# Stable maps: scaling limits of random planar maps with large faces


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Conformal maps from probability to physics
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# Planar maps

## Definition

A planar map is a proper embedding of a connected graph in the two-dimensional sphere, considered up to direct homeomorphisms of the sphere.



- A rooted map: an oriented edge ($e$) is distinguished
- A pointed map: a vertex ($v_*$) is distinguished
- Notations:
  - $V(\mathbf{m})$ set of vertices
  - $F(\mathbf{m})$ set of faces
  - $d_{\mathrm{gr}}$ the graph distance

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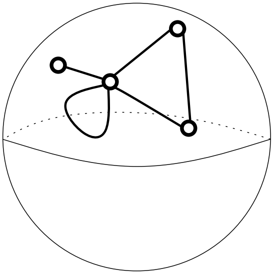

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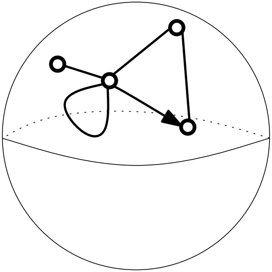

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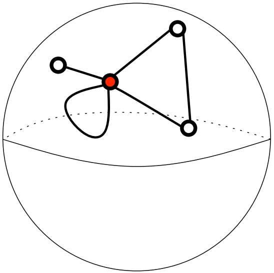

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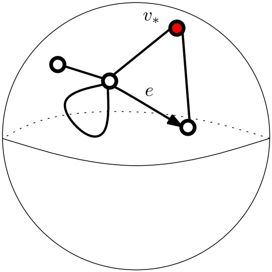


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# Natural ways of picking a map at random

All maps we consider are rooted.

- pick a $p$-angulation with $n$ vertices, uniformly at random (ex $p = 3$ triangulation, $p = 4$ quadrangulation)
- From now on we only consider bipartite plane maps (with faces of even degree)
- Boltzmann distribution: let $q = (q_k, k \geq 1)$ a non-negative sequence. Define a measure on the set of (rooted) planar maps by

$$W_q(\mathbf{m}) = \prod_{f \in F(\mathbf{m})} q_{\deg(f)/2} \, .$$

- Let

$$P_q(\cdot) = \frac{W_q(\cdot)}{Z_q} \, ,$$

where $Z_q = \sum_{\mathbf{m}} W_q(\mathbf{m})$ is finite iff there exists $x > 1$ such that

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- Under $P_q$, it holds that the degrees of the faces of $\mathbf{m}$ form an independent, identically distributed family of random variables, when these faces are explored in an appropriate way to be explained later. The common law is that of a typical face $f^*$, e.g. the face incident to the root edge of the map.
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defining a probability measure.

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- Let $M_n$ be a uniform $2p$-angulation with $n$ vertices. Endow the set $V(M_n)$ of its vertices with the usual graph distance $d_{\mathrm{gr}}$. Then ([Chassaing-Schaeffer], for $p = 2$) it holds that typical distances are of order $n^{1/4}$ as $n \to \infty$.
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for some constant $c_p > 0$, where $(S, d)$ is a random metric space, the Brownian map.

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*For every increasing sequence in $\mathbb{N}$, there exists a sub-sequence along which the convergence (1) holds in distribution for the Gromov-Hausdorff topology on compact metric spaces. The limit $(S, d)$ is a.s. homeomorphic to $\mathbb{S}_2$, and has Hausdorff dimension a.s.*

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- When $q$ is ‘regular enough’ (e.g. decreasing sufficiently fast), then degrees of typical faces of a $W_q^n$-sampled map are exponentially tight. Intuitively, faces remain small:
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- But if for some $a \in (3/2, 5/2)$, we have

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# Main result

- We assume $q_k = c\, q_k^\circ$ where $q_k^\circ \sim k^{-a}$ for some $a \in (3/2, 5/2)$, $c > 0$ a critical value (explicit in terms of $q^\circ$).
- Let $\alpha = a - 1/2 \in (1, 2)$.
- Let $M_n$ be a map with distribution $W_q^n$.

**Theorem**

*For every increasing sequence, there exists a subsequence along which*

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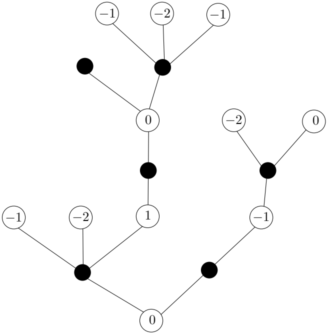
# Main tool: bijective methods

- A now standard tool to attack scaling limits problems on random maps is to use bijective encodings of maps by tree structures whose scaling limit is easier to determine.
- We use the Bouttier-Di Francesco-Guitter (BDG) bijection between rooted, pointed bipartite maps and mobiles.




A mobile is a pair $(\mathcal{T}, (\ell(v)_{v \in \mathcal{T}^\circ}))$ where

- $\mathcal{T}$ a rooted plane tree: vertices $\mathcal{T}^\circ$ at even generations are white, others are black $\mathcal{T}^\bullet$.
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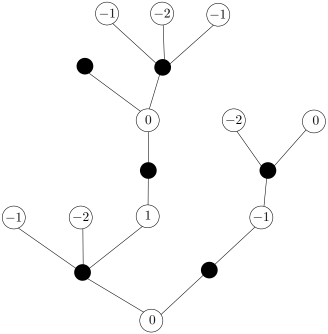


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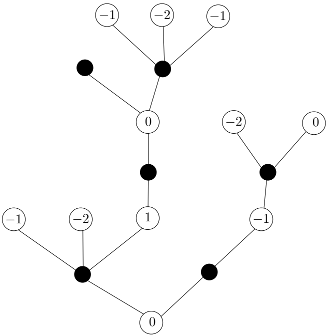


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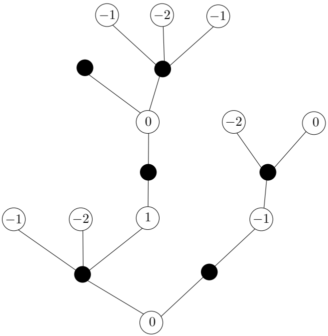


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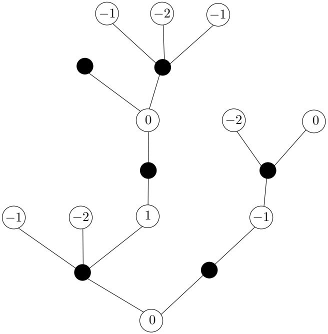


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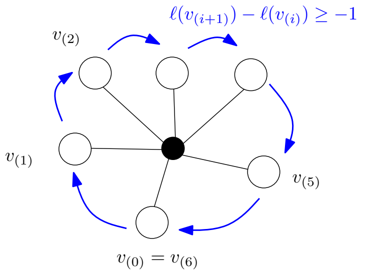


Discrete bridge

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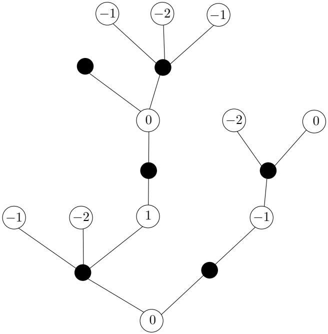
# The BDG bijection

- Start from a mobile $\theta = (\mathcal{T}, \ell)$ with $n+1$ vertices.
- Let $v_0^\circ = \mathrm{root}, v_1^\circ, v_2^\circ, \ldots, v_{n-1}^\circ$ be the contour exploration of white vertices, extended by periodicity to $v_i^\circ, i \geq 0$.
- Add a vertex $v_*$ not in $\mathcal{T}$, set $v_\infty^\circ = v_*$ by convention.
- For every $i \geq 0$, draw an edge between $v_i^\circ$ and $v_{\phi(i)}^\circ$ where

$$\phi(i) = \inf\{j \geq i : \ell(v_j^\circ) = \ell(v_i^\circ) - 1\} .$$

- Root the graph at the edge from $v_{\phi(0)}^\circ$ to $v_0^\circ$.
- Remove edges incident to $\mathcal{T}^\bullet$.



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- Start from a mobile $\theta = (\mathcal{T}, \ell)$ with $n+1$ vertices.
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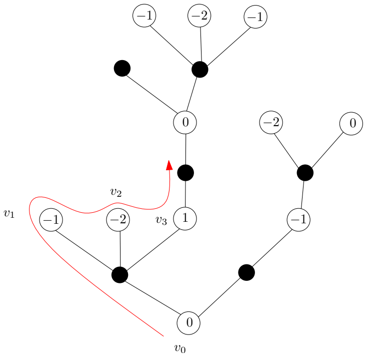

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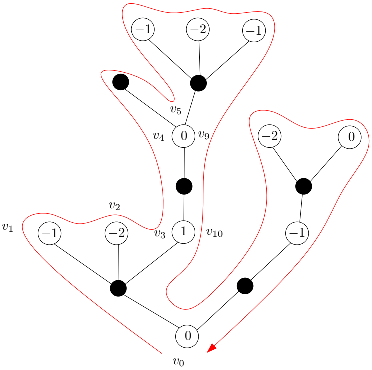

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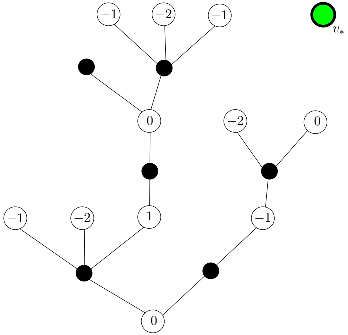

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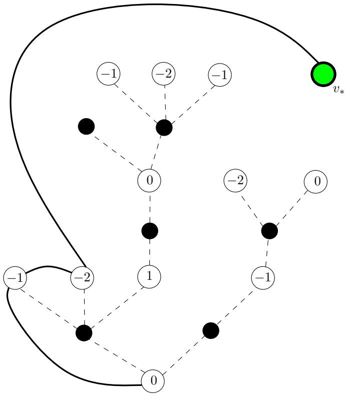

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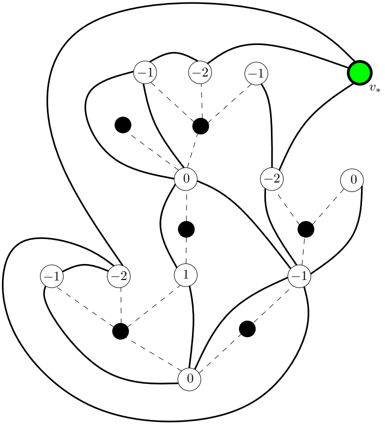


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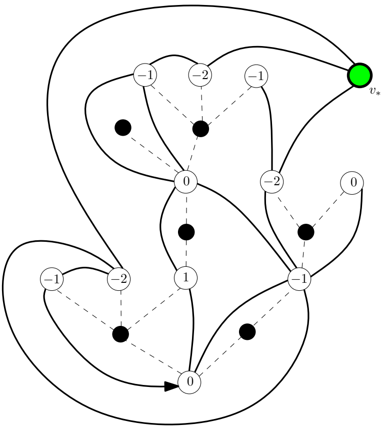

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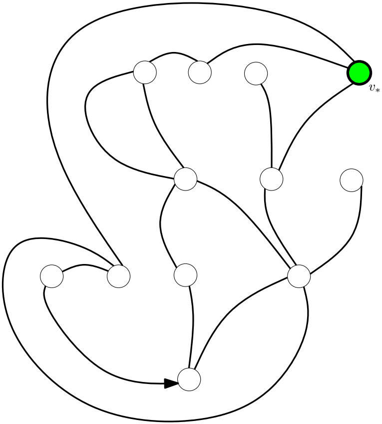

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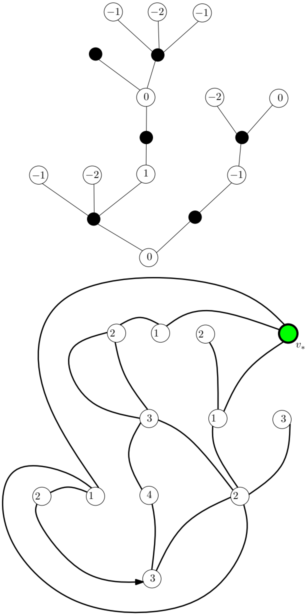


## Proposition

*This yields a bijection between*

1. *Mobiles $\theta = (\mathcal{T}, \ell)$, and*
2. *bipartite, rooted and pointed maps $(\mathbf{m}, v_*, e)$ such that (positivity)*

$$d_{\mathrm{gr}}(v_*, e_-) = d_{\mathrm{gr}}(v_*, e_+) - 1\,.$$

- *A vertex $v \in \mathcal{T}^\circ$ corresponds to a vertex $v \in V(\mathbf{m}) \setminus \{v_*\}$ such that*

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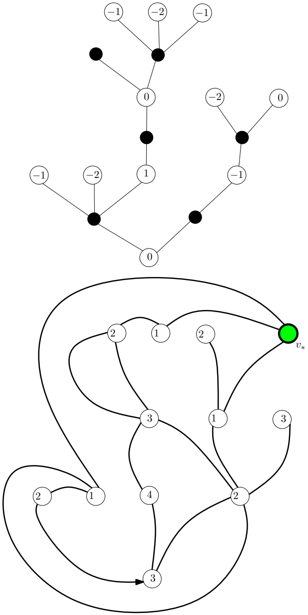


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# Boltzmann distributions and the BDG bijection

Assume $(M, v_*, e)$ has a Boltzmann distribution

$$P_q(\mathbf{m}, v_*, e) = Z_q^{-1} \prod_{f \in F(\mathbf{m})} q_{\deg(f)/2} \cdot$$

Let $\theta = (\mathcal{T}, \ell)$ be the random mobile associated with $M$.

## Proposition

- *The tree $\mathcal{T}$ is a Galton-Watson tree with two alternating types, and respective (white, black) offspring distributions* $\mu_0(k) = Z_q^{-1}(1 - Z_q^{-1})^k, k \geq 0$, *and*

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# Large faces in scaling limits

- For an appropriate choice of $q_k$, in the form

$$q_k = c\beta^k q_k^\circ \,, \qquad q_k^\circ \sim k^{-a} \,,$$

  $a \in (3/2, 5/2)$, the tree $\mathcal{T}$ is critical and

$$\mu_1([k, \infty)) \sim C_q k^{-\alpha} \,, \qquad k \to \infty \,,$$

  $\alpha = a - 1/2$. This says that the degree of a typical face of $M$ (the offspring distribution of a typical vertex of $\mathcal{T}^\bullet$) is in the domain of attraction of a stable($\alpha$) random variable.
- Conditioning on the number of vertices of $M$ to be $n + 1$ ($n$ the number of vertices of $\mathcal{T}^\circ$), the largest faces will have degrees of order $n^{1/\alpha}$ and follow a Poissonian-like repartition.

# Key result

Let $M_n$ have distribution $W_q^n$, $v_*$ a uniformly chosen vertex in $M_n$, $\theta_n = (\mathcal{T}_n, \ell_n)$ the associated mobile, $v_0^\circ, v_1^\circ, \ldots$ the contour sequence.

$$\Lambda_i^{\theta_n} = \ell_n(v_i^\circ)\,, \qquad i \geq 0$$

(0 for $i \geq \#\mathcal{T}$). Recall that $\ell_n$ measures distances in $M_n$:

$$d_{\mathrm{gr}}(v_i^\circ, v_*) = \ell_n(v_i^\circ) - \min \ell_n + 1 = \Lambda_i^{\theta_n} - \underline{\Lambda}^{\theta_n} + 1\,.$$

**Proposition**

*As $n \to \infty$, we have the following convergence in distribution in the Skorokhod space:*

$$\left(n^{-1/2\alpha}\Lambda_{[nt]}^{\theta_n}, t \geq 0\right) \xrightarrow[n\to\infty]{(d)} (D_t, t \geq 0)\,,$$

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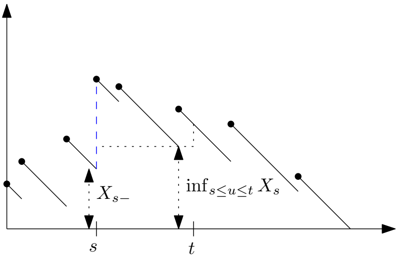
# The continuous distance process

- Let $(X_t, 0 \leq t \leq 1)$ be the standard excursion above its minimum of a stable($\alpha$) Lévy process with only positive jumps.




A simplifying picture (making as if $X$ were of finite variation)

- With each jump of $X$, say $s$ such that $\Delta X_s = X_s - X_{s-} > 0$, associate an independent Brownian bridge

$$(b_s(u), 0 \leq u \leq \Delta X_s)$$

with duration $\Delta X_t$.

- Set

$$D_t = \sum_{0<s\leq t} b_s\left(\left(\inf_{s\leq u\leq t} X_u - X_{s-}\right)^+\right)$$

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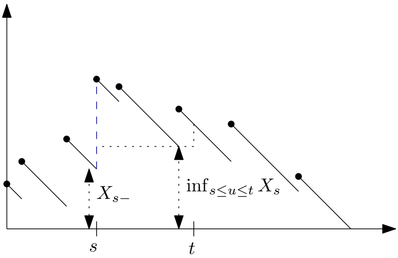


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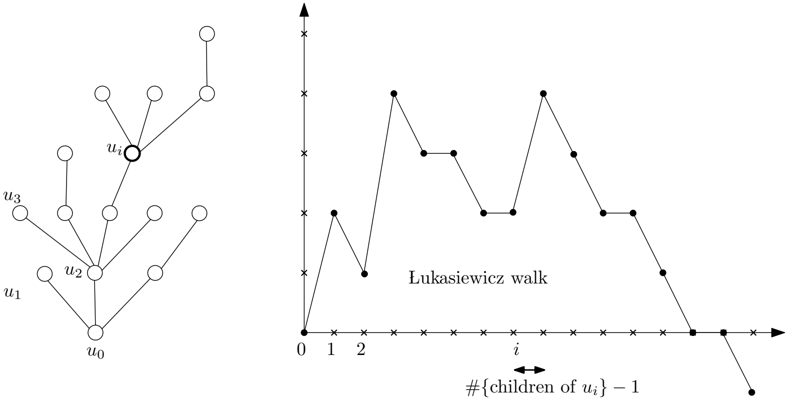


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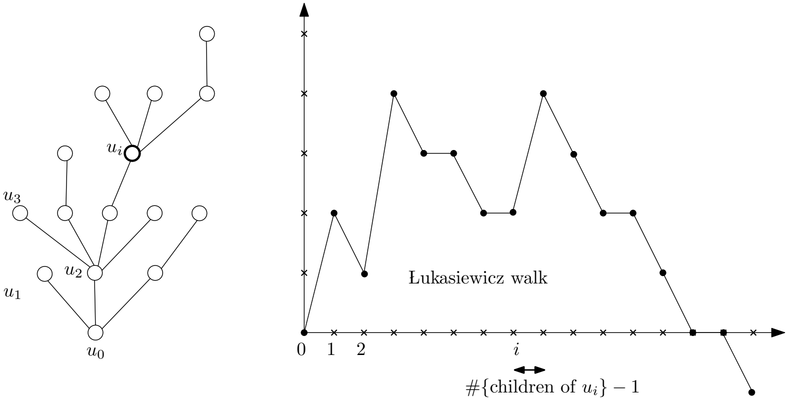


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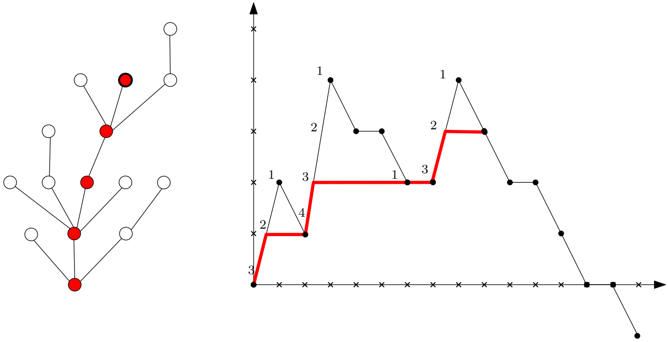


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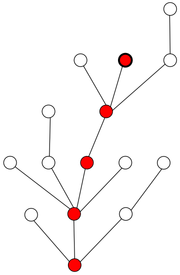

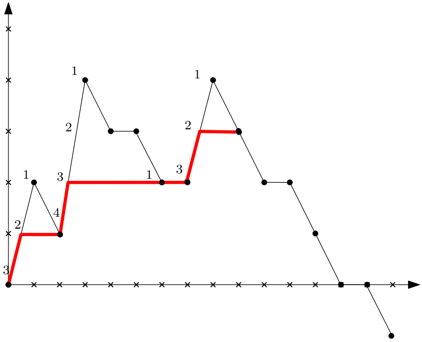


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# A motivation from physics: $O(N)$ models

Let **q** be a rooted quadrangulation, i.e. a rooted (planar) map with faces all of degree 4.

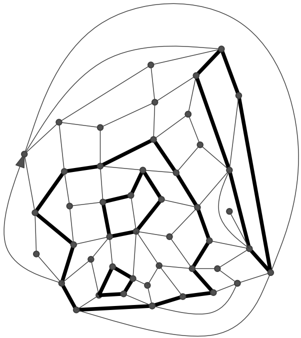
A loop configuration on **q** is a collection $\mathcal{L} = \{c_1, \ldots, c_k\}$, where

- $c_1, \ldots, c_k$ are simple cycles,
- the $c_i$'s are non-intersecting

Set

$$\#\mathcal{L} = k \quad \text{and} \quad \lg(\mathcal{L}) = \sum_{i=1}^{k} \lg(c_i),$$

where $\lg(c_i)$ is the number of edges in the path $c_i$.



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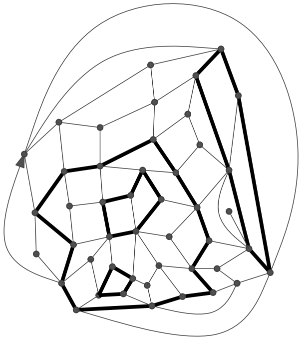


# O(N) measure on random quadrangulations

- Let $N \geq 0$ be fixed. Let $\beta, x > 0$ be positive numbers.
- On the set of pairs $(\mathbf{q}, \mathcal{L})$, where
  - $\mathbf{q}$ is a rooted quadrangulation
  - $\mathcal{L}$ is a loop configuration on $\mathbf{q}$,

  we define a $\sigma$-finite measure by

$$W_{O(N)}(\mathbf{q}, \mathcal{L}) = e^{-\beta \# F(\mathbf{q})} x^{\lg(\mathcal{L})} N^{\#\mathcal{L}} \,,$$

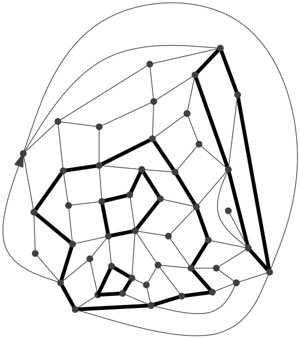
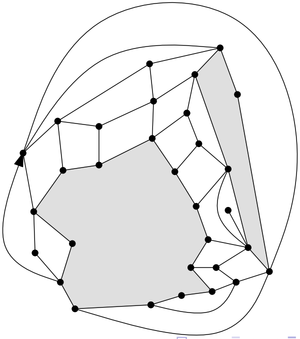
  the annealed $O(N)$ measure on random quadrangulations.
- When the total mass is finite:

$$Z_{O(N)}(\beta, x) = W_{O(N)}(1) < \infty \,,$$

  we define a probability measure $P_{O(N)}$ by renormalizing $W_{O(N)}$ by $Z_{O(N)}(\beta, x)$.

# Exterior gaskets

- Let $(\mathbf{q}, \mathcal{L})$ be a configuration. A cycle $c \in \mathcal{L}$ has an interior (the component of $\mathbb{S}_2 \setminus c$ not containing the face incident to the root)
- Deleting the interior of all cycles $c \in \mathcal{L}$, get the external gasket of $\mathcal{E}(\mathbf{q}, \mathcal{L})$.
- The map $\mathcal{E}(\mathbf{q}, \mathcal{L})$ has two types of faces: native quadrangles $Q(\mathbf{m})$ and holes $H(\mathbf{m})$ of any degree (shaded), with simple and mutually avoiding boundaries.

# Boltzmann laws induced by $O(N)$ measures

- The law of the exterior gasket of an $O(N)$-model on quadrangulations is

$$W_{O(N)}(\{\mathcal{E}(\mathbf{q},\mathcal{L})=\mathbf{m}\}) = e^{-\beta\#Q(\mathbf{m})}\prod_{f\in H(\mathbf{m})} q_{\deg f/2}\,,$$

  where

$$q_k = x^{2k} Z^{\partial}_{O(N),k}(\beta,x)\,,$$

  where $Z^{\partial}_{O(N),k}(\beta,x)$ is the partition function for the $O(N)$-model with a boundary of length $2k$.
- This can be seen as a kind of Boltzmann distribution on random maps, similar to the ones studied before.

# Prediction from Physics

- Expect (see e.g. surveys by Duplantier), for $N = 2\cos(\pi\theta)$ with $\theta \in (0, 1/2)$, that there exists $x_c(\beta)$ a positive function, and $\beta_c > 0$ such that
  - for given $\beta > \beta_c$, $x = x_c(\beta)$, then as $k \to \infty$

$$Z^{\partial}_{O(N),k}(\beta, x) \approx k^{-2+\theta}$$

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  respectively called dense and dilute phases.

- This should correspond to our models with $\alpha \in \{3/2 - \theta, 3/2 + \theta\}$. Note the conjectured coexistence when $\theta = 0, N = 2$.
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- Expect (see e.g. surveys by Duplantier), for $N = 2\cos(\pi\theta)$ with $\theta \in (0, 1/2)$, that there exists $x_c(\beta)$ a positive function, and $\beta_c > 0$ such that
  - for given $\beta > \beta_c$, $x = x_c(\beta)$, then as $k \to \infty$
  $$Z^{\partial}_{O(N),k}(\beta, x) \approx k^{-2+\theta}$$
  - for $\beta = \beta_c$, $x = x_c(\beta_c)$, then as $k \to \infty$
  $$Z^{\partial}_{O(N),k}(\beta, x) \approx k^{-2-\theta}$$

  respectively called dense and dilute phases.
- This should correspond to our models with $\alpha \in \{3/2 - \theta, 3/2 + \theta\}$. Note the conjectured coexistence when $\theta = 0, N = 2$.
- This should be related to conformal loop ensembles (Sheffield and Werner), and the KPZ formula linking models on random maps and regular lattices.

# The case of the Ising model

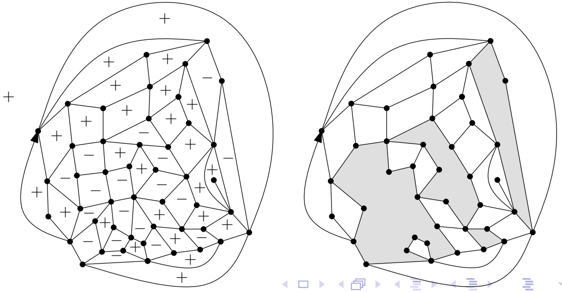
An Ising configuration is now a pair $(\mathbf{q}, \sigma)$ where $\mathbf{q}$ is a rooted quadrangulation, and

$$\sigma = (\sigma_f, f \in F(\mathbf{q})) \in \{-1, +1\}^{F(\mathbf{q})}$$

The (annealed) Ising measure is ($J$ a real parameter)

$$W_I(\mathbf{q}, \sigma) = e^{-\beta \# F(\mathbf{q})} \exp\left( J \sum_{f \sim f'} \sigma_f \sigma_{f'} \right) ,$$

and define exterior gaskets in a similar fashion as for $O(N)$ models — Note that this time, the boundaries are only weakly avoiding



# Predictions

- Predictions from physics [Kazakov] identify $J_c = \ln 2$ as critical
- Expects that respectively for $J = J_c$ or $J < J_c$ (and the appropriate values of $\beta$), the Ising model has the same scaling limit as the dilute and dense phases of the $O(N = 1)$ model
- These correspond to $\theta = 1/3$ and $\alpha \in \{11/6, 7/6\}$. Need to compute generating functions for Ising model with boundary.

# Open problems and perspectives

1. Uniqueness of the limit laws.
2. Equivalent question: joint laws of mutual distances between $k$ randomly sampled points.
3. Other geometric aspects of the limit (“random Sierpinsky gasket”).
4. Adding topological constraints on faces (self and mutually avoiding).