Quantum Hall transitions and conformal restriction

Ilya A. Gruzberg
University of Chicago

Collaborators: E. Bettelheim (Hebrew University)
A. W. W. Ludwig (UC Santa Barbara)

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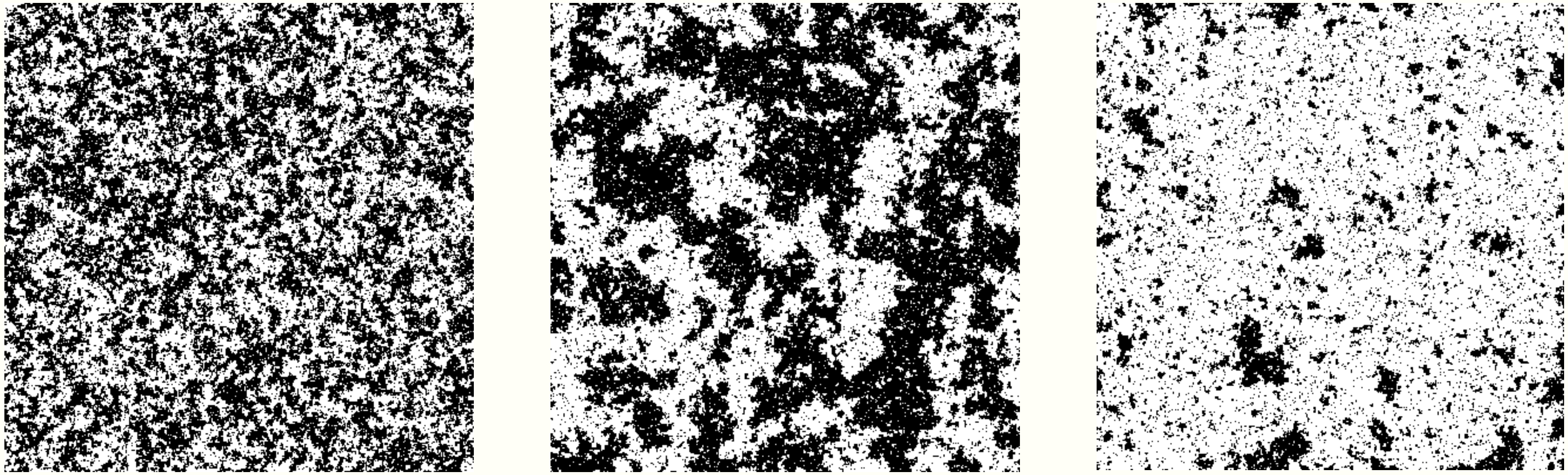
General comments

- SLE and statistical mechanics
  - Clean 2D systems at critical points
  - Example: Ising model

$$H = -\sum_{kl} J\sigma_k\sigma_l, \qquad W(\{\sigma\}) = e^{-\beta H}, \qquad Z = \sum_{\{\sigma\}} W(\{\sigma\})$$



$\beta < \beta_c$ $\qquad$ $\beta = \beta_c$ $\qquad$ $\beta > \beta_c$

General comments: disorder

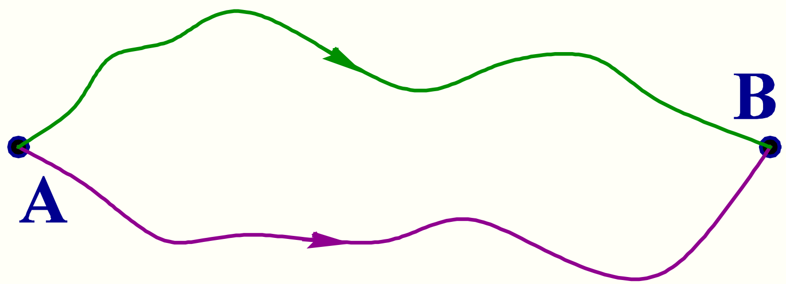
- Disordered systems with critical points
  - Example: random bond Ising model $H = -\sum_{kl} J_{kl}\sigma_k\sigma_l$
  - Random bonds J_{kl}
  - Correlation functions are random variables
  - Average correlators over distribution $P(J_{kl})$

$$\mathbb{E}[\langle O\rangle] = \mathbb{E}\left[\frac{\sum O e^{-\beta H}}{\sum e^{-\beta H}}\right] \neq \frac{\mathbb{E}\left[\sum O e^{-\beta H}\right]}{\mathbb{E}\left[\sum e^{-\beta H}\right]}$$

- Domain Markov property is lost: SLE does not seem to apply

General comments: quantum mechanics

- Qualitative semiclassical picture



- Individual path amplitudes are complex: probability theory does not apply
- Superposition: add probability amplitudes, then square

$$\mathrm{Prob}[\mathrm{A} \to \mathrm{B}] = \Big| \sum_{\mathrm{paths}\, i} A_i \Big|^2 = \sum_i |A_i|^2 + \sum_{i \neq j} A_i A_j^*.$$

- Interference term is purely quantum and can lead to Anderson localization

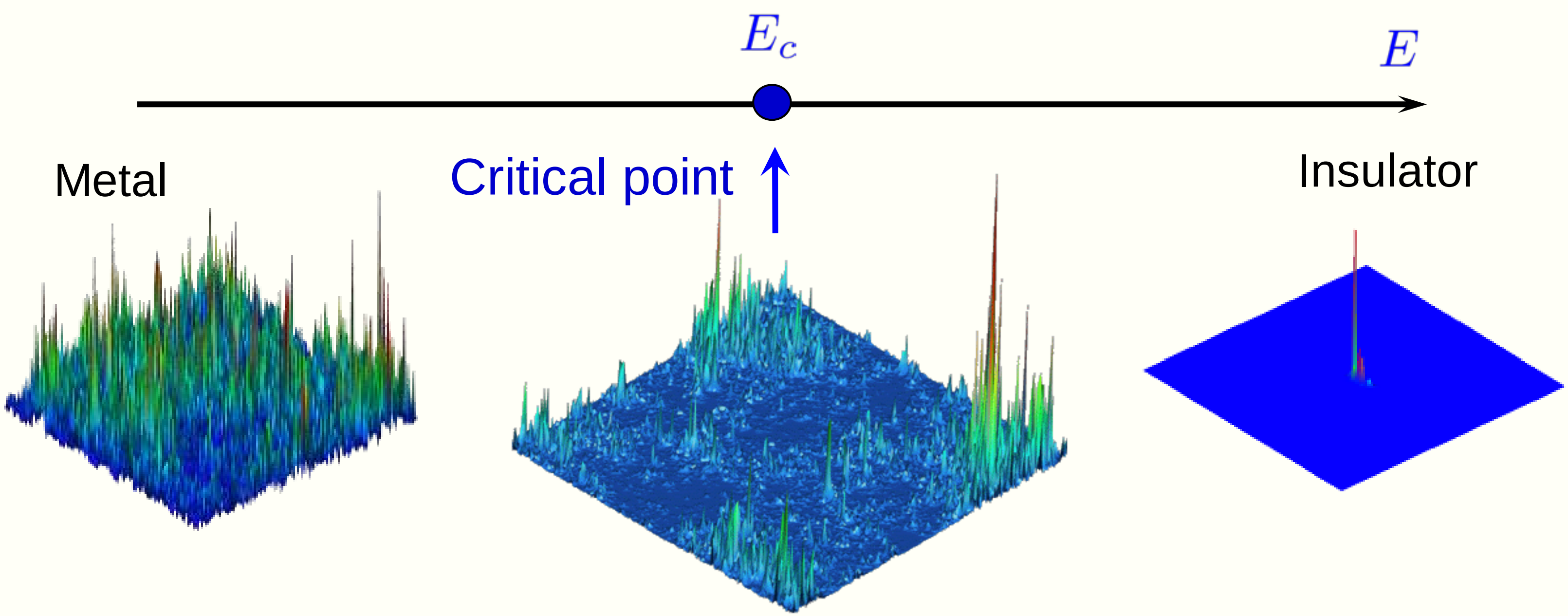
Disordered electronic systems and Anderson transitions

- Electron in a random potential $U(r)$ (and possibly magnetic field)
- Schrodinger equation for wave function $\psi(r)$ at energy E

$$\left[\left(i\nabla + \boldsymbol{A}\right)^2 + U(r)\right]\psi(r) = E\psi(r)$$

- Ensemble of disorder realizations: statistical treatment
- Metal-insulator transitions driven by disorder: Anderson transitions seen in electrical transport (current, voltage, resistance, etc.)
- Both disorder and quantum interference: seems hopeless
- Today consider only 2D systems: expect conformal invariance

Wave functions across Anderson transition

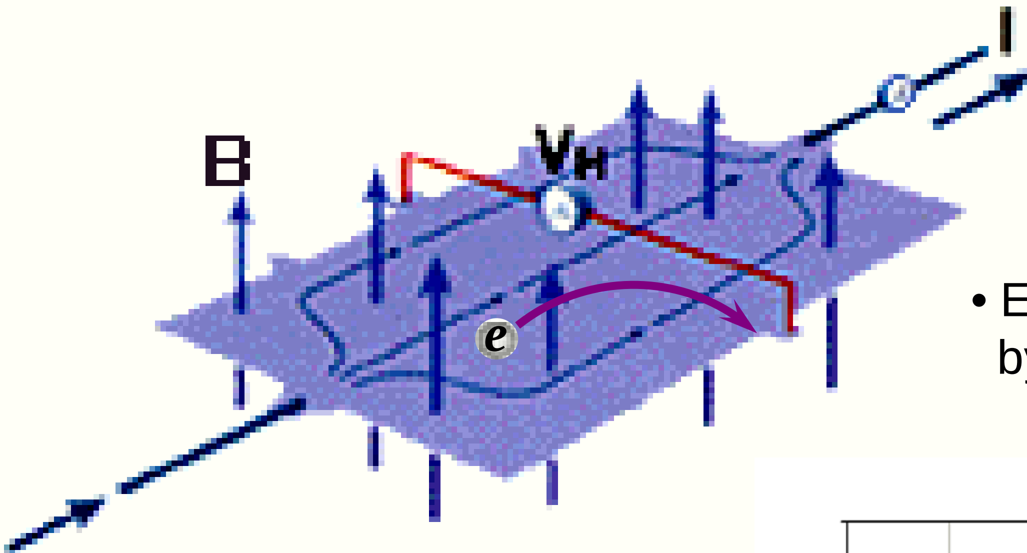


Extended $\psi(r)$ Critical multifractal $\psi(r)$ Localized $\psi(r)$

- Spatial extent of localized states: localization length $\xi(E) \propto |E - E_c|^{-\nu}$
- Critical $\psi(r)$ defines a random conformally-invariant multifractal measure

$$d\mu(r) = |\psi(r)|^2 d^2 r$$

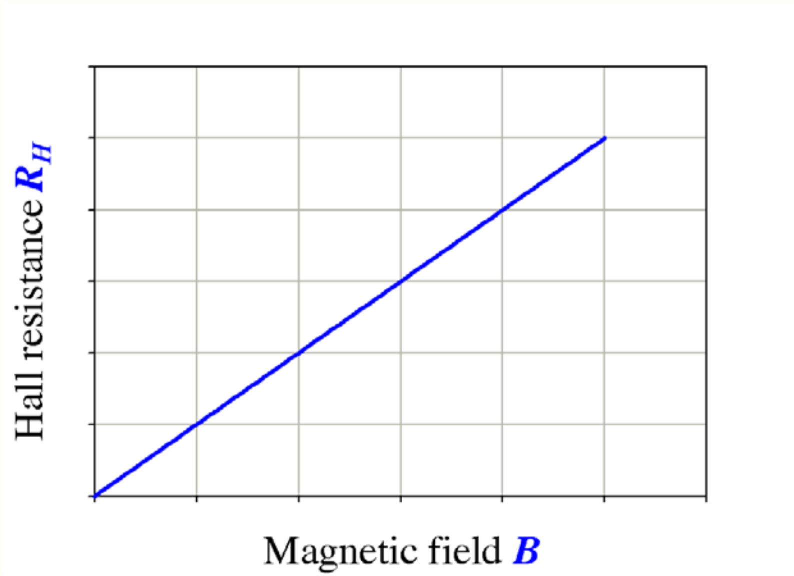
Classical Hall effect



- Electron trajectories bent by magnetic field

- Classical Hall resistance

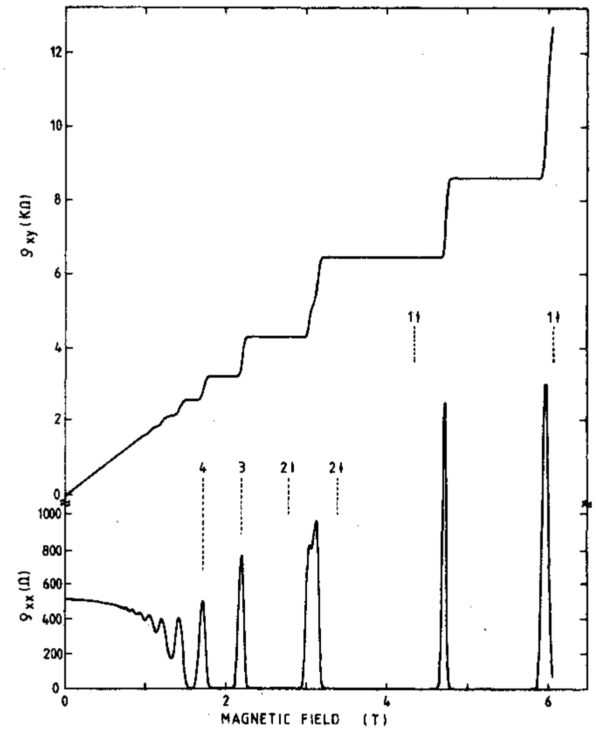
$$R_H = \frac{V_H}{I} \propto B$$



Integer quantum Hall effect

- Two-dimensional electron gas
- Strong magnetic field, low temperature
- Hall resistance shows plateaus

$$R_H = \frac{1}{n}\frac{h}{e^2}$$



K. v. Klitzing, Rev. Mod. Phys. 56 (1986)

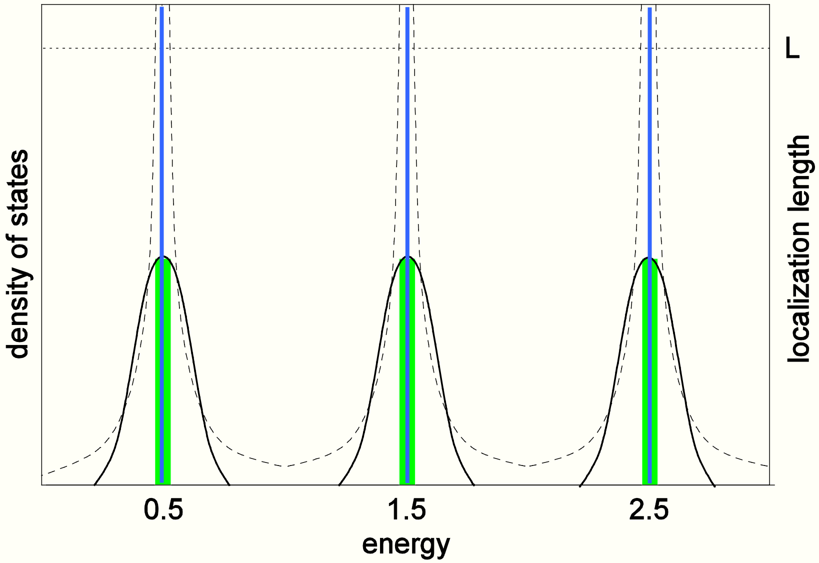
IQH and localization in strong magnetic field

- Single electron in a magnetic field and a random potential

$$\left[(i\nabla + \boldsymbol{A})^2 + U(r)\right]\psi(r) = E\psi(r)$$

- Without disorder: Landau levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \quad \omega_c = \frac{eB}{mc}$$



- Disorder broadens the levels and localizes most states

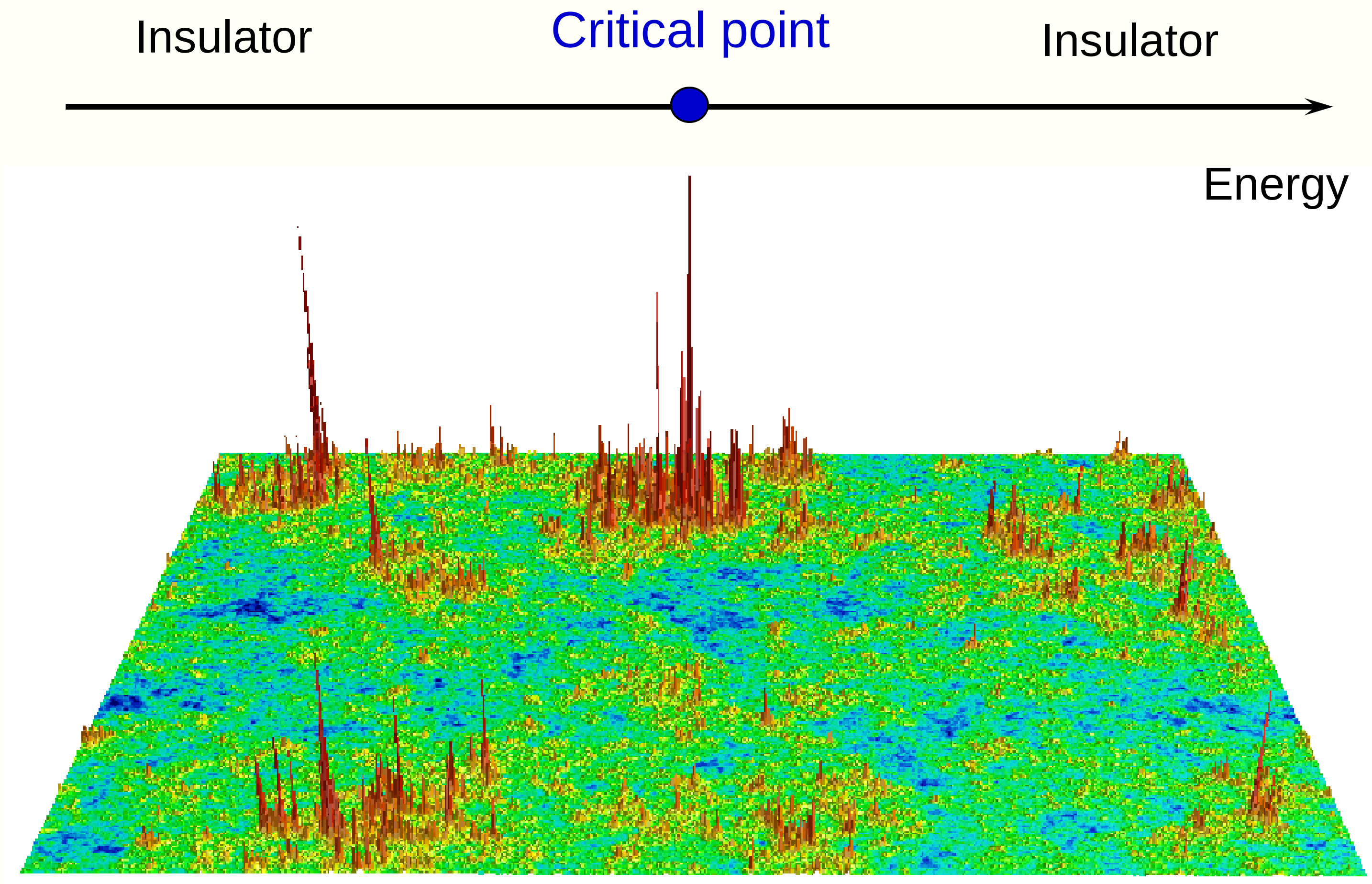
- Extended states at E_n

- Localization length diverges near E_n

$$\xi(E) \propto |E - E_c|^{-\nu}$$

- Transition between QH plateaus upon varying E_F or B

Wave function at quantum Hall transition



Theory of IQH plateau transition

- Goals for a theory of the transition:

  - Critical exponents
  - Scaling functions
  - Correlation functions at the transition

- No analytical description of the critical region so far

- Conformal invariance at the transition in 2D should help

- Plenty of numerical results (confirming conformal invariance)

- A variant of this problem ("spin quantum Hall effect") maps to *classical* bond percolation on square lattice

IAG, A. W. W. Ludwig, N. Read, 1999
E. J. Beamond, J. Cardy, J. T. Chalker, 2002

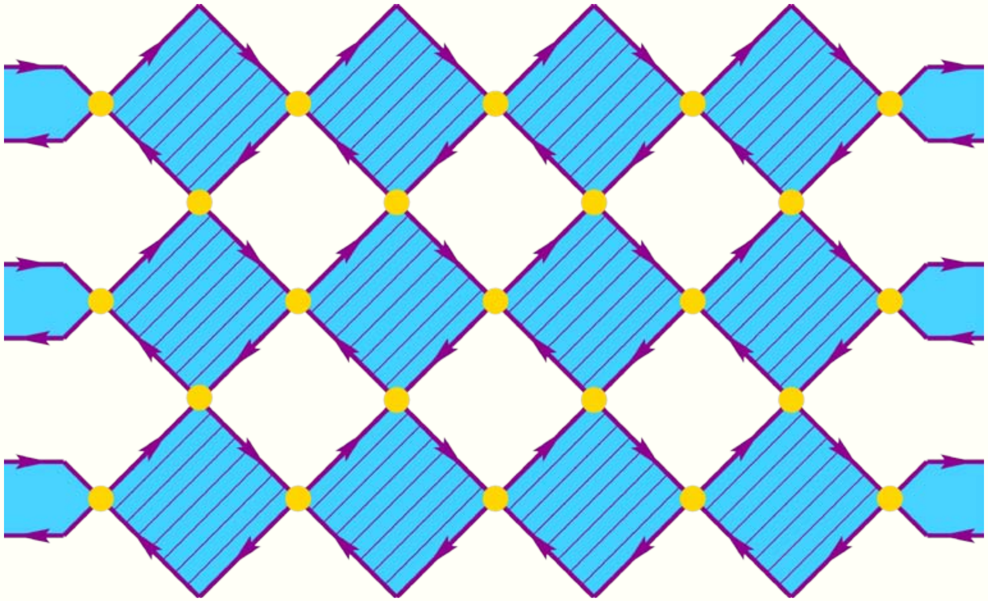
Our approach

- New approach using ideas of conformal stochastic geometry
    - Conformal restriction
    - Schramm-Loewner evolution (SLE)
- These ideas apply to non-random classical statistical mechanics problems but seem useless for disordered and/or quantum systems
- We work with the Chalker-Coddington network model
- Map average point contact conductances (PCC) to a *classical* problem
- Establish crucial restriction property
- Assume conformal invariance in the continuum limit and obtain PCC in a finite system with various boundary conditions

Chalker-Coddington network model

J. T. Chalker, P. D. Coddington, 1988

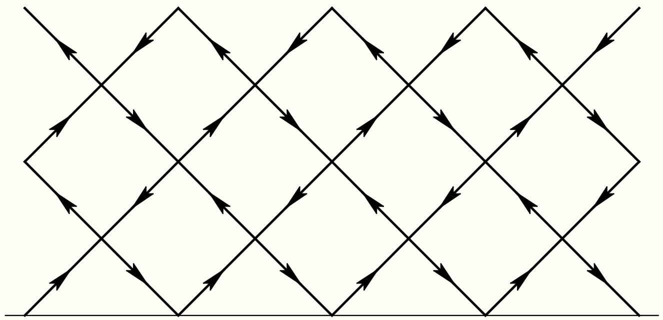
• Obtained from semi-classical drifting orbits in smooth potential



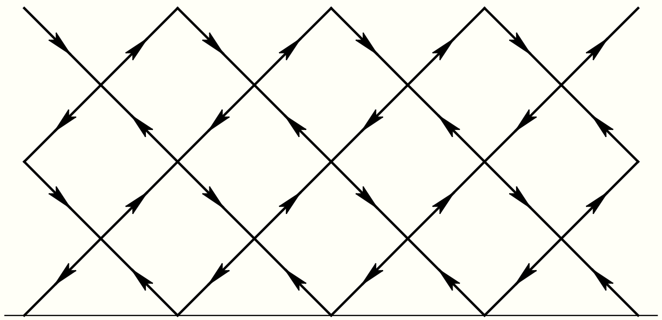
• Fluxes (currents) on links, scattering at nodes

• The model is designed to describe transport properties

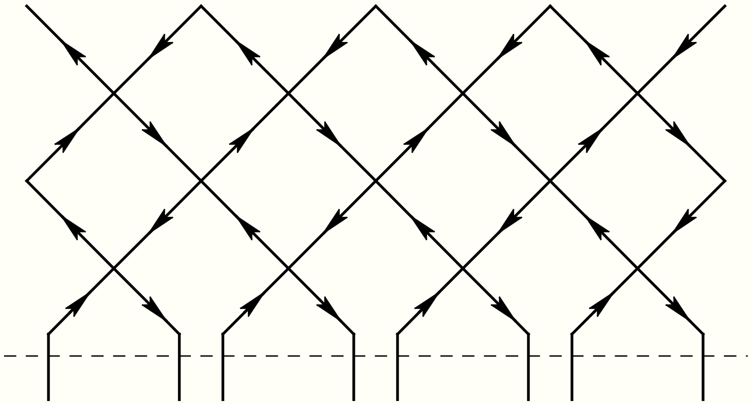
Boundary conditions



- Reflecting (right)

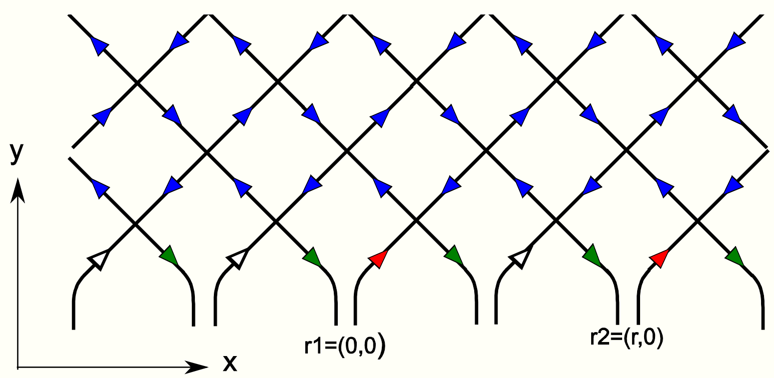


- Reflecting (left)

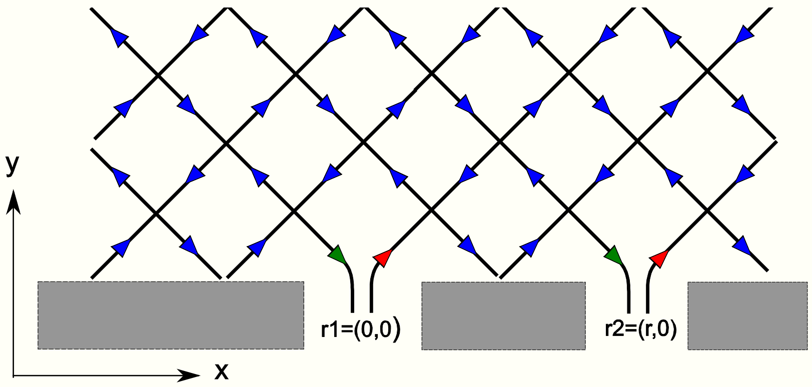


- Absorbing: boundary nodes are the same as in the bulk

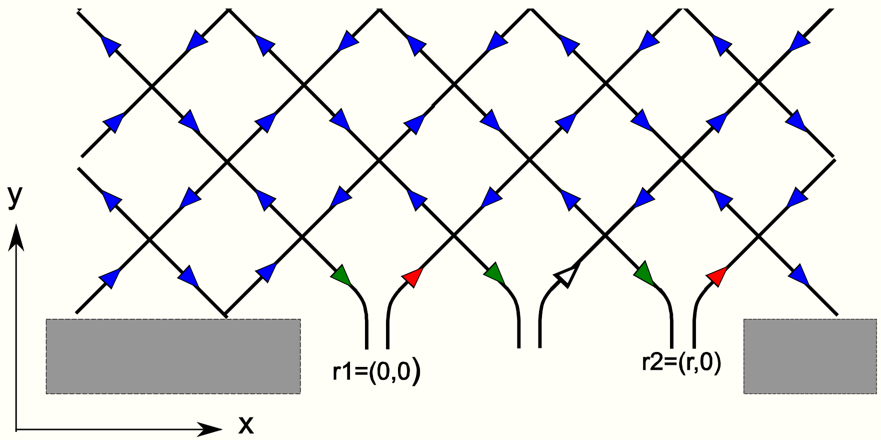
Boundary PCC and boundary conditions



- Absorbing



- Reflecting (left or right)



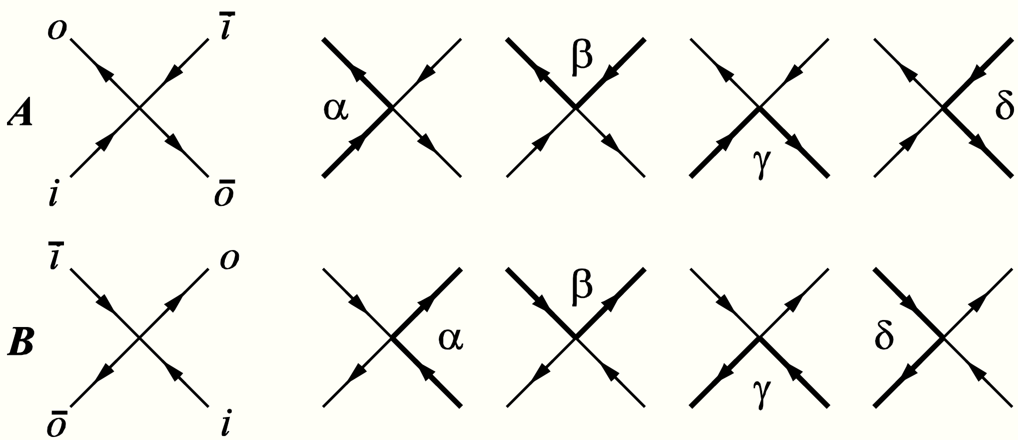
- Mixed

Chalker-Coddington network model

- States of the system specified by $Z \in \mathbb{C}^{N_l}$

  N_l the number of links

- Evolution (discrete time) specified by a random $U \in \mathrm{U}(N_l)$



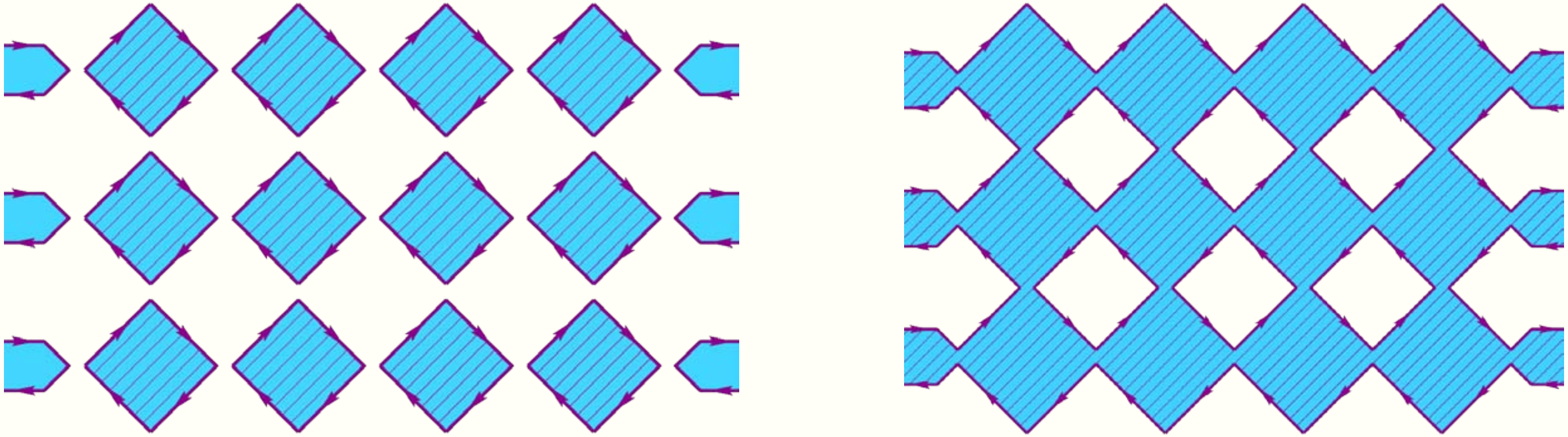
$$\begin{pmatrix} z_o \\ z_{\bar{o}} \end{pmatrix} = \mathcal{S} \begin{pmatrix} z_i \\ z_{\bar{\imath}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_i \\ z_{\bar{\imath}} \end{pmatrix}$$

Chalker-Coddington network model

- Choice of disorder

$$\mathcal{S} = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \begin{pmatrix} \sqrt{1-t^2} & t \\ -t & \sqrt{1-t^2} \end{pmatrix} \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix}$$

- Extreme limits and duality: $t_c = 1/\sqrt{2}$



$t = 0$ Insulator $\qquad$ $t = 1$ Quantum Hall

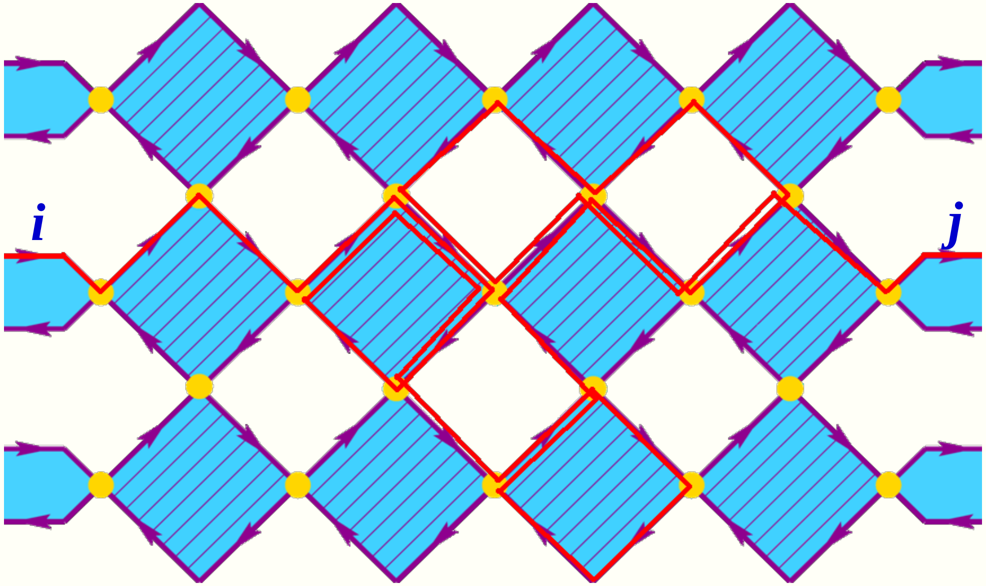
Chalker-Coddington network model

- Propagator (resolvent matrix element) $G_{ij} = \langle i | (1 - e^{-\eta} U)^{-1} | j \rangle$

- Graphical representation in terms of a sum over (Feynman) paths

$$G_{ij} = \sum_{f:i \to j} A_{ij}, \qquad A_{ij} = \prod_{\text{nodes} \in f} S_{ab}$$

$A_{ij} =$



PCC and mapping to a classical problem

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

- Average point contact conductance (PCC)

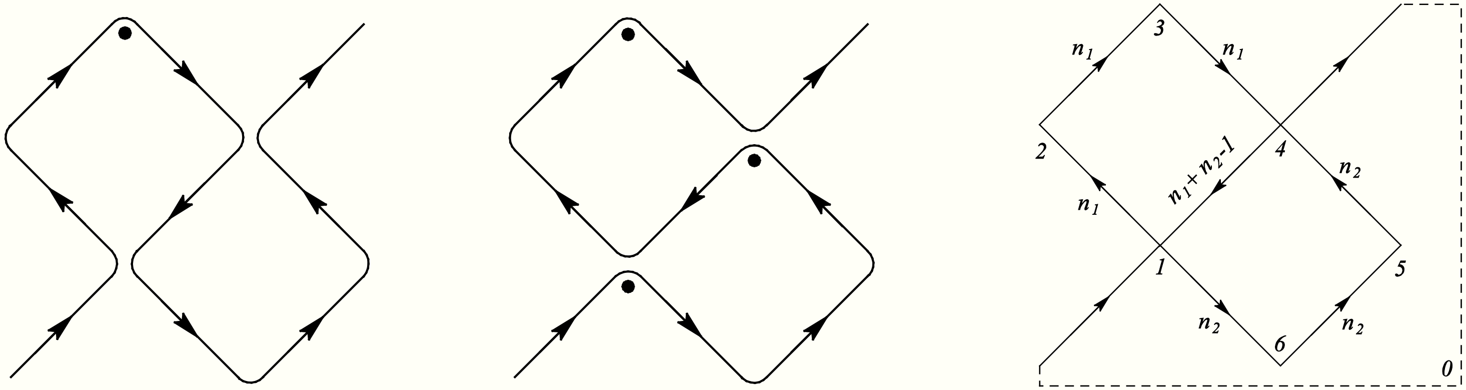
$$\overline{g_{ij}} = \overline{|G_{ij}|^2} = \sum_p W(p)$$

- $W(p)$ are intrinsic positive weights of “pictures” p

- This representation is valid at and away from the critical point as well as for anisotropic variants of the model

Pictures and paths

- Picture is obtained by “forgetting” the order in which links are traversed



$$W(p) = S^2(p), \qquad S(p) = 2^{-N(p)/2} \sum_{f \in F(p)} (-1)^{N_-(f)}$$

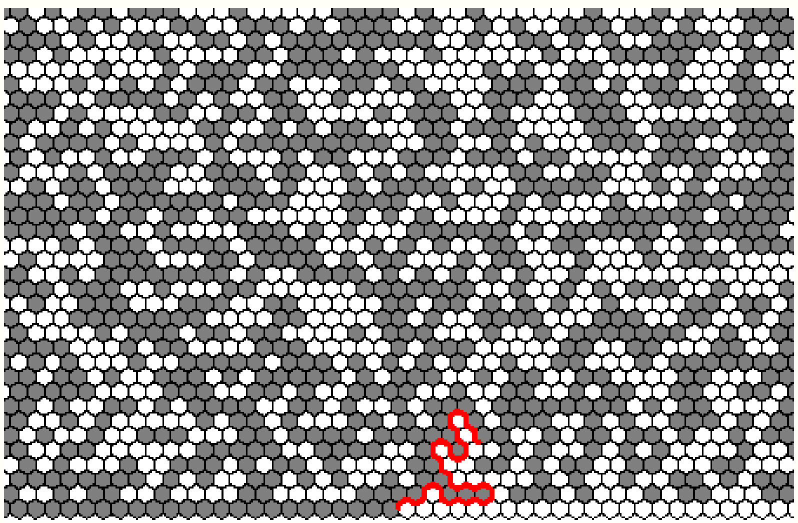
- We know how to enumerate paths giving rise to a picture
- Detailed analysis of the weights $W(p)$ may lead to a complete solution
- We try to go to continuum directly using restriction property

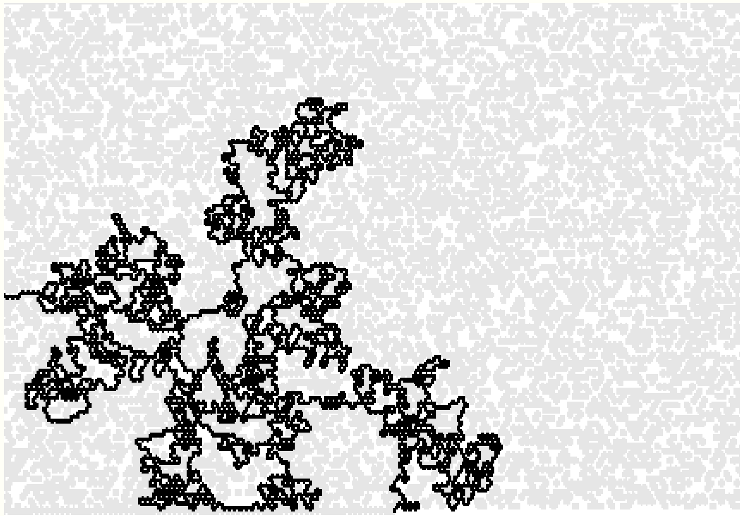
Stochastic geometry and conformal invariance

- Schramm-Loewner evolution (SLE_κ)

O. Schramm, 2000

- Precise geometric description of classical conformally-invariant 2D systems
- Complementary to conformal field theory (CFT)
- Focuses on extended random geometric objects: cluster boundaries





- Powerful analytic and computational tool

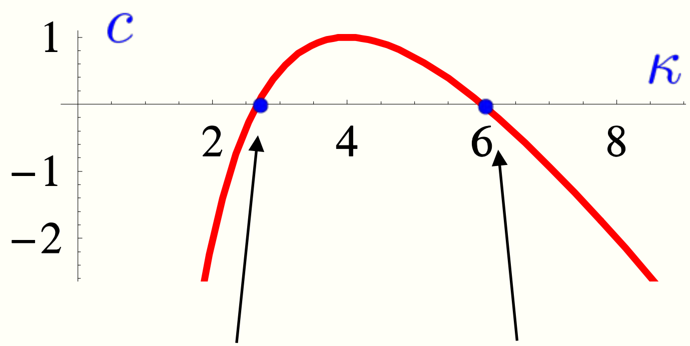
Stochastic geometry and conformal invariance

- SLE does not seem to apply to our case

- Pictures are neither lines nor clusters in a local model

- SLE_κ corresponds to CFT with

$$c_\kappa = \frac{(8-3\kappa)(\kappa-6)}{2\kappa}$$



- CFTs for Anderson transitions in 2D should have $c = 0$ $\Rightarrow$ $\kappa = 8/3$ $\kappa = 6$

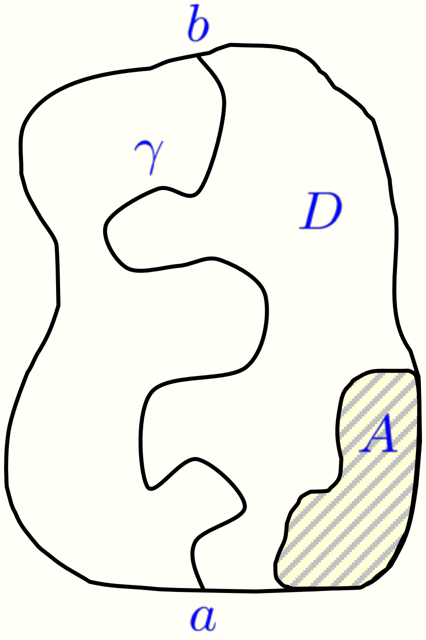
- Not enough for all 2D Anderson transitions and other $c = 0$ theories

- Appropriate stochastic/geometric notion is conformal restriction

G. Lawler, O. Schramm, W. Werner, 2003

Conformal restriction

- Consider an ensemble of curves γ in a domain D and a subset $A \subset D$ "attached" to boundary of D



- From ensemble of curves γ in D we can get an ensemble in $D \setminus A$ in two ways:

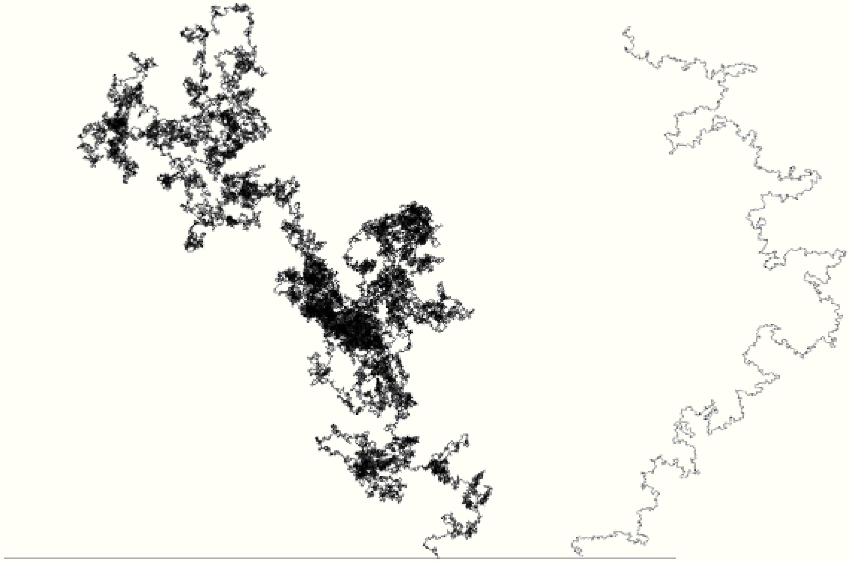
  - conditioning (keep only curves in the subset)
  - conformal transformation $f : D \to D \setminus A$

- If the two ways give the same result, the ensemble is said to satisfy (chordal) conformal restriction G. Lawler, O. Schramm, W. Werner, 2003

- Essentially, any *intrinsic* probability measure on curves satisfies restriction

Restriction measures

- More general sets than curves satisfying conformal restriction
- (Filled in) Brownian excursions, self-avoiding random walks, conditioned percolation hulls

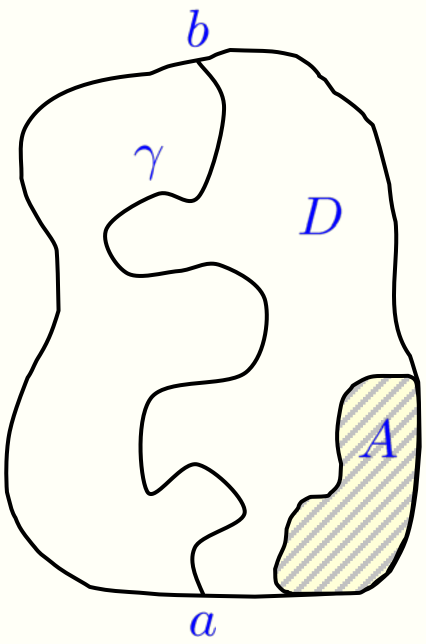


Restriction measures

- Restriction exponent

$$\boldsymbol{P}[\gamma \subset D \setminus A] = |f'_A(a)|^h |f'_A(b)|^h$$

$$f_A : D \setminus A \to D, \quad f_A(a) = a, \quad f_A(b) = b$$



Brownian excursion $h = 1$

SAW $h = 5/8$

- One-sided versus two-sided
- Every two-sided is also one-sided, but not vice versa
- Interpretation in terms of absorbing boundary

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

Restriction measures

- Statistics of a restriction measure is fully determined by the statistics of its boundaries

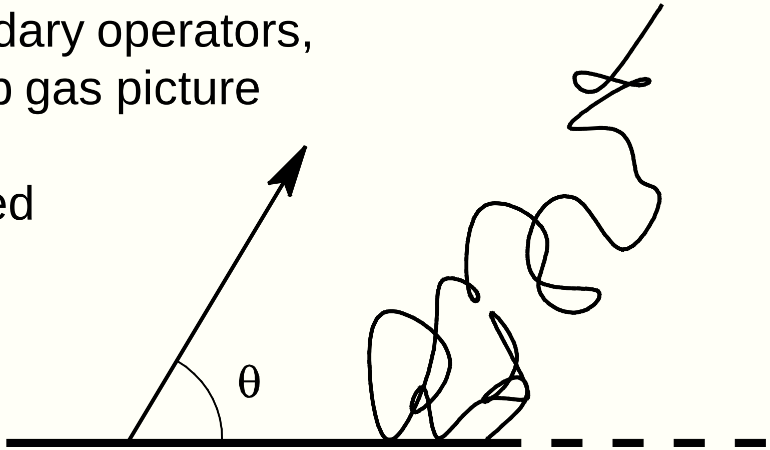
- Boundary of a restriction measure is a variant of SLE: $\mathrm{SLE}(8/3, \rho)$

$$h(\rho) = \frac{(3\rho + 10)(2 + \rho)}{32}, \qquad \rho(h) = \frac{2}{3}\sqrt{24h + 1} - \frac{8}{3}$$

- In CFT h are dimensions of boundary operators, and ρ are charges in the Coulomb gas picture

- General construction using reflected Brownian motions

$$h = 1 - \frac{\theta}{\pi}$$



IQH transition and restriction

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

- Weights of pictures $W(p)$ are *intrinsic*: their ensemble satisfies restriction property with respect to *absorbing* boundaries

- Assume conformal invariance, then can use conformal restriction theory

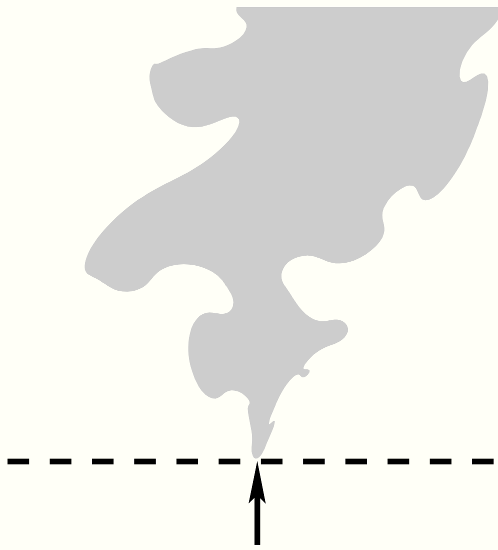
- Current insertions are *primary* CFT operators

- Important to know their dimensions

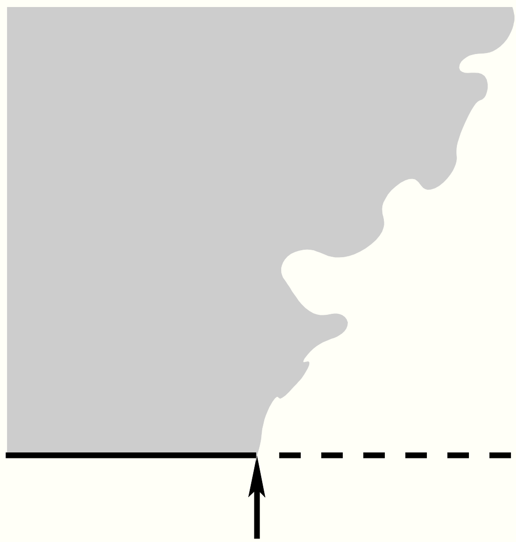
- Explicit analytical results for average PCC with various boundaries

Boundary operators and dimensions

E. Bettelheim, IAG, A. W. W. Ludwig, 2010



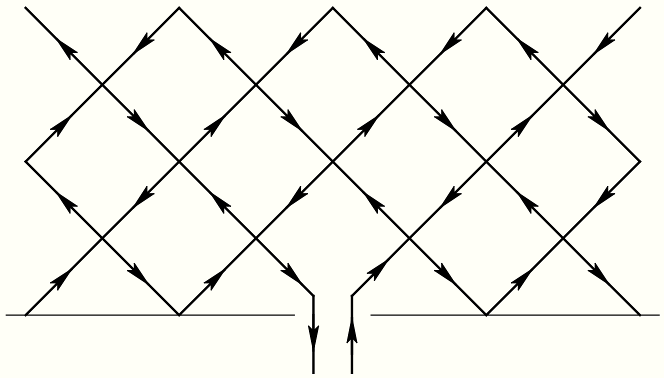
$h_a = 1$ (exact ?)



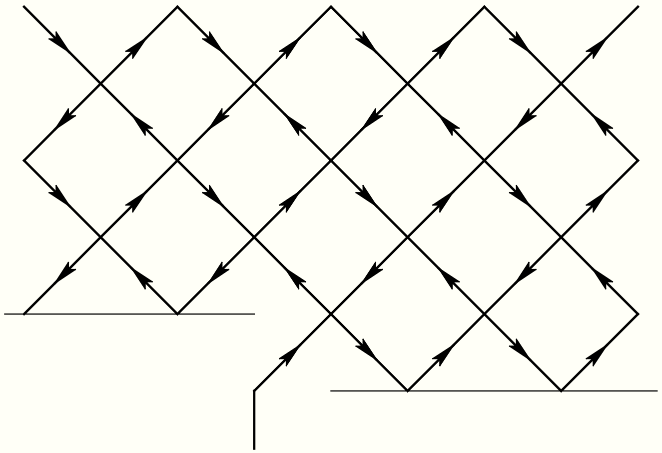
$h_m^{L,R}$

Boundary operators and dimensions

E. Bettelheim, IAG, A. W. W. Ludwig, 2010



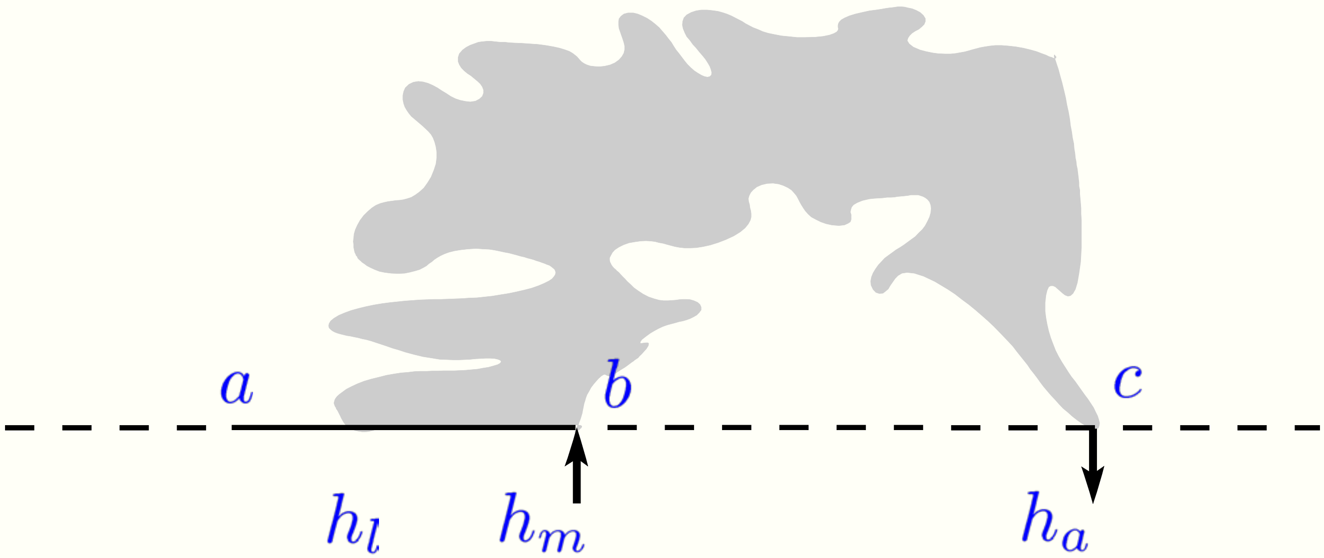
h_r



$h'_r = 0$ (exact)

- Other boundary operators
- All dimensions known numerically and some exactly

Explicit exact results for PCC



$$\overline{g_{bc}} \propto \frac{(b-a)^{1+h_a-h_m-h_l}}{(c-b)^{2h_a}}$$

$$\times\, {}_2F_1\left(h_a+h_l-h_m, 1+h_a-h_m-h_l; 2+h_a-h_m-h_l; -\frac{b-a}{c-b}\right)$$

- Conformal restriction theory gives $h_l = 1$

Other systems and conformal restriction

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

- Same approach applies to other disordered systems in 2D:

  - Spin QH transition where we have an exact mapping to 2D percolation

    IAG, A. W. W. Ludwig, N. Read, 1999
    E. J. Beamond, J. Cardy, J. T. Chalker, 2002

    In this case all dimensions are known analytically

  - The classical limit of CC model (diffusion in strong magnetic fields)

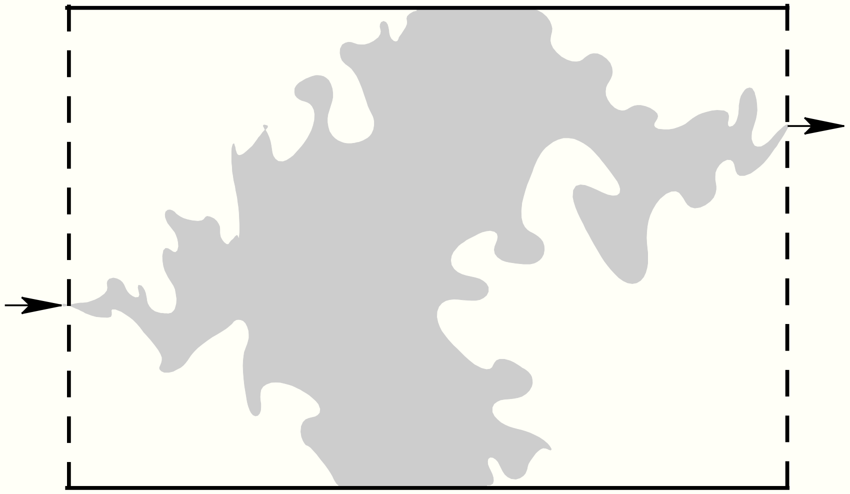
  - Metal in class D

    In these cases all dimensions are known analytically
    in terms of the Hall angle

Conclusions and future directions

- Conformal restriction and SLE: a new approach to quantum Hall transitions

- Other boundary conditions and (degenerate) operators



- Numerical studies

S. Bera, F. Evers, H. Obuse, in progress

Conclusions and future directions

- Conformal restriction in the bulk (radial and whole plane cases): bulk-boundary and bulk-bulk PCCs

- "Massive" (off-critical) restriction: exponent ν and scaling functions

- Other disordered systems

- Plenty of opportunities and work for everybody!