Optimized Schwarz methods in spherical geometry with an overset grid system

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Summary. In recent years, much attention has been given to domain decomposition methods for solving linear elliptic problems that are based on a partitioning of the domain of the physical problem. More recently, a new class of Schwarz methods known as optimized Schwarz methods was introduced to improve the performance of the classical Schwarz methods. In this paper, we investigate the performance of this new class of methods for solving the model equation $(\eta - \Delta)u = f$, where $\eta > 0$, in spherical geometry. This equation arises in a global weather model as a consequence of an implicit (or semi-implicit) time discretization. We show that the Schwarz methods improved by a non-local transmission condition converge in a finite number of steps. A local approximation permits the use of the new optimized methods on a new overset grid system on the sphere called the Yin-Yang grid.

1 Introduction

Meteorological operational centers are using increasingly parallel computer systems and need efficient strategies for their real-time data assimilation and forecast systems. This motivates the present study, where parallelism based on domain decomposition methods is analyzed for a new overset grid system on the sphere introduced by Kageyama and Sato [2004] called the Yin-Yang grid.

We investigate domain decomposition methods for solving $(\eta - \Delta)u = f$, where $\eta > 0$, in spherical geometry. The key idea underlying the optimal Schwarz method has been introduced in Hagstrom et al. [1988] in the context of non-linear problems. A new class of Schwarz methods based on this idea was then introduced in Charton et al. [1991] and further analyzed in Nataf and Rogier [1995] and Japhet [1998] for convection diffusion problems. For the case of the Poisson equation, see Gander et al. [2001], where also the terms optimal and optimized Schwarz were introduced. Optimal Schwarz methods have non-local transmission conditions at the interfaces between subdomains, and are therefore not as easy to use as classical Schwarz methods. Optimized Schwarz methods use local approximation of the optimal, non-local transmission conditions of optimal Schwarz at the interfaces and are therefore as easy to use as classical Schwarz, but have a greatly enhanced performance.

In Section 2, we introduce the model problem on the sphere and the tools of Fourier analysis, we also recall briefly some proprieties of the associated Legendre functions, which we will need in our analysis. In Section 3, we present the Schwarz algorithm for the model problem on the sphere with a possible overlap. We show that asymptotic convergence is very poor in particular for low wave-number modes. In Section 4, we present the optimal Schwarz algorithm for the same configuration. We prove convergence in two iterations for the two subdomain decomposition with non-local convolution transmission conditions. We then introduce a local approximation which permits the use of the new method on a new overset grid system on the sphere called the Yin-Yang grid which is pole-free. In Section 5 we illustrate our findings with numerical experiments.

2 The problem setting on the sphere

Throughout this paper we consider a model problem governed by the following equation

$$\mathcal{L}(u) = (\eta - \Delta)(u) = f, \qquad \text{in} \quad S \subset \mathbb{R}^3, \tag{1}$$

where S is the unit sphere centered at the origin. Using spherical coordinates, equation (1) can be rewritten in the form

$$\mathcal{L}(u) = \left(\eta - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} - \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial}{\partial\phi})\right)(u) = f, \tag{2}$$

where ϕ stands for the colatitude, with 0 being the north pole and π being the south pole, and θ is the longitude. For our case on the surface of the unit sphere, we consider solutions independent of r, e.g., $r = 1$, which simplifies (2) to

$$\mathcal{L}(u) = \left(\eta - \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2} - \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial}{\partial\phi})\right)(u) = f. \tag{3}$$

Our results are based on Fourier analysis. Because u is periodic in θ, it can be expanded in a Fourier series,

$$u(\phi,\theta) = \sum_{m=-\infty}^{\infty} \hat{u}(\phi,m)e^{im\theta}, \quad \hat{u}(\phi,m) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}u(\phi,\theta)d\theta.$$

With the expanded u, equation (3) becomes a family of ordinary differential equations. For any positive or negative integer m, we have

$$-\frac{\partial^2 \hat{u}(\phi,m)}{\partial \phi^2} - \frac{\cos\phi}{\sin\phi}\frac{\partial \hat{u}(\phi,m)}{\partial \phi} + (\eta + \frac{m^2}{\sin^2\phi})\hat{u}(\phi,m) = \hat{f}(\phi,m). \tag{4}$$

By linearity, it suffices to consider only the homogeneous problem, $\hat{f}(\phi,m) = 0$, and analyze convergence to the zero solution. Thus, for m fixed, the homogeneous problem in (4), can be written in the following form

$$\frac{\partial^2 \hat{u}(\phi,m)}{\partial \phi^2} + \frac{\cos\phi}{\sin\phi}\frac{\partial \hat{u}(\phi,m)}{\partial \phi} + (\nu(\nu+1) - \frac{m^2}{\sin^2\phi})\hat{u}(\phi,m) = 0, \tag{5}$$

where $\nu = -1/2 \pm 1/2\sqrt{1-4\eta}$. Note that the solution of equation (5) is independent of the sign of m, and thus, for simplicity, we assume in the sequel that m is a positive integer. Equation (5) is the associated Legendre equation and admits two linearly independent solutions with real values, namely $P_\nu^m(\cos\phi)$ and $P_\nu^m(-\cos\phi)$, see e.g., Gradshteyn and Ryzhik [1981], where $P_\nu^m(\cos\phi)$ is called the conical function of the first kind.

Remark 1. The associated Legendre function can be expressed in terms of the hypergeometric function and one can show that the function $P_\nu^m(\cos\phi)$ has a singularity at $\phi = \pi$ and is monotonically increasing in the interval $[0,\pi]$. Furthermore, the derivative of the function $P_\nu^m(z)$ with respect to the variable z is given by

$$\frac{\partial P_\nu^m(z)}{\partial z} = \frac{1}{1-z^2}\left(-mzP_\nu^m(z) - \sqrt{1-z^2}P_\nu^{m+1}(z)\right). \tag{6}$$

3 The classical Schwarz algorithm on the sphere

We decompose the sphere into two overlapping domains as shown in Fig. 1 on the left. The Schwarz method for two subdomains and model problem (1) is then given by

$$\begin{aligned} \mathcal{L}u_1^n &= f, \text{ in } \Omega_1, \quad u_1^n(b,\theta) = u_2^{n-1}(b,\theta), \\ \mathcal{L}u_2^n &= f, \text{ in } \Omega_2, \quad u_2^n(a,\theta) = u_1^{n-1}(a,\theta), \end{aligned} \tag{7}$$

and we require the iterates to be bounded at the poles of the sphere. By linearity it suffices to consider only the case $f = 0$ and analyze convergence to the zero solution.
Taking a Fourier series expansion of the Schwarz algorithm (7), and using the condition on the iterates at the poles, we can express both solutions using the transmission conditions as follows

$$\hat{u}_1^n(\phi,m) = \hat{u}_2^{n-1}(b,m)\frac{P_\nu^m(\cos\phi)}{P_\nu^m(\cos b)}, \quad \hat{u}_2^n(\phi,m) = \hat{u}_1^{n-1}(a,m)\frac{P_\nu^m(-\cos\phi)}{P_\nu^m(-\cos a)}. \tag{8}$$

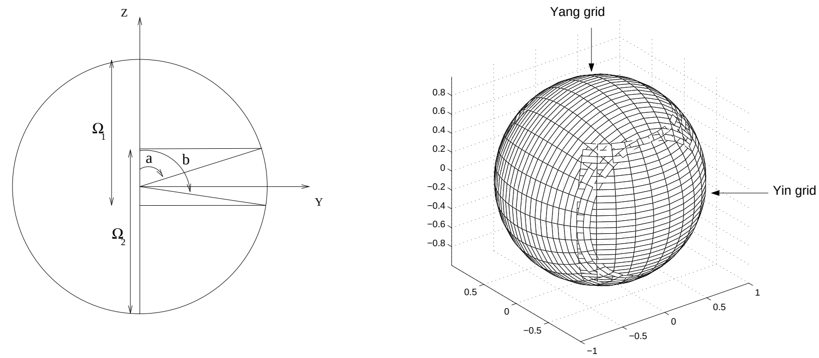

Fig. 1. Left: Two overlapping subdomains. **Right:** The Yin-Yang grid system.

Evaluating the second equation at $\phi = b$ for iteration index $n-1$ and inserting it into the first equation, evaluating this latter at $\phi = a$, we get over a double step the relation

$$\hat{u}_1^n(a,m) = \frac{P_\nu^m(-\cos b) P_\nu^m(\cos a)}{P_\nu^m(-\cos a) P_\nu^m(\cos b)} \hat{u}_1^{n-2}(a,m). \tag{9}$$

Therefore, for each m, the convergence factor $\rho(m,\eta,a,b)$ of the classical Schwarz algorithm is given by

$$\rho_{cla} = \rho_{cla}(m,\eta,a,b) := \frac{P_\nu^m(-\cos b) P_\nu^m(\cos a)}{P_\nu^m(-\cos a) P_\nu^m(\cos b)}. \tag{10}$$

A similar result also holds for the second subdomain and we find by induction

$$\hat{u}_1^{2n}(a,m) = \rho_{cla}^n \hat{u}_1^0(a,m), \qquad \hat{u}_2^{2n}(b,m) = \rho_{cla}^n \hat{u}_2^0(b,m). \tag{11}$$

Because of Remark 1, the fractions are less than one and this process is a contraction and hence convergent. We have proved the following

Proposition 1. *For each m, the Schwarz iteration on the sphere partitioned along two colatitudes $a < b$ converges linearly with the convergence factor*

$$\rho_{cla} = \rho_{cla}(m,\eta,a,b) := \frac{P_\nu^m(-\cos b) P_\nu^m(\cos a)}{P_\nu^m(-\cos a) P_\nu^m(\cos b)} \leq 1.$$

The convergence factor depends on the problem parameters η, the size of the overlap $L = b - a$ and on the frequency parameter m. Fig. 2 on the left, shows the dependence of the convergence factor on the frequency m for an overlap $L = b - a = \frac{1}{100}$ and $\eta = 2$. This shows that for small values of m the rate of convergence is very poor, but the Schwarz algorithm can damp high frequencies very effectively.

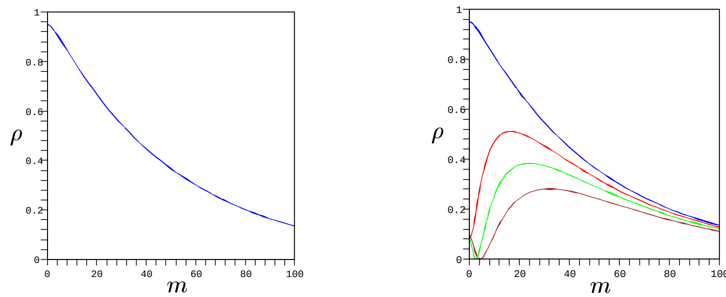

Fig. 2. Left: Behavior of the convergence factor ρ_{cla}. **Right**: Comparison between ρ_{cla} (top curve), ρ_{T0} (2^{nd} curve), ρ_{T2} (3^{th} curve) and ρ_{O0} (bottom curve). In both plots $a = \pi - L/2$ and the overlap is $L = b - a = \frac{1}{100}$ and $\eta = 2$.

4 The optimal Schwarz algorithm

Following the approach in Gander et al. [2001], we now introduce a modified algorithm by imposing new transmission conditions,

$$\begin{array}{ll} \mathcal{L}(u_1^n) = f, \text{ in } \Omega_1, & (S_1 + \partial_\phi)(u_1^n)(b,\theta) = (S_1 + \partial_\phi)(u_2^{n-1})(b,\theta), \\ \mathcal{L}(u_2^n) = f, \text{ in } \Omega_2, & (S_2 + \partial_\phi)(u_2^n)(a,\theta) = (S_2 + \partial_\phi)(u_1^{n-1})(a,\theta), \end{array} \tag{12}$$

where S_j, $j = 1, 2$, are operators along the interface in the θ direction. As for the classical Schwarz method, it suffices by linearity to consider the homogeneous problem only, $f = 0$, and to analyze convergence to the zero solution. Taking a Fourier series expansion of the new algorithm (12) in the θ direction, we obtain

$$\begin{array}{l} (\sigma_1(m) + \partial_\phi)(\hat{u}_1^n)(b,m) = (\sigma_1(m) + \partial_\phi)(\hat{u}_2^{n-1})(b,m), \\ (\sigma_2(m) + \partial_\phi)(\hat{u}_2^n)(a,m) = (\sigma_2(m) + \partial_\phi)(\hat{u}_1^{n-1})(a,m), \end{array} \tag{13}$$

where σ_j, $j = 1, 2$, denotes the symbol of the operators S_j, $j = 1, 2$, respectively. To simplify the notation, we introduce the function

$$q_{\nu,m}(x) = \frac{P_\nu^{m+1}(\cos x)}{P_\nu^m(\cos x)}.$$

As in the case of the classical Schwarz method, we have to choose $P_\nu^m(\cos\phi)$ as solution in the first subdomain and $P_\nu^m(-\cos\phi)$ as solution in the second subdomain. Using the transmission conditions and the definition of the derivative of the Legendre function in (6), we find the subdomain solutions in Fourier space to be

$$\begin{aligned} \hat{u}_1^n(\phi,m) &= \frac{\sigma_1(m) + m\cot b - q_{\nu,m}(\pi - b)}{\sigma_1(m) + m\cot b + q_{\nu,m}(b)} \frac{P_\nu^m(\cos\phi)}{P_\nu^m(\cos b)} \hat{u}_2^{n-1}(b,m), \\ \hat{u}_2^n(\phi,m) &= \frac{\sigma_2(m) + m\cot a + q_{\nu,m}(a)}{\sigma_2(m) + m\cot a - q_{\nu,m}(\pi - a)} \frac{P_\nu^m(-\cos\phi)}{P_\nu^m(-\cos a)} \hat{u}_1^{n-1}(a,m). \end{aligned} \tag{14}$$

Evaluating the second equation at $\phi = b$ for iteration index $n-1$ and inserting it into the first equation, we get after evaluation at $\phi = a$,

$$\hat{u}_1^n(a,m) = \rho_{opt}(m,a,b,\eta,\sigma_1,\sigma_2)\hat{u}_1^{n-2}(a,m), \tag{15}$$

where the new convergence factor ρ_{opt} is given by

$$\rho_{opt} := \frac{\sigma_1(m) + m\cot b - q_{\nu,m}(\pi - b)}{\sigma_1(m) + m\cot b + q_{\nu,m}(b)} \frac{\sigma_2(m) + m\cot a + q_{\nu,m}(a)}{\sigma_2(m) + m\cot a - q_{\nu,m}(\pi - a)}\rho_{cla}. \tag{16}$$

As in the classical case, we can prove the following

Proposition 2. *The optimal Schwarz algorithm (12) on the sphere partitioned along two colatitudes $a < b$ converges in two iterations provided that σ_1 and σ_2 satisfy*

$$\sigma_1(m) = -m\cot b + q_{\nu,m}(\pi - b) \quad \textit{and} \quad \sigma_2(m) = -m\cot a - q_{\nu,m}(a). \tag{17}$$

This is an optimal result, since convergence in less than two iterations is impossible, due to the need to exchange information between the subdomains. In practice, one needs to inverse transform the transmission conditions involving $\sigma_1(m)$ and $\sigma_2(m)$ from Fourier space into physical space to obtain the transmission operators S_1 and S_2, and hence we need

$$S_1(u_1^n) = \mathcal{F}_m^{-1}(\sigma_1(\hat{u}_1^n)), \qquad S_2(u_2^n) = \mathcal{F}_m^{-1}(\sigma_2(\hat{u}_2^n)).$$

Due to the fact that the σ_j contain associated Legendre functions, the operators S_j are non-local. To have local operators, we need to approximate the symbols σ_j with polynomials in im. Inspired by the results for elliptic problems in two-dimensional Cartesian space, we introduce the following ansatz

$$q_{\nu,m}(\phi) \approx \frac{\sin(\phi)\sqrt{\eta + m^2}}{1 + \cos(\phi)}. \tag{18}$$

Based on this ansatz we can expand the symbols $\sigma_j(m)$ in (17) in a Taylor series,

$$\begin{aligned} \sigma_1(m) &= \tfrac{\sin(b)\sqrt{\eta}}{-\cos(b)+1} + \tfrac{\sin(b)m^2}{2(-\cos(b)+1)\sqrt{\eta}} + \mathcal{O}(m^4), \\ \sigma_2(m) &= -\tfrac{\sin(a)\sqrt{\eta}}{\cos(a)+1} - \tfrac{\sin(a)m^2}{2(\cos(a)+1)\sqrt{\eta}} + \mathcal{O}(m^4). \end{aligned}$$

A zeroth order Taylor approximation $T0$ is obtained by using only the first terms in the Taylor expansion of σ_j, while a second order approximation $T2$ is obtained by using both terms from the expansion. In Fig. 2 on the right, we compare the convergence factor ρ_{cla} of the classical Schwarz method with the convergence factor ρ_{T0} of the zeroth order Taylor method and the convergence factor ρ_{T2} of the second order Taylor method. Numerically, we find the optimized Robin conditions, namely $\sigma_1 \approx 1.4$ and $\sigma_2 \approx -1.4$, and we compare the corresponding convergence factor ρ_{O0} to the other methods.

5 Numerical experiments

We perform two sets of numerical experiments, both with $\eta = 1$. In the first set we consider our model problem on the sphere using a longitudinal co-latitudinal grid, where we adopt a decomposition with two overlapping subdomains as shown in Fig. 1 on the left. In this case, we combine a spectral method in the θ-direction with a finite difference method in the ϕ-direction. We use a discretization with 6000 points in ϕ, including the poles, and spectral modes from -10 to 10. The decomposition is done in the middle and the overlap is chosen to be $[0.49\pi, 0.51\pi]$, see Fig. 3 on the left, where the curves with (circle) and without (square) overlap of optimal Schwarz are on top of each other. In the second experiment, we solve the model problem on the Yin-Yang grid. This is a composite grid, which covers the surface of the sphere with two identical rectangles that partially overlap on their borders. Each grid is an equatorial sector having a different polar axis but uniform discretization, see Fig. 1 on the right. The Ying-Yang grid system is free from the problem of singularity at the poles, in contrast to the ordinary spherical coordinate system. In Fig. 3 on the right we show some screenshots of the

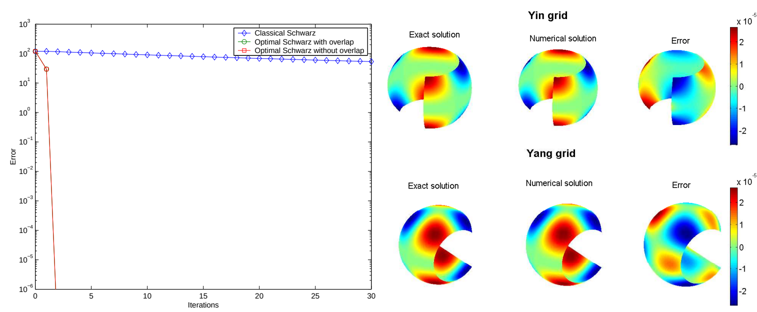

Fig. 3. Left: Convergence behavior for the methods analyzed for the two subdomain case. **Right:** Screenshots of solutions and the error for the Yin-Yang grid system. In both plots $\eta = 1$.

	Classical Schwarz		Taylor 0 method		Taylor 2 method		Optimized 0 method	
h	$L = 1/50$	$L = h$	$L = 1/50$	$L = h$	$L = 1/50$	$L = h$	$L = 1/50$	$L = h$
1/50	184	184	22	22	16	16	12	12
1/100	184	284	22	27	16	19	12	16
1/150	183	389	21	31	15	21	11	19
1/200	184	497	22	36	16	24	12	22

Table 1. Number of iterations of the classical Schwarz method compared to the optimized Schwarz methods for the Yin-Yang grid system with $\eta = 1$.

exact and numerical solutions for the Yin-Yang grid using optimized Robin conditions with $\sigma_1 = -1.4$ and $\sigma_2 = 1.4$. In Table 1 we compare the classical Schwarz method to the optimized methods in the Yin-Yang grid system.

Conclusion

In this work, we show that numerical algorithms already validated for a global latitude/longitude grid can be implemented, with minor changes, for the Yin-Yang grid system. In the future we will implement optimized second order interface conditions in order to improve the convergence of the elliptic solver.

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