2d Field Theory and Random Planar Sets: past and future

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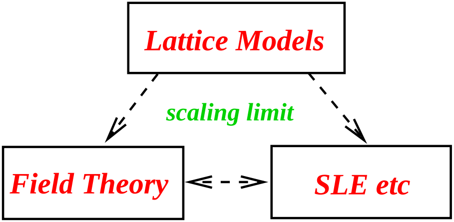
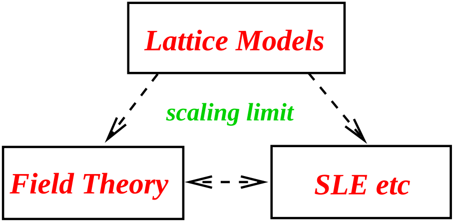
Conformal Maps from Probability to Physics
Ascona, May 2010


Lattice Models
scaling limit
Field Theory
SLE etc



- 2d field theory is a rich source of conjectures for SLE-type results

2d field theory, *c.* 1991

- [1960s] Scaling limits of lattice models: limit as lattice spacing $a \to 0$ at fixed correlation length ξ should exist

$$\lim_{a \to 0} a^{-x_1 \ldots - x_n} \, \mathbb{E}[\phi_1^{\mathrm{lat}}(z_1) \cdots \phi_n^{\mathrm{lat}}(z_n)] = \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle$$

  and be given by correlators satisfying axioms of a euclidean QFT.
- when $\xi^{-1} = 0$ (critical point) this implies scale covariance:

$$\langle \phi_1(bz_1) \cdots \phi_n(bz_n) \rangle_{b\mathcal{D}} = b^{-x_1 \ldots - x_n} \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_{\mathcal{D}}$$

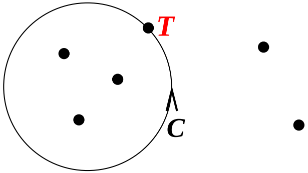
- [Polyakov 1970]: this should extend to covariance under *conformal mappings $z \to f(z)$:*

$$\langle \phi_1(f(z_1)) \cdots \phi_n(f(z_n)) \rangle_{f(\mathcal{D})} = \prod_{j=1}^{n} |f'(z_j)|^{-x_j} \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_{\mathcal{D}}$$

Conformal Field Theory (CFT)

- [Belavin, Polyakov, Zamolodchikov 1984]: important role played by fields whose correlators are holomorphic in z, in particular the *stress tensor* $T(z)$ which implements infinitesimal conformal mappings $z \to z + \alpha(z)$ via *conformal Ward identity*:

$$\sum_{z_j \text{ inside } C} \langle \delta\phi_j(z_j) \cdots \rangle = \frac{1}{2\pi i} \oint_C \alpha(z) \langle T(z) \phi_j(z_j) \cdots \rangle dz + \text{c.c.}$$



Virasoro and all that

$$T(z) \cdot \phi_j(z_j) = \sum_{n \leq n_{\max}} (z - z_j)^{-2-n} L_n \phi_j(z_j)$$

$$[L_n, L_m] = (n-m)L_{n+m} + (c/12)n(n^2-1)\delta_{n,-m} \qquad \text{(Vir)}$$

- there are two independent copies $(\text{Vir}, \overline{\text{Vir}})$ corresponding to $T(z)$ and $\overline{T}(\bar{z})$
- to each *primary* field ϕ_j such that $L_n\phi_j = 0$ for all $n \geq 1$ corresponds a set of descendants:

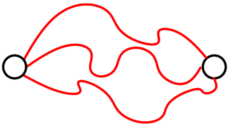
$$\begin{array}{ll} \phi_j & \\ L_{-1}\phi_j & (= \partial_z \phi_j) \\ L_{-2}\phi_j,\ L_{-1}^2\phi_j & \\ \vdots & \end{array}$$

- sometimes these are *degenerate* , e.g. at level 2

$$L_{-2}\phi_j = (\kappa/4)L_{-1}^2\phi_j = (\kappa/4)\partial_z^2\phi_j$$

- by choosing $\alpha(z) \propto (z - z_j)^{-1}$ we can use the conformal Ward identity to show that in these cases the correlators of ϕ_j satisfy (2nd order) linear PDEs wrt z_j
- [JC 1984] all these ideas extend to *boundary* fields with $z_j \in \partial\mathcal{D}$, with the identification $\mathrm{Vir} = \overline{\mathrm{Vir}}$

- Coulomb gas methods [Nienhuis, den Nijs, early 1980s]: many properties of 2d critical systems (e.g. scaling dimensions x_j) follow from conjectured relationship to modified gaussian free field (GFF) compactified on circle radius $\propto \kappa^{-1/2}$
- [Duplantier, 1980s] local scaling fields ϕ can also describe sources for N mutually avoiding Brownian curves and also in conjectured scaling limit of $\mathrm{O}(n)$ model and hulls of FK clusters in Q-state Potts model



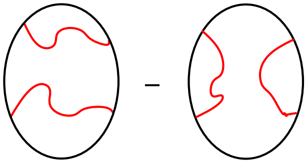
$$Z \propto (\epsilon/r)^{2x_N}$$

- scaling dimensions conjectured from CFT and Coulomb gas methods, e.g. in O(n) model

$$x_N^{\text{bulk}} = \frac{N^2}{2\kappa} - \frac{(\kappa-4)^2}{8\kappa}, \qquad x_N^{\text{boundary}} = \frac{N(N+2)}{\kappa} - \frac{N}{2}$$

where $n = -2\cos(4\pi/\kappa)$.

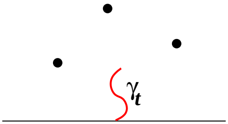
- in particular ϕ_N^{boundary} is degenerate at level $N+1$.
- [JC 1991] boundary fields for percolation hulls are degenerate (at level 2) and so their 4-point correlators satisfy 2nd order PDE $\Rightarrow$ percolation crossing formula



Then SLE came along. . .

- [Schramm 2000]: if percolation hull exploration process converges to SLE_6, crossing formula follows
- [Smirnov 2001]: crossing formula holds for scaling limit of triangular lattice percolation $\Rightarrow$ exploration process converges to SLE_6
- and much more. . .

SLE and ϕ_1^{boundary} fields in CFT



- [Bauer-Bernard, Friedrich-Werner 2002]: CFT correlators have martingale property

$$\begin{aligned}\langle \mathcal{O}\,\phi_1(0)\rangle_{\mathbb{H}} &= \mathbb{E}\left[\langle \mathcal{O}\,\phi_1(\text{tip}_t)\rangle_{\mathbb{H}\setminus\gamma_t}\right] \\ &= \mathbb{E}\left[\langle g_t(\mathcal{O})\, g_t(\phi_1)(0)\rangle_{\mathbb{H}}\right]\end{aligned}$$

- infinitesimal Loewner map

$$\alpha(z) = 2dt/z - \sqrt{\kappa}dB_t \quad \Rightarrow \quad 2dt\,L_{-2} - \sqrt{\kappa}dB_t\,L_{-1}$$

$$g_t(\phi_1)(0) = e^{-\int_0^t (2L_{-2}dt' - \sqrt{\kappa}L_{-1}dB_{t'})}\phi_1(0)$$

$$\mathbb{E}\left[g_t(\phi_1)(0)\right] = e^{-\int_0^t (2L_{-2} - (\kappa/2)L_{-1}^2)dt'}\phi_1(0)$$

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- [Bauer-Bernard-Kytola]: conditioned CFT partition functions $\Rightarrow$ variants like multiple SLEs and $\text{SLE}(\kappa, \rho)$
- if we know CFT partition functions in other domains $\mathcal{D}$ we can deduce corresponding Loewner driving process - however in general these are not known!

Can we get the whole of CFT from SLE (or CLE)?

- [Friedrich-Werner 2002, Doyon-Riva-JC 2005]: identification of stress tensor T in SLE setting
- when conformal restriction on curves γ holds

$$T(z) \propto \lim_{\epsilon \to 0} \epsilon^{-2} \int d\theta e^{-2i\theta} \mathbf{1}_{\gamma \text{ separates } (z \pm \epsilon e^{i\theta})}$$

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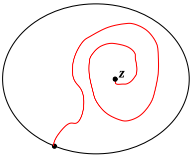
- more generally for $c \neq 0$, T can be defined by the notion of conformal derivative [Doyon 2010]

Holomorphic fields

- [Smirnov, Riva-JC, Rajabpour-JC, Ikhlef-JC]: in many lattice models, local observables of curves γ can be identified which are *discretely holomorphic*, e.g.

$$\psi_\sigma(z) \propto \int d\theta e^{-i\sigma\theta}\, \mathbf{1}_{\gamma \text{ ends at } z \text{ with winding angle } \theta}$$



- in the cases where convergence of $\langle\psi_\sigma(z)\rangle$ to a continuous holomorphic function can be proved with suitable boundary conditions this implies convergence of γ to SLE_κ with $\sigma = (6-\kappa)/2\kappa$ (e.g. Ising [Chelkak-Smirnov])

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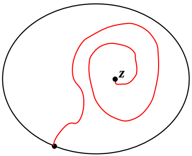

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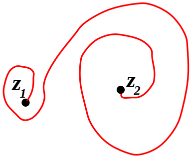
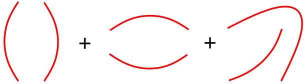
- the existence of discretely holomorphic observables appears to be linked to *integrability* of lattice models

Other correlators of holomorphic fields



- 2-point function in $\mathbb{R}^2$

$$\langle \psi_\sigma(z_1)\psi_\sigma(z_2)\rangle \sim (z_1 - z_2)^{-2\sigma}$$



- 4-point function: Ising case

$$\langle \psi_{\frac{1}{2}}(z_1)\psi_{\frac{1}{2}}(z_2)\psi_{\frac{1}{2}}(z_3)\psi_{\frac{1}{2}}(z_4)\rangle_{\mathbb{R}^2} \propto \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right)$$

- for general κ, conjectured scaling limit of these Smirnov observables corresponds to holomorphic CFT fields which are degenerate at level 2 and so we know their higher-order correlators
- in general the solution space has dimension > 1 and they have non-trivial monodromy, e.g.

$$\langle \psi_\sigma(z_1) \cdots \psi_\sigma(z_4) \rangle = \left(\frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{41}} \right)^{2\sigma} (A_1 F_1(\eta) + A_2 F_2(\eta))$$

where $\eta = z_{12} z_{34} / z_{13} z_{24}$ and $F_j(\eta)$ are hypergeometric functions

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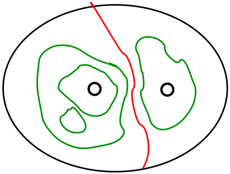
- these correlators can be considered as multi-particle wave functions of a quantum system in 2+1 dimensions
- non-Abelian fractional statistics, can be used in principle to make a quantum computer!

Other degenerate bulk fields

- [Gamsa-JC, Simmons-JC]: in the conjectured CFT description of the O(n) model 'twist' fields are also degenerate at level 2 and so their correlators satisfy 2nd order PDEs

$$\phi^{\text{twist}}(z,\bar{z}) \propto (-1)^{\text{number of curves separating } z \text{ and } z_0}$$



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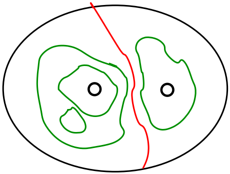
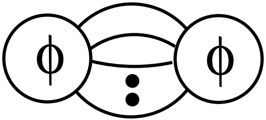


- gives 2-point information about $SLE_{8/3}$
- in all these examples of *bulk* level 2 degenerate fields, is there a stochastic calculus interpretation?

Off-critical scaling limits

- if p is a parameter of the lattice model coupling to a local quantity $\phi^{\text{lattice}}(z)$ with scaling dimension is x, in order to get a non-trivial off-critical scaling limit we need to keep the correlation length $\xi \propto a|p-p_c|^{-1/(2-x)}$ fixed as lattice spacing $a \to 0$, i.e. $|p-p_c| \propto a^{2-x} \to 0$
- in 2d QFT much progress has been made in the *integrable* case: out of the infinite number of conserved local fields made from the stress tensor and its descendants $(T(z), T(z)^2, (\partial_z T(z))^2, \ldots)$ in the CFT, a smaller infinity survives
- this allows the computation of *form factors* of local fields $\phi(r)$:



$$\langle\phi(r)\phi(0)\rangle = \sum_{N=1}^{\infty}\prod_{j=1}^{N}\int_{-\infty}^{\infty} d\theta_j |F_N(\{\theta_j\})|^2 e^{-|r/\xi|\sum_j \cosh\theta_j}$$

- these computations are actually carried out in Minkowski space where $ds^2 = dx^2 - dt^2$ and analytically continued back to $\mathbb{R}^2$
- however the 'particles' $j = 1, \ldots, N$ can probably be interpreted as 'dressed' non-intersecting curves
- sum over N is rapidly convergent and in practice only $N \leq 2$ need be kept

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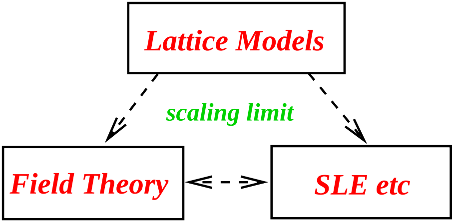
- example: mean size of finite clusters in percolation

$$\sum_r \langle \phi(r)\phi(0) \rangle \sim \Gamma^{\pm}|p - p_c|^{-\gamma} \quad \text{as } p \to p_c\pm$$

  where $\phi(r) =$ magnetization of Potts model as $Q \to 1$
- amplitudes $\Gamma^{\pm}$ are not universal but their ratio is
  - [Delfino-Viti-JC 2010]: $\Gamma^{-}/\Gamma^{+} \approx 160.2$
  - simulations [Jensen-Ziff] give 162.5 ± 2
- however these field theory results do not so far give much information about the *measure* on the random curves

Conclusions



Interactions between these 3 fields have been remarkably productive

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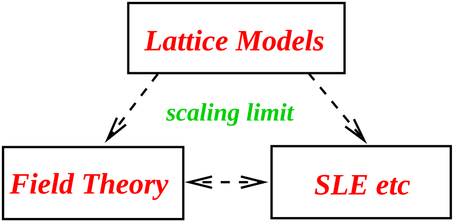

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May They Ever Flourish!