple example of the half space $\Omega = \mathbb{R}^d_+$ easily shows. The $\varepsilon-$estimate however already suffices for the proof of the $\varepsilon-$approximation of our main theorem.

1.2 The heuristics that demistifies the $\varepsilon-$estimate

As a motivation for the sequel, let us now explain in a very heuristic way, how the $\varepsilon-$estimate above is related to the bound in theorem II.1. The starting point of the method used in the proof of the latter is the resolvant equation.

1.2.1 The resolvent equation

Recall that the Green function can be defined, at least formally, as the solution of the equation $G\Delta = \Delta G = \delta$. In other words, the Green function can be thought as the resolvant operator associated to the Laplacian Δ. For any operator A et B with resolvant R_A and R_B satisfying $R_A A = AR_A = \delta$ and $R_B B = BR_B = \delta$, one has the following resolvant identity :

$$R_A - R_B = R_A(B - A)R_B.$$

Here we are dealing with the two operators $A = \delta - \mu = \Delta_\mu$ and $B = \Delta$, so that

$$G_\mu - G = G_\mu \left(\Delta - \Delta_\mu\right) G.$$

Now, by Taylor expansion, for a smooth function f on Ω, the difference $\Delta_\mu - \Delta$ acts as a third order operator :

$$\left((\delta - \mu) - \Delta\right) f = O\left(\nabla^3 f\right).$$

Thus if we want to estimate the difference $G_\mu(x_1, x_2) - G(x_1, x_2)$ where $x_1, x_2 \in \Omega$, what has to be estimated is the integral :

$$\int G_\mu(x_1, x)\nabla_x^3 G(x, x_2)dx.$$

1.2.2 Harnack inequalities

At this point, it is crucial to note that $\nabla^3 G(x, x_2) \sim |x_2 - x|^{-d-1}$ when x goes to x_2 so that the last integral diverges badly near x_2. To come up with this difficulty, since G and its derivatives vanish at the boundary, one can integrate by parts to get

$$\int G_\mu(x_1, x)\nabla_x^3 G(x, x_2)dx = \int \nabla_x G_\mu(x_1, x)\nabla_x^2 G(x, x_2)dx.$$

The Green function G being harmonic, we will see in the next section that G and its derivatives satisfy an Harnack inequality, namely

$$|\nabla_x^k G(x, x_2)| \leq CG(x, x_2)\left(\delta(x)^{-k} + |x_2 - x|^{-k}\right).$$

In a future lecture, using the theory of random walks, we will see that the discrete Green function G_μ and its derivatives also satisfy a (non trivial) Harnack inequality of the same type :

$$|\nabla_x G_\mu(x_1, x)| \leq C G_\mu(x_1, x) \left(\delta(x)^{-1} + |x_1 - x|^{-1}\right).$$

Therefore, the integral that really need to be estimated is

$$\int G_\mu(x_1, x) G(x, x_2) \left(\delta(x)^{-1} + |x_1 - x|^{-1}\right) \left(\delta(x)^{-2} + |x_2 - x|^{-2}\right) dx.$$

1.2.3 Coarse estimate

The next step in the proof is to get a control of the discrete Green function by the continuous one. In a future lecture, we will in fact establish the so called “coarse estimate"

$$G_\mu(x_1, x) \leq C G(x_1, x). \qquad \text{(coarse estimate)}$$

To bound the difference $G_\mu(x_1, x_2) - G(x_1, x_2)$, one thus have to estimate the following integral :

$$\int G(x_1, x) G(x, x_2) \left(\delta(x)^{-1} + |x_1 - x|^{-1}\right) \left(\delta(x)^{-2} + |x_2 - x|^{-2}\right) dx.$$

1.2.4 Estimates involving convolution of Green functions

All the preceding integrals are of course divergent near the singularities x_1, x_2 and $\partial\Omega$. We thus have to stay away from the boundary, hence the condition $\delta(x) > C$. The way to handle the singularities x_1, x_2 and product terms $\delta^{-1}(x)|x_2 - x|^{-2}, \delta^{-2}(x)|x_1 - x|^{-1}, |x_1 - x|^{-1}|x_2 - x|^{-2}$ is not so difficult so that we are left with the dominant term $G(x_1, x)G(x, x_2)\delta^{-3}(x)$. At any rate, if we stay away from the singularity we can replace this dominant term by $G(x_1, x)G(x, x_2)\delta^{-2-\varepsilon}(x)$ with $0 < \varepsilon < 1$, hence our $\varepsilon-$estimate :

$$\int G(x_1, x) G(x, x_2) \delta(x)^{-2-\varepsilon} dx.$$

Of course we have to modify by an ε the additional terms above to make the integral convergent $\delta(x)^{-1}|x_2 - x|^{-1-\varepsilon}$ etc. and we must prove analogue estimates for the corresponding integrals which are easier than the $\varepsilon-$estimate.

2 Potential theory in a Lipschitz domain

Let us now recall some facts concerning potential theory in Lipschitz domains, that will be the essential ingredients of the proof of the $\varepsilon-$estimate. We first recall some basic facts of potential theory in the interior of the domain, namely the maximum principle and the Harnack principle. We then explain how these notions extends at the boundary of a Lipschitz domain by stating the Carleson principle and the comparison principle.

2.1 Classical potential theory in the domain

The *maximum and classical Harnack principles* assert that if u is a positive harmonic function in the ball $B_r(x_0) := \{x \in \mathbb{R}^d, |x - x_0| < r\}$, and if u is continuous up to the boundary $\partial B_r(x_0)$, then

(i) if $y \in \overline{B_r}(x_0)$ is such that $u(y) = \sup_{x \in B_r(x_0)} u(x)$, then $y \in \partial B_r(x_0)$.

(ii) there exists a positive constant C depending only on the dimension d such that
$$C^{-1} \leq \frac{u(x)}{u(y)} \leq C, \ \forall x, y \in B_{r/2}(x_0).$$

From this, we can naturally deduce a lot more, for example if $\Omega_1 \subset \overline{\Omega}_1 \subset \Omega$, with $\overline{\Omega}_1$ compact, then

$$C_1^{-1} \leq \frac{u(x)}{u(y)} \leq C_1, \ \forall x, y \in \overline{\Omega}_1,$$

with a constant C_1 that now depends on Ω_1, Ω, but not on u. Moreover, for two positive harmonic functions u and v, we have the so called comparison result
$$\frac{u(x)}{u(y)} \approx \frac{v(x)}{v(y)},$$
meaning that there exists positive constants c and C not depending on u and v such that :
$$c\frac{u(x)}{u(y)} \leq \frac{v(x)}{v(y)} \leq C\frac{u(x)}{u(y)}, \ \forall x, y \in \overline{\Omega}_1.$$

2.2 Potential theory at the boundary

The two preceding classical notions of potential theory in the interior of a domain extends naturally at the boundary of a Lipschitz domain, and more generally at the boundary of any non tangentially accessible domain

(NTA), see [Ken94]. For simplicity, we will only consider here half space type Lipschitz domains. So let Ω be such a domain, with a Lipschitz function ϕ, $Q = (\mathbf{x}, \phi(\mathbf{x})) \in \partial\Omega$, and let us introduce the following (classical) notations :

$$A_r(Q) := (\mathbf{x}, \phi(\mathbf{x}) + r) = Q + (0, \ldots, 0, r),$$

$$T_r(Q) := \{y = (\mathbf{y}, y_d) \in \Omega,\ |\mathbf{y} - \mathbf{x}| \leq r,\ |y_d - \phi(\mathbf{x})| < cr\},$$

$$\Delta_r(Q) := \partial T_r(Q) \cap \partial\Omega.$$

Here c is a constant, large enough so that the domain $T_r(Q)$ is connected (see figure 7 below).

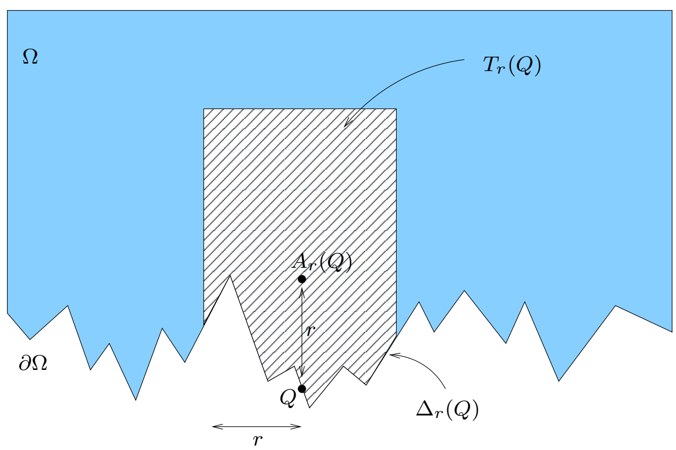

FIGURE 7: Near the boundary of a half space type Lipschitz domain.

Consider a positive harmonic function u in $T_{2r}(Q)$ that "vanishes" at the boundary $\Delta_{2r}(Q)$, *i.e* u is continuous up to the boundary $\Delta_{2r}(Q)$ and vanishes there[1]. The *Carleson principle* then says that

$$u(x) \leq Cu(A_r(Q)),\ \forall x \in T_r(Q),$$

with C only depending on $\operatorname{Lip}\Omega$. Let now u, v two positives harmonic functions in $T_{2r}(Q)$ that vanish on $\Delta_{2r}(Q)$, then we have the so called *comparison principle* :

$$C^{-1}\frac{u(A_r(Q))}{u(x)} \leq \frac{v(A_r(Q))}{v(x)} \leq C\frac{u(A_r(Q))}{u(x)}.$$

[1] This is not the correct way of interpreting the vanishing of u but it will be good enough for our purpose since everything will be done in terms of a priori inequalities.

2.3 Harmonic measure and the doubling property

Let $x \in \Omega$, the harmonic measure $h_x(d\xi)$ can be defined as the unique measure on $\partial\Omega$ such that $u(x) = \int u(\xi)h_x(d\xi)$ for any harmonic function u in Ω. This definition makes sense if Ω is bounded but not if Ω is the upper half plane $\mathbb{R}^d_+ := \{x = (\mathbf{x}, x_d) \in \mathbb{R}^d,\ x_d > 0\}$, because of the immediate counter example $u(x) = x_d$. The best general definition for the harmonic measure is via Brownian motion and it has the advantage of being natural in the context of random walks in discrete potential theory. So let $x \in \Omega$, $(B_t)_{t\geq 0}$ a Brownian motion starting from x and τ the exit time of Ω : $\tau := \inf\{t > 0,\ B_t \notin \Omega\}$. One then defines the harmonic measure as :

$$h_x(E) := \mathbb{P}_x(B_\tau \in E),\ \forall E \subset \partial\Omega \text{ measurable.}$$

This definition extends verbatim for a random walk (say a simple random walk) in $\mathbb{Z}^d$ and $\Omega \subset \mathbb{Z}^d$ except that now $E \subset \Omega^c$ and we do not talk about the boundary $\partial\Omega$ because the random walk does not see it (it jumps over !). Let us go back to half space type Lipschitz domain Ω. The basic fact there is that we have the comparison :

$$G(x_0, A_r(Q)) \approx h_{x_0}(\Delta_r(Q))r^{2-d}$$

for all $r > 0$ and $x_0 \notin T_{2r}(Q)$, and the constants in $\approx$ only depend on $\mathrm{Lip}\,\Omega$. From this, one deduces the doubling property that writes

$$h_x(\Delta_r(Q)) \approx h_x(\Delta_{2r}(Q)).$$

It is of some importance to note that all the above hold for more general domains, namely for NTA domains (see [Ken94]).

Remark II.2 — Note that the doubling property is the key point in the proof of many deep results in real analysis. For example, if μ is a measure on $\mathbb{R}^{d-1}$ that satisfies the above doubling property, then one can show that the Hardy-Littlewood operator

$$M : f \to Mf, \quad Mf(x) := \sup_{r>0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y)|\mu(dy),$$

is a bounded operator mapping $\mathbb{L}^p$, $p > 1$ into itself, *i.e.* if $f \in L^p(\mathbb{R}^{d-1})$, then $Mf \in \mathbb{L}^1_{weak}(\mathbb{R}^{d-1}) \cap \mathbb{L}^p(\mathbb{R}^{d-1})$. The doubling property is also the key point in the proof of Fatou's theorem on non tangential limits of harmonic functions at the boundary.

3 Pointwise estimate and harmonic measure integrability

3.1 A pointwise estimate : the 3−points lemma

The first corollary of the above results on potential theory in Lipschitz domains that is needed in the proof of the $\varepsilon-$estimate is a pointwise estimate, the so called 3−points lemma. This lemma is valid in dimension $d \geq 2$, but its two dimensional version can be seen as a particular case.

3.1.1 The 2−dimensional result coming from conformal mapping

Let us first state the 3−points lemma in the two dimensional case. So let $\Omega \subset \mathbb{R}^2 = \mathbb{C}$ be a simply connected domain and let x, x_1, x_2 . Let also $m(x_1, x_2, x) = |x - x_1|\delta^{-1}(x_1) \wedge |x - x_2|\delta^{-1}(x_2)$. Then, for all $a > 0$, there exists a constant $C_a > 0$ such that

$$G(x_1, x)G(x, x_2) \leq C_a G(x_1, x_2), \ \forall x \in \Omega \text{ such that } m(x_1, x_2, x) > a.$$

Of course, the homogeneity in the latter estimate is non surprising since when $d = 2$, the Green function G is conformally invariant.

3.1.2 Analogous result when $d \geq 3$

The $d \geq 3$ analogue of the latter estimate holds for general NTA domains, in particular if Ω is a Lipschitz domain, and $x, x_1, x_2 \in \Omega$ then one has

$$G(x_1, x)G(x, x_2) \leq C \times \left(|x_1 - x|^{2-d} + |x_2 - x|^{2-d} \right) G(x_1, x_2),$$

where C only depends on $\mathrm{Lip}\,\Omega$.

In other words, for $d \geq 2$, one has the following pointwise estimate, which is a consequence of the results described in section 2 :

Lemma II.1 (3−points lemma) — *Let $\Omega \subset \mathbb{R}^d$ be a half space type Lipschitz domain. There exists a constant C only depending on $Lip\,\Omega$ such that for all $x, x_1, x_2 \in \Omega$:*

$$G(x_1, x)G(x, x_2) \leq C \times \left(|x_1 - x|^{2-d} + |x_2 - x|^{2-d} \right) G(x_1, x_2).$$

3.2 Estimates involving the harmonic measure

The second main ingredient in the proof of the $\varepsilon-$estimate is in some sense a quantitative version of the fact that the harmonic measure is uniformly square integrable :

$$\forall x \in \Omega,\ h_x(d\xi) \in \mathbb{L}^2(\partial\Omega, d\xi).$$

The quantitative aspect is picked up by the notion of B_q-measure, *i.e* if $d\mu(x) = f(x)dx$:

$$\left(|I|^{-1}\int_I |f|^q dx\right)^{1/q} \leq C\left(|I|^{-1}\int_I |f| dx\right).$$

Consider the following geometric situation :

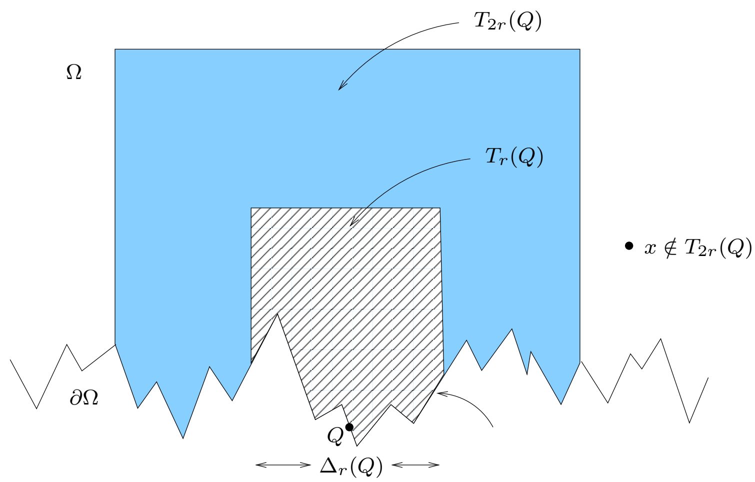

Figure 8: Configuration where a quantitative estimates of h_x can be obtained.

In this situation, one can indeed show that :

$$h_{x|\Delta_r} \in B_2,$$

with some constants that depend only on $\operatorname{Lip}\Omega$. The corollary of the above that will be needed in the proof of the $\varepsilon-$estimate is more precisely the following. Consider a positive harmonic function defined in $T_{2r}(Q)$, and define for $\mathbf{y} \in \Delta_r(Q)$:

$$u^*(\mathbf{y}) := \sup_{0<\rho<r} \rho^{-1}|u(\mathbf{y}, \phi(\mathbf{y}) + \rho)|.$$

Then one has

$$||u^*||_{\mathbb{L}^2(\Delta_r(Q))} \leq Cr^{(d-3)/2}u(A_r(Q)).$$

In the next lecture, we will see that the 3−point lemma combined with the above $\mathbb{L}^2$ norm estimate for u^* lead us to the ε−estimate, and according to the heuristic of section 1.2, to the theorem II.1

Lecture III

On the proof of the $\varepsilon-$estimate

Contents of the lecture

In this lecture, we use the results of potential theory in Lipschitz domains stated in section 2 of lecture II, associated to an ad hoc decomposition of the domain, to derive the $\varepsilon-$estimate needed to establish the $\varepsilon-$approximation of the main theorem. We first describe the decomposition of the domain into "constellations", *i.e.* we cover the whole domain with balls of geometrically increasing diameter. We then use the estimates of section 2.1 in lecture II to control the behavior of the integrant in the $\varepsilon-$estimate when the covering balls are in the interior of the domain. The behavior near the boundary is finally controlled thanks to the estimates of section 2.2.

1 Decomposition of the domain into constellations

Let us first describe the decomposition into "constellations" of our half space type Lipschitz domain. This decomposition consists in covering the whole domain by balls with geometrically increasing radii. The number of balls that will intersect a given ball will be uniformly bounded. Our decomposition can be thought as a simplified version of Whitney decomposition.

1.1 First : make it flat !

As explain in the introduction of this lecture, the key point in the proof of the $\varepsilon-$estimate

$$\int_\Omega G(x_1,x)G(x,x_2)\delta(x)^{-2-\varepsilon}dx \leq \ldots,$$

or more generally in the proof of estimates of integrals of the following type

$$\int_\Omega G(x_1,x)G(x,x_2)|x_1-x|^{-\alpha}|x_2-x|^{-\beta}\delta(x)^{-\gamma}dx \leq \ldots,$$

is to use "harmonic function estimates" as the ones stated in section 2 of lecture II on each component of an appropriate decomposition of the half space type domain Ω. To simplify, everything will be done in a "bilipschitz invariant manner", *i.e.* we transform Ω to the "real" half space via a bilipschitz map $\Phi : \mathbb{R}^d \to \mathbb{R}^d$:

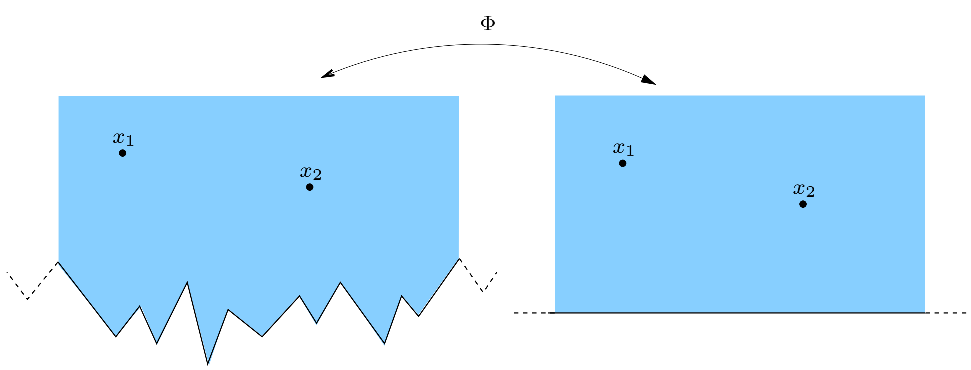

FIGURE 9: From half space type Lipschitz domains to half space.

1.2 Two points complement constellations

To explain our decomposition, let us start with the whole space $\mathbb{R}^d - \{x_1, x_2\}$ and decompose it into three *constellations* (see figure 10 below) :

(i) the x_1-constellation concern points in a neighbourhood of $\{x_1, x_2\}$ that are closer from x_1 than from x_2 ;

(ii) the x_2-constellation concern points in a neighbourhood of $\{x_1, x_2\}$ that are closer from x_2 than from x_1 ;

(iii) the outer constellation concern points far from the neighbourhood of $\{x_1, x_2\}$, their distance to x_1 and x_2 is thus comparable.

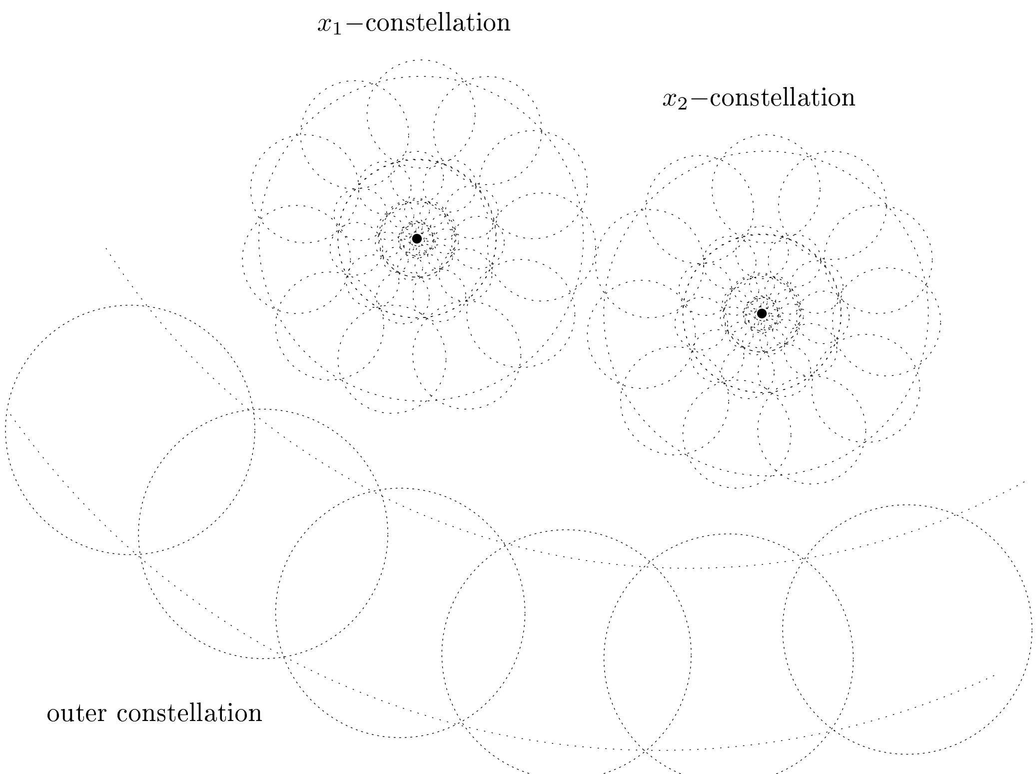

FIGURE 10: Constellations in $\mathbb{R} - \{x_1, x_2\}$.

The constellations are made of balls that cover annuli of geometrically increasing "diameters". Each ball intersects the other ones a finite, uniformaly bounded, number of time. The step $1/\rho$ of the geometrical growth is supposed very small so that each of the covering ball B_j constructed has a radius $r_j \approx \rho^{-j} << dist(B_j, x_1) \wedge dist(B_j, x_2)$.

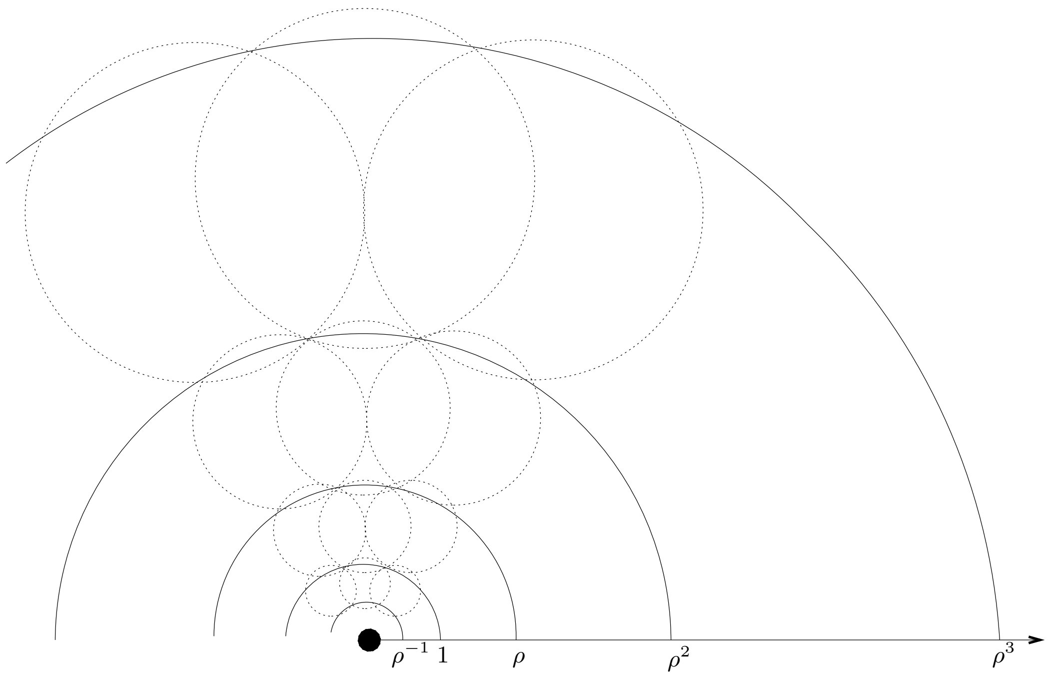

FIGURE 11: Geometrical growth of the constellation near x_1 and x_2.

Now we go back to the previous image and delete small disks D_1 and D_2 centered at x_1 and x_2 with small diameters, say

$$diam(D_i) \leq 10^{-10} dist(x_i, \partial\Omega) = 10^{-10}\delta(x_i), \quad i = 1, 2.$$

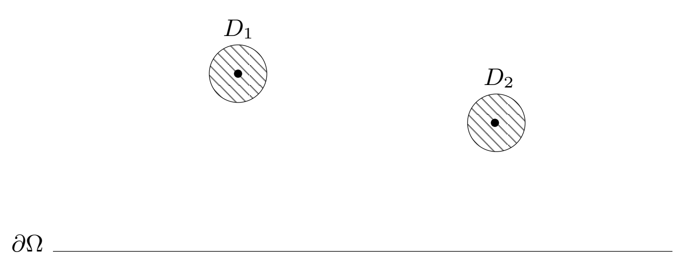

FIGURE 12: Remove small disk at x_1 and x_2;

We restrict the above covering to $\Omega' = \Omega - (D_1 \cup D_2)$, *i.e.* we ignore the balls that lie outside Ω'. This is essentially the covering that we shall need to derive the ε−estimate in the next section. Yet, we have to modify a little bit the covering near the boundary. This modification is explained in the next paragraph.

1.3 Slight change near the boundary

The modification concern balls that are close or intersect the boundary $\partial\Omega$. The latter covering balls will be replaced by slightly larger ones that are centered on $\partial\Omega$. By doing that, we are able to ignore the balls B_j that are

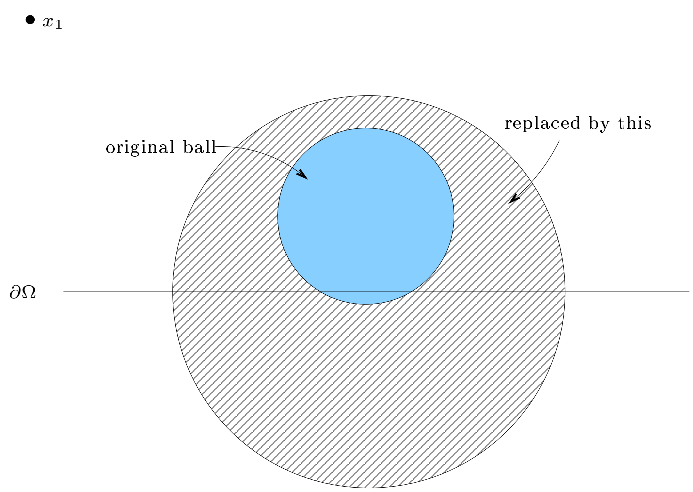

FIGURE 13: Covering balls at the boundary.

inside Ω and for which we have not $r_j << dist(B_j, \partial\Omega)$. So we end up with two kind of covering balls :

(i) the ones inside Ω with radius $r_j << dist(B_j, \partial\Omega)$;

(ii) the ones that intersect the boundary.

2 Deriving estimates on the constellations

Let us now explain how the above decomposition, combined with the estimates of potential theory in Lipschitz domains, leads to the $\varepsilon-$estimate.

2.1 Estimate in the domain

Consider a covering ball B_j with radius r_j in the preceding decomposition, that lie inside Ω. By applying the Harnack inequality in this ball, the integrant in the $\varepsilon-$estimate is essentially constant, that is

$$G(x_1, x)G(x, x_2)\delta(x)^{-2-\varepsilon} \approx G(x_1, y_j)G(y_j, x_2)r_j^{-2-\varepsilon}, \ \ \forall x \in B_j.$$

In particular, by the 3−points lemma (lemma II.1), the integrant on such an interior ball is controlled by

$$G(x_1,x_2)\left(|x_1-y_j|^{2-d}+|x_2-y_j|^{2-d}\right)r_j^{-2-\varepsilon}\approx G(x_1,x_2)r_j^{-d-\varepsilon}.$$

For estimating the integral, we just multiply by $vol(B_j)\approx r_j^d$ and get

$$\int_{B_j} G(x_1,x)G(x,x_2)\delta(x)^{-2-\varepsilon}dx\le CG(x_1,x_2)r_j^{-\varepsilon}.$$

2.2 Estimates near the boundary

The same type of estimate holds for balls near the boundary of the domain. To see this, recall the Carleson estimate and the harmonic measure estimate of last section 2.2. We first write

$$\int_{B_j} G(x_1,x)G(x,x_2)\delta(x)^{-2-\varepsilon}dx=\int_{B_j}\underbrace{\frac{G(x_1,x)}{\delta(x)}}_{u_1(x)}\underbrace{\frac{G(x,x_2)}{\delta(x)}}_{u_2(x)}\delta(x)^{-\varepsilon}dx.$$

By taking the supremum (on vertical segment) of u_1 and u_2, the last integral is controlled by

$$\int_{\Delta_j} u_1^*(x)u_2^*(x)r_j^{1-\varepsilon}dx.$$

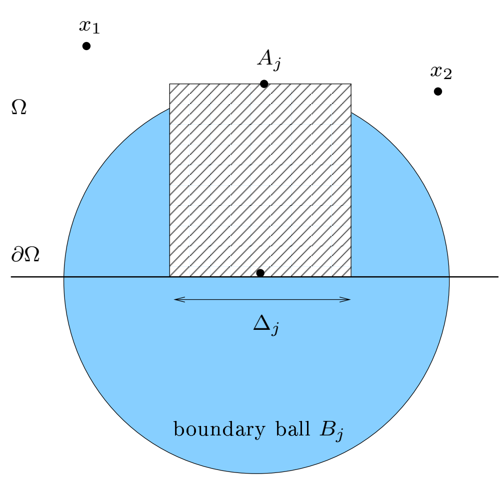

FIGURE 14: Estimates near the boundary.

Now applying Hölder inequality, we get

$$\int_{B_j} G(x_1,x)G(x,x_2)\delta(x)^{-2-\varepsilon}dx\le C||u_1^*||_{\mathbb{L}^2(\Delta_j)}||u_2^*||_{\mathbb{L}^2(\Delta_j)}r_j^{1-\varepsilon},$$

that is

$$\int_{B_j} G(x_1, x)G(x, x_2)\delta(x)^{-2-\varepsilon}dx \le CG(x_1, A_j)G(A_j, x_2)r_j^{(d-3)/2}r_j^{(d-3)/2}r_j^{1-\varepsilon},$$

or

$$\int_{B_j} G(x_1, x)G(x, x_2)\delta(x)^{-2-\varepsilon}dx \le CG(x_1, A_j)G(A_j, x_2)r_j^{d-2-\varepsilon}.$$

We thus obtain the following estimate :

$$\begin{aligned} \int_{\Omega} G(x_1, x)G(x, x_2)\delta(x)^{-2-\varepsilon}dx &\le \sum_j \int_{B_j} G(x_1, x)G(x, x_2)\delta(x)^{-2-\varepsilon}dx \\ &\le CG(x_1, x_2)\sum r_j^{-\varepsilon}. \end{aligned}$$

Now we pass to the final property of our decomposition. The geometrical growth of the radii essentially says that $r_{j+1}/r_j > 1+\delta$. To be more precise, we have to modify things slightly so that $r_1 \le r_2 \le \ldots$ and $r_{j+100}/r_j > 1+\delta$. This allows us to estimate the sum

$$\sum r_j^{-\varepsilon} \approx \inf\{r_j\}^{-\varepsilon} \approx (\delta(x_1) \wedge \delta(x_2))^{-\varepsilon},$$

hence the result.

Lecture IV

On the discrete Harnack inequality

Contents of the lecture

What has been said in the past lectures brings us to the issue of the Harnack estimates in the discrete setting, *i.e.* Harnack inequality associated to random walks. We give here an overview of a proof of this result based on the Edgeworth expansion. In fact, we first prove the estimate for the Green function in the whole space, then in a ball. The case of a general harmonic function can then be easily derived using representation formulae. For the curious reader, an alternative proof can be found in [Law91].

1 Harnack inequality for the global discrete Green function

First of all, let us clarify what we mean by Harnack estimate in a discrete setting. Typically for a classical harmonic function u in ball centered at the origin $B_r = \{x \in \mathbb{R}^d,\ |x| \leq r\}$, we have

$$|\nabla u(0)| \leq Cr^{-1}||u||_\infty = Cr^{-1} \sup_{x \in B_r} |u(x)|,$$

and more generally

$$|\nabla^k u(0)| \leq C_k r^{-k}||u||_\infty.$$

The constants C and C_k only depend on the dimension. From this, we obtain easily the Moser estimate, that is, if u is a positive harmonic function in B_r, then

$$C^{-1} \leq \frac{u(x)}{u(0)} \leq C, \quad x \in B_{r/2}.$$

The Moser estimate is indeed automatic if we strengthen the gradient estimate to

$$\nabla u(0) \leq Cr^{-1}u(0).$$

It is then only a matter of integrating $\int \nabla \log u(x)$. The corresponding estimate in the discrete setting, *i.e.* for random walks, holds verbatim

$$|\delta_{i_1}\delta_{i_2}\ldots\delta_{i_k}u(0)| \leq Cr^{-k}u(0)$$

for a $\mu-$harmonic function in B_r where $\delta_i u(x) = u(x+e_i) - u(x)$ is the difference operator with respect to the unit coordinate vector $e_i = (0, 0, 1, 0, \ldots, 0)$. In the rest of this lecture, we will outline the steps of a proof of this result in a relatively self-contained manner. The first set of steps of this proof consist in proving the Harnack inequality for the Green function on the whole space. This is done using

(i) the Edgeworth expansion ;

(ii) the coarse gaussian estimate ;

(iii) upper and lower estimates for the Green function ;

(iv) Harnack estimate for Green function :

$$u(x) = G(x, y), \quad x \in B_r,\ y \notin B_{2r}.$$

1.1 Comments on the Edgeworth expansion

The starting point of our proof is the Edgeworth expansion, which deals with the convergence of the density of a well-renormalized sum of random variables to the gaussian density. A good treatment of the subject can be found in [Fel71]. Suppose that $\mu^{\star n}$ the n^{th} convolution of a measure μ has a density :

$$\mu^{\star n}(dx) = \phi_n(x)dx.$$

The Edgeworth expansion says that

$$\phi_n(x) = \sum_{j=0}^{m} n^{-j/2} P_j\left(\frac{x}{\sqrt{n}}\right) \times n^{-d/2} \exp\left(-\frac{|x|^2}{c_0 n}\right) + O\left(n^{-(m+d)/2}\right), \quad \text{(IV.1)}$$

where the P_j are Hermite polynomials (up to multiplicative constants). Notice that the remainder term is uniform in x.

1.2 The coarse Gaussian estimate

The second step of the proof is a coarse gaussian estimate, namely :

$$\phi_n(x) \leq C n^{-d/2} \exp\left(-\frac{|x|^2}{cn}\right). \quad \text{(IV.2)}$$

This is tricky to prove if μ is not symmetric. Even for μ symmetric, it is quite difficult to prove in that form but at least, it has the merit to exist in the litterature. What one can on the other hand prove using classical method of probability theory is a coarse estimate of the form :

$$\phi_n(x) \leq C n^{100+d} \exp\left(-\frac{|x|^2}{cn}\right).$$

Combining equations (IV.1) and (IV.2), we obtain the remainder term in (IV.1) is in fact

$$O\left(n^{-(m+d)/2+1} \exp\left(-\frac{|x|^2}{cn}\right)\right).$$

1.3 Upper and lower bound for the Green function

Now we can take a geometric average of the two above estimates. If we sum over n and compare the sum with the integrals

$$\int_0^{+\infty} t^{-a} \exp\left(-\frac{|x|^2}{t}\right) dt = |x|^{2-2a},$$

we obtain an asymptotic development of the Green function :

$$\sum \phi_n(x) = |x|^{2-d} + C(x,n)|x|^{1-d} + \dots$$

and at any rate for sure $|x|^{2-d} + O\left(|x|^{1-d}\right)$. From the Edgeworth expansion, for x small enough, we also have a lower bound for $\phi_n(x)$, namely

$$\phi_n(x) \geq cn^{-d/2}, \quad for\ |x| \leq c\sqrt{n}.$$

By integating, we thus get

$$G(x) \geq c|x|^{2-d}.$$

1.4 Harnack estimate at a respectable distance

Now the difference operator δ can be applied directly on the Edgeworth expansion and summed. We obtain thus the Harnack estimate for the function $u(x) := G(x,y)$ where $x \in B_r$ and $y \notin B_r$. It is important here to note that the difference operator is applied before summing. The reason is that

$$\nabla_x \left[P_j \left(\frac{x}{\sqrt{n}} \right) \exp\left(-\frac{|x|^2}{ct} \right) \right] \approx \frac{1}{\sqrt{t}} \left[\widetilde{P}_j \left(\frac{x}{\sqrt{n}} \right) \exp\left(-\frac{|x|^2}{ct} \right) \right],$$

i.e. we gain a factor $t^{-1/2}$ so that after summation :

$$\nabla G(x) = O\left(|x|^{2-d-1}\right) = O\left(|x|^{1-d}\right).$$

We thus can compare $\nabla G(x)$ with $G(x)$ thanks to the lower bound for G.

2 The Harnack estimate for the Green function in a ball

Now the issue is to deduce the Harnack estimate for the Green function defined in ball B_r only. As before, this is done in several steps, using the preceding estimate for the Green function on the whole space.

2.1 Comparison between Green functions

Let $G_r(x,y)$ denote the Green function in the ball of radius r centered at the origin B_r. For $a > 0$, we then have the following comparison where G is the Green function on the whole space :

$$C^{-1}G(x,y) \leq G_{ar}(x,y) \leq CG(x,y).$$

Only lower estimate needs to be proved, this is done using the two estimates

$$G(x,y) \leq cr^{2-d}, \quad \text{and} \quad G_{ar}(x,y) \geq G(x,y) - \sup_{\xi \in \partial B_{ar}} G(x,\xi)$$

Here the supremum is no more than $((a-1)r)^{2-d}$, provided a is choosen large enough.

2.2 Gradient estimate at a respectable distance

We now use the “representation" formula :

$$G_{ar}(x,y) = G(x,y) - \int_{\partial B_{ar}} G(x,\xi) h_y(d\xi)$$

By applying the difference operator acting on x, we get

$$|\nabla_x G_{ar}(x,y)| \leq |\nabla_x G(x,y)| + \sup_{\xi \in \partial B_{ar}} \nabla_x G(x,\xi).$$

We then use the Harnack estimate for the whole space Green function G to dominate the right side by

$$r^{-1}\left(G(x,y) + \sup_{\xi \in \partial B_{ar}} G(x,\xi)\right).$$

We can then use the Moser estimate to move ξ to y. We have then

$$|\nabla_x G_{ar}(x,y)| \leq Cr^{-1} G(x,y)$$

and thanks to the comparison of the preceding paragraph :

$$|\nabla_x G_{ar}(x,y)| \leq Cr^{-1} G_{ar}(x,y). \tag{IV.3}$$

2.3 Uniform estimate in the annulus

The third step consists in proving the estimate (IV.3) uniformly in y in the annulus A between $2r$ and ar. For this, use the representation formula

$$G_{ar}(x,y) = \int_{\partial A} G_{ar}(x\xi) h_y(d\xi, A)$$

This “barycenter" the problem to $y \in \partial B_{2r}$. What remains to prove is the last exit decomposition the harmonic function $h_x(\xi)$ when ξ is just outside the ball B_{ar}.

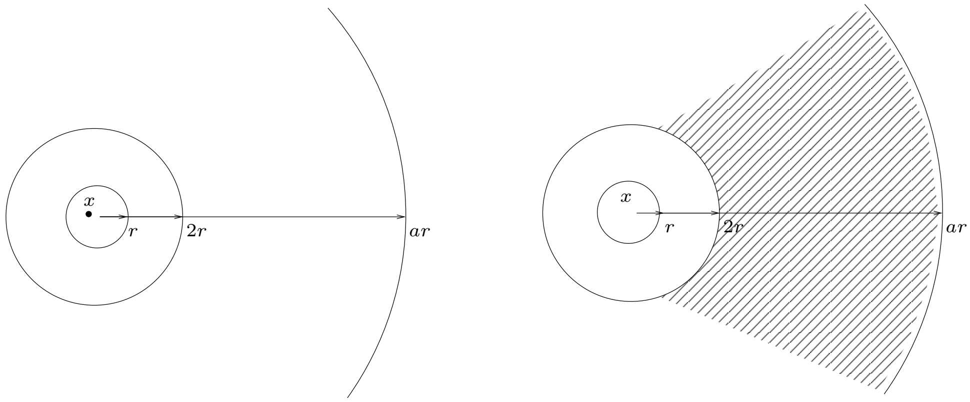

FIGURE 15: From the boundary to the whole annulus.

But $h_x(\xi)$ is the measure of all the paths that start at x and exit B_{ar} in ξ. There are finitely many possibilities for the point just before ξ, say $\zeta_1, \ldots, \zeta_N$. But the paths that exit the domain via ζ_i are measured by $G(x, \zeta_i)$, so that

$$h_x(\xi) = \sum_{i=1}^{N} \lambda_i G(x, \zeta_i), \quad 0 < \lambda_i < 1.$$

This give the barycenter that is needed.

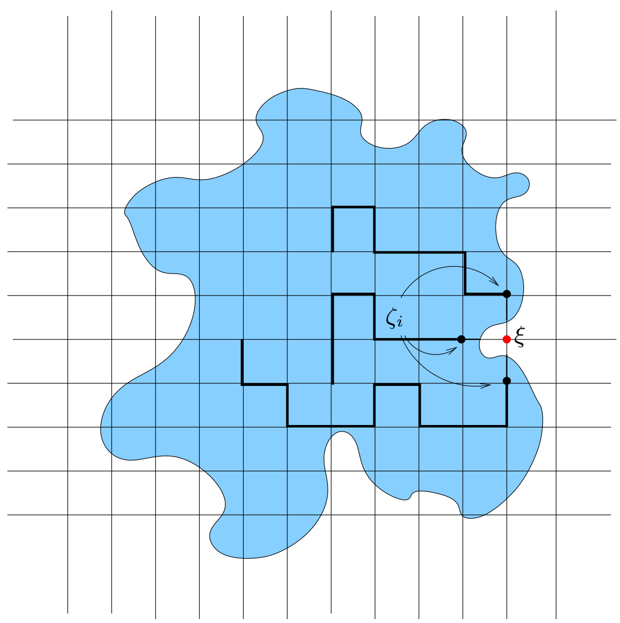

FIGURE 16: Exit points in the domain.

Lecture V

On the proof of the coarse estimate

Contents of the lecture

This last lecture is dedicated to the proof of the coarse estimate, *i.e.* the control of the discrete Green function by the continuous one, namely for a constant $C > 0$ large enough :

$$G_\mu(x, y) \leq CG(x, y).$$

In fact, the continuous Green function can also be controlled by the discrete one, so that we have

$$C^{-1}G(x, y) \leq G_\mu(x, y) \leq CG(x, y).$$

We will concentrate here only on the second inequality which is the one that is needed in the proof of the main theorem and its $\varepsilon-$approximation.

1 Super-harmonicity of the perturbed Green function

1.1 Perturbation of the Green function

The key in the proof of the coarse estimate lies in a small perturbation of the euclidian Green function, and the use of the super-harmonicity of this perturbation. Let us thus consider the perturbation H of the continuous Green function :

$$H(x) := G(x, x_0) + \omega(x),$$

where

$$\omega(x) := \int_{|y-x_0|>a\delta(x_0)} G(x,y)G(y,x_0)\left[\widetilde{\delta}^{-2-\varepsilon}(y) + |y-x_0|^{-2-\varepsilon}\right],$$

and $\widetilde{\delta}$ is a smooth approximation of δ such that

$$\delta(x) \approx \widetilde{\delta}(x), \quad |\nabla^k \widetilde{\delta}(x)| \le C\delta^{1-k}(x), \;\; x \in \Omega, \;\; k \ge 0.$$

The reason of this smoothing is that we will be led to take derivatives of $\omega(x)$

$$\nabla^k \omega(x) = \int \nabla^k G(x,y)\,(\ldots)\,.$$

Near the singularity x, the Green function $G(x,y)$ is of the order $|x-y|^{2-d}$ (+ correcting harmonic function) so that we are dealing with

$$\int \nabla^k |x-y|^{2-d} G(y,x_0)\left[\widetilde{\delta}^{-2-\varepsilon}(y) + |y-x_0|^{-2-\varepsilon}\right] dy.$$

It is thus essential that $\widetilde{\delta}$ is smooth so as to interpret the last integral in the distribution sense. The upshot is that, with computations analoguous to what we already did, we have the control

$$\nabla^k \omega(x) \le CG(x,x_0)\left[\delta(x)^{-\varepsilon} + \delta(x_0)^{-\varepsilon} + |x-x_0|^{-\varepsilon}\right]\left[\delta(x)^{-k} + |x-x_0|^{-k}\right].$$

Now we go back to $\omega(x)$ and we need to observe in addition

$$\Delta\omega(x) = G(x,x_0)\left[\widetilde{\delta}(x)^{-\varepsilon} + |x-x_0|^{-\varepsilon}\right],$$

because $\Delta G(x,y) = \delta_x(y)$

1.2 Super-harmonicity of the perturbation

Once we have the perturbation, the second step is to prove its superharmonicity namely

$$\Delta_\mu H = (\delta - \mu)H(x) > 0,$$

provided that $\delta(x) > C$, $\delta(x_0) > C$, and $|x - x_0| > C$. As before, this estimate is established by considering the resolvent equation and the Taylor expansion to get the control

$$(\Delta - \Delta_\mu)H = O\left(\nabla^3 H\right),$$

so that

$$\Delta_\mu H = \Delta H + O\left(\nabla^3 H\right).$$

We use then the following estimate for G :

$$\nabla^3 G = O\left(G(x, x_0)\left[\delta^{-3}(x) + |x - x_0|^{-3}\right]\right),$$

and for $\nabla^3\omega$, we use the previous estimate. Now from the $\Delta_\mu-$superharmonicity of the perturbation H, we deduce that $H(x) \geq CG(x, x_0)$ in a slightly smaller domain included in Ω.

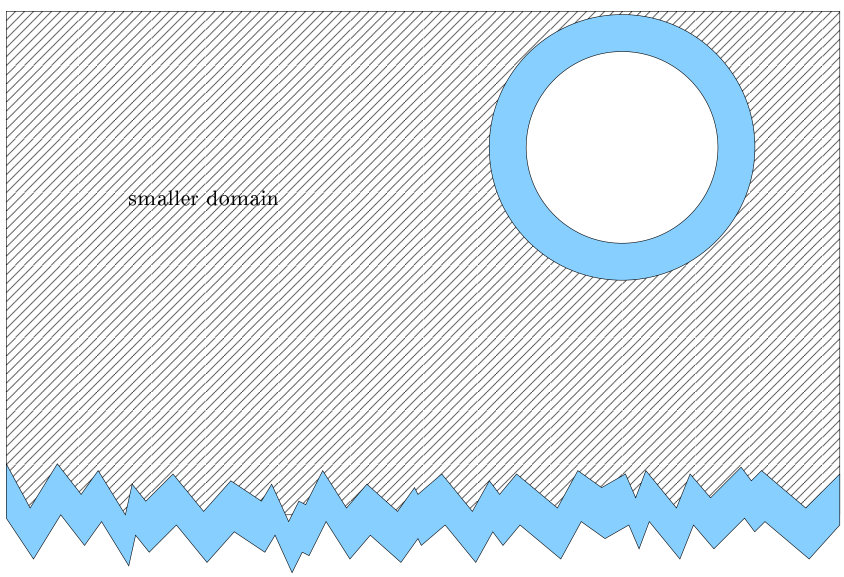

FIGURE 17: Remaining domain consisting in the union of the belt and the annulus.

To conclude, it is enough to verify the estimate in the region left. To see this, we use an idea that is very specific to the Lipschitz case.

2 Translation of Lipschitz domains

2.1 Moving upward the domain

The idea here is very simple : we translate upwards Ω at a distance λ. We then consider the discrete Green function G_μ in the domain Ω_λ and extend it to be zero in the belt.

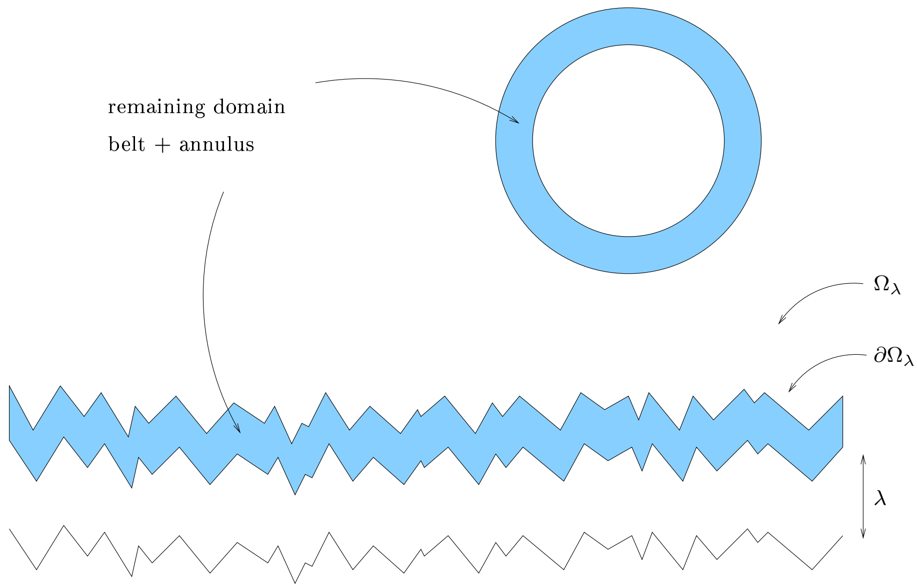

Figure 18: Remaining domain consisting in the union of the belt and the annulus.

Therefore, we have

(i) $H(x) \geq CG_\mu^{\Omega_\lambda}(x, x_0)$, for x in the belt ;

(ii) in the annulus, we use upper and lower estimates for G and G_μ and we have again $H(x) \geq CG_\mu^{\Omega_\lambda}(x, x_0)$, for x in the annulus.

Here we also use the fact that since $\delta(x_0) > C$, $G(x, x_0) \approx |x - x_0|^{2-d}$ and also $G_\mu(x, x_0) \approx |x - x_0|^{2-d}$.

2.2 End of the proof

The Δ_μ−harmonicity of G_μ is now used to extend the inequality

$$H(x) \geq cG_\mu^{\Omega_\lambda}(x, x_0)$$

in the shaded region. One final twist is used. Namely we use the fact that $G^\lambda(x, y) \approx G(x, y)$ if x and y are far out, that is $\delta(x), \delta(y) \geq C_\lambda$.

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