ON V. I. SMIRNOV DOMAINS

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Abstract. We show that complement of a non-V. I. Smirnov domain, coming from the Duren-Shapiro-Shields or Kahane construction must be a V. I. Smirnov domain. There is therefore a negative answer to the old question: need a complement of a non-V. I. Smirnov domain be a non-V. I. Smirnov domain itself?

The purpose of this note is to prove a general, elementary theorem that provides an answer to an old problem on V. I. Smirnov domains. Recall that if Γ is a rectifiable closed Jordan curve and Ω_+ is the bounded component complementary to Γ, Ω_+ is a V. I. Smirnov domain if F'_+ is an outer function. Here F_+ is any choice of conformal map of $\mathbb{D}$ onto Ω_+. The class of such domains was introduced by V. I. Smirnov in [S] in connection with some questions of approximation theory. Consult the expository paper [D] of P. Duren for their properties and further references.

Several authors (starting with P. Duren, H. S. Shapiro, and A. L. Shields, see [DSS]) in the 1960's found examples of Γ and Ω_+, with F'_+ a singular inner function,

$$F'_+ \ = \ \exp\{-(\mu + i\tilde{\mu})\} \tag{1}$$

where μ is a positive measure on the circle that is singular with respect to Lebesgue measure, $d\mu \perp d\theta$. When Condition (1) holds, Ω_+ is <u>not</u> of V. I. Smirnov type (existence of non-Smirnov domains was established earlier by M. V. Keldysh and M. A. Lavrentiev in [KM]). If we let Ω_- denote the unbounded domain complementary to Γ, we have a similar definition of V. I. Smirnov (and non-V. I. Smirnov) domains.

Theorem. *Let Γ, Ω_+, Ω_-, be as above. Then if F_+ satisfies Condition* (1), *F_- (the conformal map of $\{|z| > 1\}$ to Ω_-) satisfies*

$$|F'_-(z)| \ \geq \ c \ > \ 0 \ , \quad |z| > 1 \ . \tag{2}$$

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Corollary. *If Ω_+ is a non-V. I. Smirnov domain satisfying Condition* (1), *Ω_- is a V. I. Smirnov domain.*

There is therefore a negative answer to the old question (see [T1, T2] for dicsussion and references): *Does Ω_+ V. I. Smirnov imply Ω_- V. I. Smirnov?*

Proof of Theorem. Let $c_1 := |F'_+(0)|$ and let ω_+ denote harmonic measure for Ω_+ (on $\partial\Omega_+$) with respect to $F_+(0)$. Then by Condition (1),

$$d\omega_+ \ = \ \frac{1}{2\pi c_1}\, ds \ ,$$

where $ds = d\mathcal{H}^1$ is one dimensional Hausdorff measure on Γ. An immediate consequence of this is the inequality

$$\omega_+(D(x,r)) \ \geq \ \frac{r}{\pi c_1} \ , \quad x \in \Gamma \ , \tag{3}$$

where $D(x,r)$ is the disk centered at x with radius $r < \operatorname{diam}(\Gamma)$ (we use the notation $\omega_+(E) := \omega_+(E \cap \Gamma)$).

Now we invoke the result of C. Bishop, L. Carleson, J. Garnett, P. Jones (see [BCGJ])

$$\omega_+(D(x,r))\ \omega_-(D(x,r)) \ \leq \ c_2 r^2 \ , \tag{4}$$

where ω_- is harmonic measure for Ω_- with respect (say) to ∞. Here the constant c_2 depends on $F_+(0)$, but not on $x \in \Gamma$, $r < \operatorname{diam}(\Gamma)$. This result is valid for harmonic measures on any two disjoint, simply connected domains Ω_+, Ω_-, and its proof is quite elementary. Now from (3) and (4) we obtain

$$\omega_-(D(x,r)) \ \leq \ c_3 r \ , \quad x \in \Gamma \ , \quad r < \operatorname{diam}(\Gamma) \ . \tag{5}$$

It is an easy exercise that Condition (5) implies Condition (2) (one can look at the harmonic measure of $D(x, 2\operatorname{dist}(x,\Gamma))$), and that F'_- is an outer function. ■

Remark. *In certain cases, one can draw a stronger conclusion, than* (2). *For example, let μ come from Kahane's construction in* [K]. *Let $K_+ \subset S^1$ be the closed support of μ. Then $K_- := F_-^{-1}(F_+(K_+)) \subset S^1$ satisfies*

$$\text{Box Dimension}\,(K_-) \ < \ 1 \ . \tag{6}$$

On the other hand, general results (the sharp version is due to N. Makarov, [M]*) show that if a set $K \subset S^1$ has zero measure for the Hausdorff gauge function $t\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$, then there is no Riemann mapping defined on $\mathbb{D}$ with the properties $F' \in H^1$ (the Hardy space),*

$$F' \ = \ G\ \exp\{-(\mu + i\tilde{\mu})\} \ ,$$

where G is outer, $d\mu \perp d\theta$, and μ is supported on K.

To verify (6) one can argue as in the following sketch. One first proves that for all $e^{i\theta} \in K_+$ and $\frac{1}{2} \leq r < 1$, there is $R \in (r, \frac{1}{2} + \frac{r}{2})$ such that

$$|SF_+(z)| \ \geq \ \varepsilon\ (1-|z|)^{-2} \ , \quad z \ := \ Re^{i\theta} \ ,$$

where S denotes the Schwarzian derivative. Here $\varepsilon > 0$ is independent of Γ and θ. An easy normal families argument then yields

$$\beta_\Gamma(x,r) \ \geq \ \delta \ , \quad x \in F_+(K_+) \ , \quad r < \mathrm{diam}(\Gamma) \ . \tag{7}$$

Here β is the usual measure of "deviation from flatness" for the set $\Gamma \cap D(x,r)$ (See e.g. [J]). Then estimates on harmonic measure (similar to those proving Condition (4), but we get a stronger result because of the "twisting", provided by Condition (7)) yield

$$\omega_+(D(x,r))\ \omega_-(D(x,r)) \ \leq \ c\ r^{2+\alpha} \ , \quad x \in F_+(K_+) \ .$$

Fortunately the argument for this inequality is given in the paper [R] of S. Rohde. S. Rohde proves his result for $K_+ = S^1$, but one sees easily, that his argument will work in our case.

Combining the last inequality with (3) one obtains

$$\omega_-(D(x,r)) \ \leq \ c\ r^{1+\alpha} \ , \quad x \in F_+(K_+) \ . \tag{8}$$

Now cover $F_+(K_+)$ by disks of harmonic measure comparable to 2^{-n}. Applying the Besikovitch covering lemma and pulling back by F_- yields

$$\text{Box Dimension}\,(F_-^{-1}(F_+(K_+))) \ < \ \frac{1}{1+\alpha} \ .$$

Here one must use that Γ has finite length. We remark that applications of estimates like (8) are the main point of S. Rohde's paper.

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