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Symbolic dynamics and Collet-Eckmann conditions

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1 Introduction

In this paper we prove that unicritical polynomials with metrically generic combinatorics of the critical orbit satisfy Collet-Eckmann condition. Here metrically generic means except for a set of zero Hausdorff dimension, and combinatorics can be understood in either of the following senses: Markov partition itineraries, kneading sequences, external angles, or harmonic measure on the Mandelbrot set.

Particularly, except for a set of Hausdorff dimension zero of angles all external rays for the degree d Mandelbrot set land at parameters c such that polynomial $z^d + c$ is Collet-Eckmann. This statement is in a sense best possible, and is much stronger than saying that almost every c with respect to harmonic measure corresponds to a Collet-Eckmann polynomial.

Some of the theorems can be generalized to polynomials with many critical points and rational functions.

1.1 Historical perspective

The perspective we take in this paper is to search for properties of a typical non-hyperbolic polynomial or rational function. Many nice properties of hyperbolic polynomials can be generalized to various classes of "weakly hyperbolic" ones, e.g. critically preperiodic, Misiurewicz, Collet-Eckmann, etc. For polynomials from all classes mentioned above Julia sets enjoy regular geometry and have Hausdorff dimension strictly less than 2, see [5, 11, 25]. On the other hand, there are many non-hyperbolic polynomials with "bad" properties, and it makes sense to ask what kind of behavior, nice or bad, is the typical one.

Of course, there are different ways to define the class of "good" rational functions. We will study rational functions satisfying the Collet-Eckmann condition and its variations, see the last section 1.3 of the Introduction for the definitions and discussion. It makes sense to study this particular class, since such rational functions enjoy many nice properties (see [11, 10, 24, 25, 26]) and are typical in many senses. On the other hand, smaller classes of functions, like Misiurewicz maps, turn out to be non-typical in the situations under consideration.

Also there are different ways to interpret the word "typical." One possible way is to look at the metrically generic polynomials or rational functions, and it was successfully exploited in two cases:

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Real quadratics. The Theorem of M. Jakobson [14] states, that there is a positive Lebesgue measure set of real parameters c, such that $z^2 + c$ admits an invariant measure, absolutely continuous with respect to Lebesgue measure on the real line. It is interesting, that it was shown that there are many non-hyperbolic polynomials by exhibiting a lot of them with nice properties. Moreover, by a theorem [18] of M. Lyubich there is no other way to do it, since for Lebesgue almost every real c quadratic map $z^2 + c$ is either hyperbolic or stochastic (i.e. like those in Jakobson's theorem).

Latter, M. Benedicks and L. Carleson extended results of M. Jakobson, proving in [1, 2], that there is a positive Lebesgue measure set of real c such that $z^2 + c$ satisfies the Collet-Eckmann condition. Note, that parameters c corresponding to Misiurewicz maps have zero length by a result [28] of D. Sands.

Rational functions. A parallel result for the rational functions is due to M. Rees [27], who showed that there is a positive Lebesgue measure set of rational functions (for a fixed degree we can parameterize them by $\mathbb{C}^N$), admitting an invariant measure, absolutely continuous with respect to Lebesgue area on $\hat{\mathbb{C}}$, and whose Julia set is the whole complex sphere $\hat{\mathbb{C}}$. It seems plausible that a positive Lebesgue measure set of those rational functions actually satisfies the Collet-Eckmann condition.

Another possible way is to look at topologically generic polynomials:

Complex quadratics. M. Shishikura has shown in [32], that for topologically generic c in the boundary of the Mandelbrot set the Hausdorff dimension of the Julia set J_{z^2+c} is 2, and deduced that the Hausdorff dimension of the boundary of the Mandelbrot set is 2 as well.

This shows some difficulties in establishing analogues of Jakobson's theorem for complex quadratics: it is not clear, what is the canonical Hausdorff measure on the boundary of the Mandelbrot set, and there appears to be many "bad" parameters c.

Our original idea was to establish an analogue of Jakobson's result for complex quadratics with respect to harmonic measure on the boundary of the Mandelbrot set. Harmonic measure in this context plays special role, since Riemann uniformization map has dynamical meaning, see [4]. Roughly speaking, it measures angles of external rays landing on the Mandelbrot set and these angles carry information about topological structure of the corresponding Julia sets. These investigations resulted into Theorem 4, and later the project expanded to include investigation of rational functions such that encoding of the critical orbit is generic with respect to some symbolic dynamics.

1.2 Results

Consider the space Σ_p of all one-sided p-symbol sequences $x = x_1 x_2 \ldots$, $x_j \in \{0, \ldots, p-1\}$, equipped with canonical shift dynamics $T : x_1 x_2 \ldots \mapsto x_2 x_3 \ldots$. We denote by $n(x, y)$ the minimal number n such that $x_n \neq y_n$ (if no such number exist, we set $n(x, y) := \infty$), then $d(x, y) := p^{-n(x,y)}$ is the canonical p-adic metric on Σ_p.

One method to study dynamics F on the Julia set is to somehow (semi)conjugate it to shift dynamics and thus introduce symbolic dynamics on the Julia set. We will show that for three canonical constructions of such conjugation polynomials with metrically generic itinerary of the critical point are Collet-Eckmann (see 1.3 for the definition). Namely we will consider symbolic dynamics arising from Markov partitions, kneading sequences, and Riemann uniformization map. Note, that all unicritical polynomials with disconnected Julia set are hyperbolic (hence Collet-Eckmann), and therefore connectedness condition can be dropped from the Theorems below.

We say, that a polynomial $z^d + c$ has critical symbolic dynamics σ, if there exists a Markov partition (similar to the first level of Yoccoz puzzle, see the Section 3 for details), such that itinerary of the critical orbit coincides with the sequence σ.

Theorem 1 (Markov partitions) *Let E_σ be the set of symbolic sequences σ such that there exists a non-Collet-Eckmann polynomial $z^d + c$ with connected Julia set and critical symbolic dynamics σ. Then Hausdorff dimension of E_σ is equal to* 0.

Kneading sequence is the itinerary of the critical orbit with respect to the partition of the Julia set by the critical point (see Section 2 for details).

Theorem 2 (Kneading sequences) *Let E_κ be the set of kneading sequences κ such that there exists a non-Collet-Eckmann polynomial $z^d + c$ with connected Julia set and kneading sequence κ. Then Hausdorff dimension of E_κ is equal to* 0.

There is an easily described map, carrying external angle of the critical value (see the Section 4) to the corresponding kneading sequence. Its (multi-valued) inverse does not increase Hausdorff dimensions (see the Proposition 1), and hence we arrive at the following

Theorem 3 (External angles) *Let E_θ be the set of angles θ, such that there exists a non-Collet-Eckmann polynomial $z^d + c$ with connected Julia set and external ray with an angle θ lands at the critical value c. Then Hausdorff dimension of E_θ is equal to* 0.

Let $\mathcal{M}_d$ denote the Mandelbrot set (locus of connectivity, i.e. the set of all c with Julia set of $z^d + c$ connected) for the family $z^d + c$. There is an intuitive principle, according to which angle of (dynamical plane) external ray landing at the critical value c in the Julia set coincides with angle of (parameter plane) external ray landing at the point c in the Mandelbrot set. It can be made precise in many situations, and in the Section 4 we show that it can be made rigorous enough to conclude the following

Theorem 4 (Mandelbrot set) *Let $E_{\mathcal{M}}$ be the set of angles θ such that corresponding external ray R_θ does not land at a Collet-Eckmann point $c \in \mathcal{M}_d$ (i.e. corresponding to a Collet-Eckmann polynomial $z^d + c$). Then Hausdorff dimension of $E_{\mathcal{M}}$ is equal to* 0.

Intuitive meaning of this theorem is that looking at the Mandelbrot set from outside, one sees only Collet-Eckmann parameters. This statement is stronger, than saying that almost all with respect to harmonic measure points on the boundary of the Mandelbrot set are Collet-Eckmann (similar theorem is also proven by J. Graczyk and G. Świątek by different methods in

[12]. Earlier J. Graczyk, G. Świątek, and the author showed that almost all points satisfy weaker Summability Condition). Our theorem is in a sense best possible: infinitely renormalizable polynomials cannot satisfy the Collet-Eckmann condition by [10], and it was shown in [19] that the corresponding set of angles has positive logarithmic capacity (and even positive Hausdorff measure with gauge function $|\log r|^{-\omega}$ for $\omega \in (1, \log 5/\log 4)$).

Note also that in all the models under consideration Misiurewicz (i.e. critically non-recurrent) polynomials are not generic, which is easy to see since the corresponding sequences are also non-recurrent (i.e. some block of digits appears only in the beginning).

Remark 1 (Multicritical case) *We abstain from working with multicritical polynomials and rational functions in order to obtain more elegant formulations of our theorems. However, the same methods would prove analogues of Theorems 1 and 2 in the multicritical case as well.*

Roughly speaking, one can show, that there is a set E of symbolic sequences of Hausdorff dimension zero, such that if there is a partition of a Julia set by repelling periodic points, critical points, and finitely many of their preimages, and the corresponding rational function satisfies Topological Collet-Eckmann condition, then itinerary of at least one of the critical points should belong to E.

1.3 Variations of the Collet-Eckmann condition

Classical Collet-Eckmann condition asserts that expansion on the critical orbits grows exponentially:

Definition 1 (CE) *We say that a rational function satisfies* Collet Eckmann condition *if there exist constants $Q > 1$ and $C > 0$, such that for every positive integer n and every critical point c belonging or accumulating to the Julia set, one has*

$$|(F^n)'(Fc)| \geq C\, Q^n .$$

We will mostly work with the following related definition, which involves only topological (not metrical) considerations and repeats one in the Section 4 of [23]. Denote by $\mathrm{Comp}_a F^{-j} B$ the component of connectivity of the preimage $F^{-j}B$, containing the point a, and by Crit the set of the critical points of F.

Definition 2 (TCE) *We say that a rational function satisfies* Topological Collet Eckmann condition, *or* TCE, *if for some $P > 1$ there exist $M > 0$ and $r > 0$, such that for every $x \in J$ there is a sequence of increasing integers $\{n_j\}$ with $n_j \leq Pj$ and*

$$\#\left\{ i:\ 0 \leq i < n_j,\ \mathrm{Comp}_{F^i(x)} F^{-n_j - i} B(F^{n_j}(x), r) \text{ intersects Crit} \right\} \leq M .$$

This condition is also called *finite criticality*, and it means that for large (i.e. $\geq 1/P$) proportion of the iterates $F^i(x)$ a ball of the radius r, when pulled back along the corresponding branch of

F^{-i}, hits critical points at most M times. Sometimes, e.g. in [26], it is formulated accordingly: by requiring a fixed proportion of times i to be "good," i.e. have criticality bounded by M.

In [25] it is shown that Collet-Eckmann condition always implies TCE, and Julia sets of TCE rational functions enjoy nice geometry. Furthermore, for unicritical polynomials TCE and Collect-Eckmann conditions are equivalent, as was shown by F. Przytycki in [23]. But it is not the case for multicritical ones, see [26].

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2 Kneading sequences

When the Julia set J of a polynomial $F(z) = z^d + c$ with no attracting or indifferent cycles is connected and locally connected, there is an external ray, landing at the critical value c (consult Section 4 for the definition). A priori there can be several of them, generating different kneading sequences, which all will correspond to our polynomial. Preimages of this ray will be d external rays, landing at the critical point 0 and cutting the Julia set (without 0) in d "kneading components" $K_1, K_2, \ldots, K_d$. For a point $x \in J$ the *kneading sequence* $\kappa(x)$ is defined as the itinerary of the orbit of x with respect to this partition.

We want to be able to work with non-locally connected Julia sets, so we generalize the definition above, employing terminology of *fibers*, as defined in [30]. Consider a polynomial $F(z) = z^d + c$ with no (super)attracting or indifferent cycles and connected Julia set. Fiber of a point $a \in J$ is the union of all points in J which cannot be separated from a by external rays, landing at periodic or preperiodic points (note than in our case all periodic points are repelling and there is only one Fatou component). Denote by G fiber of the critical point 0. Julia set J has d-fold rotational symmetry around 0, and removing G from J splits the latter in components of connectivity in a symmetric way. Moreover, any point $x \in J \setminus G$ and its $(d-1)$ symmetric counterparts belong to different components. Hence, one can construct d rotationally symmetric disjoint kneading components $K_1, K_2, \ldots, K_d$ whose union is $J \setminus G$ and which are closed inside $J \setminus G$.

Then for every $x \in J$ we can write the itinerary $\kappa(x) = \kappa_F(x) \in \Sigma_d$, where n-th term is equal to i if and only if $F^n(x) \in K_i$. There is a possibility (realized only for points in the preimages of G), that one of the iterates of x will be inside G (or will coincide with 0 in the locally connected situation) and then we just put an arbitrary digit in the corresponding position of x (traditionally symbol $*$ is written) – it won't affect our reasoning, since for any of the exceptional x's it happens only once. In fact, otherwise G would be periodic (images of fibers are fibers, and intersecting fibers coincide), which cannot happen since by Lemma 3.8 in [30]) polynomial F would have indifferent or (super)attracting cycle in this case.

Sequence $\kappa(F) := \kappa_F(0)$ is called the *kneading sequence of the polynomial* F (a priori it depends on the choice of the partition, but this does not affect our statements). Note also that for a fixed polynomial F the kneading map $\kappa : x \mapsto \kappa_F(x)$ conjugates dynamics F on the Julia set with the shift dynamics T on Σ_d: $\kappa(F(z)) = T(\kappa(z))$.

We do not consider polynomials F with attracting or indifferent cycles so that the critical point belongs to the Julia set, and our definition of the kneading sequence works. Nevertheless, there is a logical way to define the kneading sequence for such polynomials as well, but such kneading sequences will be periodic (see the discussion in the Section 4). Set of periodic sequences is countable and of Hausdorff dimension zero, so it does not affect our statements.

Proof of the Theorem 2. The proof will have two parts: first we will show, that if F fails the TCE condition for some $P > 1$ (with any $M > 0$, $r > 0$) then any kneading sequence of F satisfies SR (Strongly Recurrent) condition. In the second part of the proof we will estimate the size of the set of SR sequences in Σ_d.

We say that $u = u_1 u_2 \ldots \in \Sigma_d$ is *SR*, if the proportion of $u_1 \ldots u_N$, which duplicates long beginnings of u (with the exception of a rare sequence of digits) can be arbitrarily close to 1 as N grows. Namely, if for every $D > 0$ there is a D-rare set $\mathcal{R} \subset \mathbb{N}$ such that

$$\limsup_{n\to\infty} \ \#\{\text{duplicating } i : \ 0 < i \leq n\} \ / \ n \ = \ 1 \ .$$

Here for a fixed n the number i is called *duplicating*, if for some $l < i$ and $l' \geq D$ with $i \in (l, l+l']$, the digits of u in positions $(l+1), (l+2), \ldots, (l+l')$ duplicate first l' digits of sequence u with the possible exception of positions in the D-rare set $\mathcal{R}$. We say that a set $\mathcal{R}$ is *D-rare*, if for every $j \in \mathbb{N}$ one has $\#\mathcal{R} \cap [j, j+D] \ \leq \ 2$.

1st part: non-TCE kneading sequences satisfy SR. The set of periodic or eventually periodic kneading sequences has zero Hausdorff dimension, since there are only countably many of them, thus we can exclude them from consideration. Moreover, periodic kneading sequences obviously satisfy SR and if $\kappa(F)$ is eventually periodic, but not periodic, the corresponding polynomial F is non-recurrent and hence TCE. Therefore we can assume that $\kappa(F)$ is not eventually periodic. Particularly this means that G is wandering (i.e. not periodic or eventually periodic).

First we will show, that if orbits of two points are close, then so are their itineraries. Namely, we will prove the following

Lemma 2.1 *Let F be a polynomial as above. For every $L > 0$ there is $r > 0$ such that if points $x, y \in J$ belong to one component of connectivity of $F^{-n}(B_r)$ for some ball B_r of radius r, then first n digits of their itineraries coincide except for a sequence $\{k_j\}$ of positions with $k_{j+1} - k_j > L$.*

Remark 2 *If the Julia set is locally connected, this Lemma is trivial modulo backward Lyapunov stability, which holds by the Corollary 1 in the paper [17] of G. Levin.*

Proof: Consider first L iterates of the critical point: $\{F^j(0)\}_{j=0}^L$. We assumed that fiber G of the critical point is wandering, therefore these iterates belong to different fibers and we can separate them by a finite collection Γ of external rays, landing at preperiodic points. Adding several more rays, we can assume Γ to be forward-invariant. Note, that Γ depends on L, and all further parameters, including r, depend on Γ.

By our construction intersection of the closures of two different kneading components K_i, K_j lies inside G, and Γ does not intersect the latter set. Hence if $\delta > 0$ is the distance from G to Γ, then δ-neighborhood of any point in Γ intersects at most one kneading component.

Next we choose r so small, that for any ball B_r intersecting the Julia set, if for some positive integer l a component of connectivity $f^{-l}(B_r)$ of its preimage intersects Γ, then $\mathrm{diam}(f^{-l}(B_r)) < \delta$. To do so, denote by Γ' the union of (repelling) cycles in Γ, and note, that a ball B_r one of whose preimages intersects Γ, also must intersect Γ. There are finitely many branches of F^i mapping points of $\Gamma \setminus \Gamma'$ into Γ', and we can choose r' so small, that pulling back $B_{r'}$ intersecting Γ' along one of these branches produces a set of diameter less than δ. Without loss of generality (consider an iterate of F) all cycles in Γ' are fixed points. Then we can choose $r'' < r'$ so small, that pulling back by a branch of F preserving fixed points $a \in \Gamma'$ decreases diameter of the ball $B_{r''}(a)$. Finally, choose $r < r''$ so small, that any ball of radius r intersecting Γ and the Julia set must be within distance r from the preperiodic points in Γ. This r is clearly the desired one.

Now suppose that Lemma fails for this choice of r, some ball B_r, two points $x, y \in F^{-n}(B_r)$ and positions i, j with $j < i < j + L$. Then $F^{i-n}(B_r)$ must intersect Γ and the Julia set, hence its diameter is smaller than δ, it intersects only one kneading component, and i-th positions of the itineraries of x and y coincide, proving the Lemma. □

Now we will show, that if a polynomial F fails TCE, its kneading sequence is SR. To do so, we must prove the following

Lemma 2.2 *For every non-TCE polynomial F with connected Julia set, and arbitrary $D > 0$, $\tau < 1$, there are time $n \geq D$ and a D-rare set $\mathcal{R} \subset \mathbb{N}$ so that for the kneading sequence $\kappa(F)$*

$$\#\{\text{duplicating } i : 0 < i \leq n\} > \tau n .$$

Note, that we actually obtain arbitrarily big n (apply Lemma with larger and larger $D \gg D_0$, we get arbitrarily large $n \geq D \gg D_0$).

Proof: To prove the Lemma, fix a polynomial F with connected Julia set (failing TCE with a fixed $P > 1$), and constants $D > 0$, $\tau < 1$.

Choose $r > 0$ according to the Lemma 2.1 applied with $L = 2D$ and $M > 2D$ so large that $(1-\tau)\frac{M}{2}\frac{P-1}{P} > 1$. Since F fails TCE (with parameters P, M, r), there exists $x \in J$ and $N > 0$ such that at least $\frac{P-1}{P}N$ integers $l \in [1, N]$ satisfy

$$\#\left\{i : \ 0 \leq i < l, \ \mathrm{Comp}_{F^i(x)} F^{i-l} B(F^l(x), r) \ni 0\right\} \ > \ M .$$

For every $j \leq N$ define j' as the largest number $j' \leq N$, such that

$$\mathrm{Comp}_{F^j(x)} F^{j-j'} B(F^{j'}(x), r) \ni 0 \ .$$

If such number exists, we add interval $[j, j']$ to collection $\mathcal{I}$.

For every j there is at most one interval I with such a beginning, and the fact, that F fails TCE as specified above means that at least $\frac{P-1}{P}N$ integers in $[1, N]$ are covered by the collection $\mathcal{I}$ at least M times. Removing from $\mathcal{I}$ all intervals of length less than D we obtain new collection $\mathcal{I}'$, which covers at least $\frac{P-1}{P}N$ integers in $[1, N]$ at least $(M - D) > M/2$ times, and therefore

$$\sum_{I \in \mathcal{I}'} |I| \ \geq \ \frac{P-1}{P} N \, \frac{M}{2} \ > \ N \, / \, (1 - \tau) \ , \tag{1}$$

where $|I|$ denotes the number of integers in I.

Note, that for every interval $I = [i, i'] \in \mathcal{I}'$ we can apply the Lemma 2.1 to $F^i(x)$ and 0, deducing that first $(i' - i)$ digits in the itinerary of 0 duplicate igits in the $(i+1), (i+2), \ldots, i'$-th positions of the itinerary of x, except for a set $\mathcal{S}_I$. Note that the difference of any two numbers in the latter set is bigger than $2D$ by the choice of L and Lemma 2.1. It follows that for any two intervals $[i, i'], [j, j'] \in \mathcal{I}'$ part of the itinerary of 0 starting at $|j - i|$ duplicates the beginning of the same itinerary, except for positions in $\mathcal{S}_I \cap \mathcal{S}_J$.

Therefore, if there is an interval $I = [i, i'] \in \mathcal{I}'$, such that at least $\tau |I|$ integers in I are covered by righter intervals in $\mathcal{I}'$ (which is canonically ordered by left ends), then we can finish the first part of the proof. In fact, from such cover one can choose a subcover $\mathcal{J}$ which covers every point at most twice (by excluding redundant intervals). We take $(i' - i)$ to be the desired n. The set $\mathcal{R}$ can then be defined as the union $\cup_{J \in \mathcal{J}} \mathcal{S}_J \cup \mathcal{S}_I$. The set $\mathcal{R}$ is D-rare, since the difference of any two numbers in any set $\mathcal{S}_J$ is bigger than $2D$, as was noted above.

Suppose it is not so, and for every $I \in \mathcal{I}'$ at most $\tau |I|$ part of it is covered by righter intervals. But if some $l \in [1, N]$ is covered by $m(l)$ intervals from $\mathcal{I}'$, then for only one (the rightmost one) of them this number l is not covered by righter intervals. So we arrive at

$$\begin{aligned}
\sum_{I \in \mathcal{I}'} |I| \ &= \ \sum_{l=1}^{N} m(l) \ = \ \sum_{l=1}^{N} (m(l) - 1) \ + \ N \\
&\leq \ \sum_{I \in \mathcal{I}'} \# \{ i \in I : \ \text{it is covered by a righter interval} \} \ + \ N \\
&\leq \ \sum_{I \in \mathcal{I}'} \tau |I| \ + \ N \ ,
\end{aligned}$$

and deduce, that $(1 - \tau) \sum_{I \in \mathcal{I}'} |I| \ \leq \ N$. The latter contradicts (1), thus completing the first part of the proof. □

2nd part: SR kneading sequences have zero Hausdorff Dimension. In this part we forget about complex dynamics and estimate the Hausdorff dimension of the (combinatorially described) set of SR kneading sequences.

Denote by $E_n(\tau, D)$ the set of all $u \in \Sigma_d$, such that there exists a D-rare set $\mathcal{R}$ and one has

$$\#\{\text{duplicating } i: \ 0 < i \leq n\} \ \geq \ \tau n \ , \tag{2}$$

as in the definition of SR. Clearly,

$$\mathrm{SR} = \bigcap_{\tau<1,\, D>0} \limsup_{n\to\infty} E_n(\tau, D) \ .$$

For $\tau < 1$, $D > 0$ define $\delta = \delta(\tau, D)$ by

$$9 \log D/D + (1-\tau) \log d \ = \ \delta \log d/2 \ .$$

It is easy to see, that $\lim_{\tau\to 1,\, D\to\infty} \delta(\tau, D) = 0$, and hence it is sufficient to prove the following

Lemma 2.3 *For any $\tau < 1$, $D > 0$ one has* $\mathrm{HD}(\limsup_{n\to\infty} E_n(\tau, D)) \ \leq \ \delta(\tau, D)$.

Proof: To prove the Lemma, fix $\tau < 1$ and $D > 0$, denote $\delta := \delta(\tau, D)$, $E_n := E_n(\tau, D)$. First study the size of E_n for some particular n. The inequality (2) means that we can choose a collection $\mathcal{I}$ of intervals $[j, j'] \subset [1, n]$ of length greater than D, which cover at least τn integers in $[1, n]$. And also for $i \in [j, j']$, i-th and $(i-j)$-th positions in u coincide, unless i belongs to a rare set $\mathcal{R}$.

First we will estimate the number of cylinders $v = v_1 \dots v_n$ of length n, needed to cover E_n.

Take some element u satisfying the condition above, and choose (standard Besikovitch covering argument) from $\mathcal{I}$ a nice subcollection $\mathcal{I}'$, having the same property and such that no integer is covered more than twice (e.g. no interval from $\mathcal{I}'$ being covered by union of others is sufficient). Then at most 2 beginnings and at most 2 ends of intervals from $\mathcal{I}'$ belong to any interval $[i, i+D)$ (if three intervals of lengths more than D have beginnings in the same interval of length D, one of them is contained in the union of two others). Observe, that choosing beginnings and ends uniquely determines $\mathcal{I}'$ (easy induction shows, that j-th beginning from the left and j-th end from the left must correspond to the same interval, otherwise there are two intervals with $I \subset J$), and there are less than D^4 possibilities to choose at most 2 beginnings and at most 2 ends in each of the $n/D+1$ intervals $[jD, jD+D)$ for $j \in [0, n/D)$. Thus there are at most $(D^4)^{n/D+1} \ < \ \exp(n\, 5 \log D/D)$ ways to choose $\mathcal{I}'$. Also, in every interval $[i, i+D)$ there are at most two numbers of the rare set $\mathcal{R}$, so there are less than $(D+1)^{2n/D+1} \ < \ \exp(n\, 4 \log D/D)$ ways to choose $\mathcal{R}$. Hence, there are at most $\exp(n\, 9 \log D/D)$ ways to choose $\mathcal{I}'$ and $\mathcal{R}$.

But once nice collection $\mathcal{I}'$ and rare set $\mathcal{R}$ are chosen, the sequence u duplicates itself so extensively, that $(1 - \tau - \frac{1}{D})n$ digits fully determine first n digits of u (in fact, for any integer k covered by $\mathcal{I}'$, but not by $\mathcal{R}$, there is an integer $j = j(k, \mathcal{I}', \mathcal{R}) < k$ independent of u, such that $u_j = u_k$). Thus, there are at most

$$\exp(n\, 9 \log D/D)\, d^{(1-\tau)n} \ = \ \exp(n\, (9 \log D/D + (1-\tau) \log d)) \ ,$$

different ways to choose first n digits of u. To each of them corresponds a cylinder of diameter $\exp(-n\log d)$.

By the choice of δ we conclude, that it is possible to cover E_n by collection $\{C_j^n\}_j$ of cylinders with

$$\sum_j \operatorname{diam}(C_j^n)^\delta \;<\; \exp(n\,(9\log D/D + (1-\tau)\log d))\cdot\exp(-\delta n\log d) \;=\; \exp\left(-\delta\frac{\log d}{2}n\right)\;,$$

and union $\cup_{n\ge m}E_n$ by collection $\{C_j^n\}_{n\ge m,\ j}$ with

$$\sum_{n\ge m,\ j}\operatorname{diam}(C_j^n)^\delta \;<\; \sum_{n\ge m}\exp\left(-\delta\frac{\log d}{2}n\right)\;\le\;\text{const}\,\exp\left(-\delta\frac{\log d}{2}m\right)\;.$$

The desired conclusion $\mathrm{HD}(\limsup_{m\to\infty}E_m)\;\le\;\delta$ readily follows. □

3 Markov partitions

In this Section we will outline the proof of the Theorem 1. The proof is very similar to that of the Theorem 2, so we will skip many details.

Markov partitions. Suppose that the Julia set except for a set X is covered by a union of disjoint open connected domains A_j, $1\le j\le p$, so that for every j every component of connectivity of the preimage $F^{-1}A_j$ is inside one of the domains A_l. We will consider only *Markov partitions* of J when X is a finite set, containing only repelling cycles or their preimages. Constructions of the Yoccoz puzzle (see [21, 13]), and the fixed point portraits (see [8] by Y. Fisher and [9] by L. Goldberg and J. Milnor) start with such a partition.

Then every point in the Julia set, except countably many (X and its preimages), can be encoded by a sequence from Σ_p. Namely, we can define a map $\phi:\,J\setminus\cup_{j\ge0}F^{-j}X\;\to\;\Sigma_p$ by $z\mapsto x_1x_2\ldots$, where $x_j=k$ when $F^j(z)\in A_k$. Clearly so constructed ϕ conjugates dynamics F to shift dynamics T.

The proof of the Theorem 1 (as for the Theorem 2) has two steps: first we show, that if f fails the TCE condition then symbolic sequence $\phi(0)$ satisfies SR' (modified Strongly Recurrent) condition. In the second part of the proof we estimate the size of the set of SR' parameters.

We say that $u\in\Sigma_p$ is *SR'*, if the proportion of $u_1\ldots u_N$, which duplicates long beginnings of u or one of a few periodic sequences can be arbitrarily close to 1 as N grows. Namely, if there exists $T<\infty$ periodic sequences $v^1,\ldots,v^T$, such that for any $D>0$

$$\limsup_{n\to\infty}\;\#\{\text{duplicating } i:\,0<i\le n\}\,/\,n\;=\;1\,.$$

Here the number i is called *duplicating*, if for some $l<i$ and $l'\ge D$ with $i\in(l,l+l']$, the digits of u in positions $(l+1),(l+2),\ldots,(l+l')$ duplicate first l' digits of one of the sequences u, $v^1,\ldots,v^T$.

1st part: non-TCE parameters satisfy SR'. Near repelling points in X dynamics F just permutes A_j, so there exist periodic sequences $v^1, \ldots, v^T$ in Σ_p such that while orbit of a point z stays in some small R-neighborhood X_R of X, it visits A_j in the corresponding order.

Taking R to be small, we can assume that any component of $F^{-1}X_R$, which is disjoint from X, is contained in A_j for some j, and that branch of F^{-1} on X_R, which preserves X, is contracting.

Take r small enough so that $z, z' \in J$, $|z - z'| < r$ implies that either z and z' belong with their r-neighborhood to the same domain A_j for some $j \in [1, p]$, or are both inside X_R. Furthermore, we take r so small, that if a point z gets inside r-neighborhood of X, its images under D' iterations still stay inside X_R.

After that we can follow the proof of the Theorem 2 once the following analog of the Lemma 2.1 is proven:

Lemma 3.1 *If $|F^j x - F^j y| < r$ and x, y belong to the same component of connectivity of $F^{-j}B_r(x)$, then there exists $i \leq j$ such that first i digits of the sequences $\phi(x)$ and $\phi(y)$ coincide, and their digits in the positions $(i+1), \ldots, j$ duplicate first $(j-i)$ digits of the periodic sequences v^A and v^B correspondingly for some A, $B \in [1, T]$.*

Proof: Take the maximal $i \leq j$ so that $F^i(x)$, $F^i(y)$ do not belong to r-neighborhood of X. Then the second part of the statement is clear. To prove the first part, note that the component of connectivity of $F^{-1}X_R$, containing $F^i(x)$, contains also the corresponding component of $F^{i-j}B_r(f^j(x))$ and is inside A_l for some l. Pulling back, we obtain the first claim. □

2nd part: SR' parameters have zero Hausdorff Dimension. If sequences $v^1, \ldots, v^T$ are fixed, we can repeat the analogous arguments from the proof of the Theorem 2 and obtain that Hausdorff dimension of the set of SR' elements of Σ_p with such periodic sequences is zero. But there are only countably many different ways to choose a positive integer T and T periodic sequences $v^1, \ldots, v^T$, and zero dimension is preserved under countable unions, hence Hausdorff dimension of the set of SR' sequences is zero.

4 External angles

For a connected Julia set and an angle θ we introduce *external ray* (or *dynamical ray* in order to distinguish from parameter rays, defined below) R_θ, which is the Green's line starting at infinity with the angle θ, or, equivalently, image of the radius $\{re^{i\theta} : 0 < r < 1\}$ under the Riemann uniformization map ϕ from the unit disc to the domain of attraction to infinity A_∞. The latter is chosen so that it conjugates dynamics $z \mapsto z^d$ on the unit disc to F on A_∞. Note, that wherever ϕ extends to the boundary, it semiconjugates $z \mapsto z^d$ on the unit circle to F on the Julia set.

Similarly, we define *external rays* in the parameter space, or simply *parameter rays*, as Green's lines for the set $\mathcal{M}_d$, the latter being the set of all $c \in \mathbb{C}$ for which the Julia set of $z^d + c$ is connected.

4.1 From kneading to angles

Consider unit circle (which we will parameterize by angles and identify with the real line modulo 1), with dynamics $T : \theta \mapsto d\theta$. Points $\frac{\theta+j}{d}$, $j = 1, \ldots, d$ split the circle in d arcs. Hence we can prescribe to an angle θ its *kneading sequence* $\Psi(\theta) \in \Sigma_d$, which is the itinerary of the orbit $\{T^n(\theta)\}_{n=1}^{\infty}$. If $T^n(\theta)$ coincides with one of the separation points, we write $*$ in the corresponding position. The latter can happen only for countably many θ, which we can exclude from consideration without affecting dimension estimates.

Suppose, that critical value c of a polynomial $F(z) = z^d + c$ is in the accumulation set of an external ray R_θ (or, more generally, if ray R_θ accumulates inside the fiber of c). Then for any n its image $F^n(c)$ is in the accumulation set of the external ray $R_{T^k\theta}$. Removing from the plane d external rays with angles $\frac{\theta+j}{d}$, $j = 1, \ldots, d$ and fiber of the critical point 0 (which contains accumulation sets of those rays – see [30]) splits the plane into d domains in a d-fold rotationally symmetric way, and it is easy to see that for such choice of the kneading components one obtains $\kappa(F) = \kappa_F(0) = \Psi(\theta)$. Let $E_\theta \subset \mathbb{T}$ be denote the set angles θ such that there is a non-TCE polynomial for which external ray R_θ accumulates at the critical value. By the Theorem 2 the set $\Psi(E_\theta) \subset \Sigma_d$ has Hausdorff dimension zero. Then Theorem 3 follows from the following

Proposition 1 *Pulling back by the map $\Psi : \mathbb{T} \to \Sigma_d$ does not increase Hausdorff dimension.*

Proof: If one increases $\theta \in [0, 1]$, then it is easy to see that points $\frac{k}{d^n-1}$, $k \in \mathbb{Z}$ are precisely the places where n-th digit in the sequence $\Psi(\theta) \in \Sigma_d$ changes, following the periodic pattern $0, 1, \ldots, (d-1), 0, 1, \ldots$, and that this digit is equal to $(d^n - 1)\theta \bmod d$. Let σ be an element of Σ_d and let S_n denote a cylinder of depth n in Σ_d, which contains sequences, such that their first n digits coincide with those of σ. Denote by $\mathcal{I}(n)$ the collection (and sometimes the union) of intervals in $\Psi^{-1}(S_n)$. Diameter of S_n (in the d-adic metric on Σ_d) is $\frac{1}{d^n}$, so we need to prove that for any positive α and β one has

$$\sum_{I \in \mathcal{I}(n)} |I|^\alpha \leq \text{const}\, d^{-\alpha n + \beta n} .$$

But intervals in $\mathcal{I}(n)$ have length at most $\frac{1}{d^n-1}$, so it is sufficient to show that for any positive ϵ and γ one has

$$\sum_{I \in \mathcal{I}(n)} |I|^\epsilon \leq \text{const}\, d^{\gamma n} . \tag{3}$$

The latter follows from the fact that for any $\epsilon > 0$ and positive integer m one has for $k > k_0 = \text{const}(\epsilon, m)$

$$\sum_{I \in \mathcal{I}(k+m)} |I|^\epsilon \leq 2m \sum_{I \in \mathcal{I}(k)} |I|^\epsilon . \tag{4}$$

Indeed, repeatedly applying (4), we obtain by induction

$$\sum_{I \in \mathcal{I}(n)} |I|^\epsilon \leq \text{const}\, (2m)^{n/m} = \text{const}\, d^{n \log_d(2m)/m} ,$$

and (3) follows if m is large enough to satisfy $\gamma \geq \log_d(2m)/m$.

To prove the inequality (4), first consider an easier model map $\tilde{\Psi}$. Change the mesh of the partitions from $\frac{1}{d^n-1}$ to $\frac{1}{d^n}$. Namely take some sequence $\{c_n\}$, consider partition given by points $\frac{c_n+k}{d^n}$, and define a map $\tilde{\Psi} : [0,1] \to \Sigma_d$ so that n-th digit in the sequence $\tilde{\Psi}(\theta) \in \Sigma_d$ equals to $(d^n\theta - c_n) \bmod d$. For the model map one can easily see by induction, that $\tilde{\Psi}^{-1}(S_n) \mathrm{mod} \frac{1}{d^n}$ covers univalently interval $\left[0, \frac{1}{d^n}\right)$. Also it consists of at most $(n+1)$ component, because there are only n partition points modulo $\frac{1}{d^n}$, and these points are exactly $\left\{\frac{c_j}{d^j}\right\}$. Therefore $\tilde{\mathcal{I}}(n) := \tilde{\Psi}^{-1}(S_n)$ consists of at most $(n+1)$ intervals with total length $\frac{1}{d^n}$ (and thus inequality (3) for the model map is trivial).

Returning to the original map Ψ, let k_0 be so large, that $8d^{m+1-k_0\epsilon} < m-1$. First fix $k > k_0$ and an interval $J \in \mathcal{I}(k)$. We will view map Ψ as a perturbation of $\tilde{\Psi}$ with properly chosen parameters (depending on J). Denote by a the left endpoint of J and let θ_n be the smallest number $j/(d^k-1)$ which is bigger than a. Take $c_n := \theta_n$, then partitions of depth n for $\tilde{\Psi}$ and Ψ will coincide at θ_n and hence won't differ much inside J. Namely for $n < k$ partitions for Ψ and $\tilde{\Psi}$ would coincide inside J, while for $n \geq k$ any partition point for Ψ inside J and its counterpart for $\tilde{\Psi}$ would differ by at most $\left(\frac{1}{d^n-1} - \frac{1}{d^n}\right)|J|/d^{-n} < 2d^{-2k}$.

Separate intervals from $J \cap \mathcal{I}(k+m)$ into two kinds: "long," i.e. of length $> 4d^{-2k}$, and "short," i.e. of length $\leq 4d^{-2k}$. Since partition points of Ψ and $\tilde{\Psi}$ inside J differ by at most $2d^{-2k}$, every long interval corresponds to some interval from $J \cap \tilde{\mathcal{I}}(k+m)$ (and intersects it). By rescaling we see $J \cap \tilde{\mathcal{I}}(k+m)$ consists of at most $(m+1)$ intervals. Thus there are at most $(m+1)$ long intervals and they are contained in J, so we can write

$$\sum_{\text{long } I} |I|^\epsilon \leq (m+1)\,|J|^\epsilon .$$

Intervals in $\mathcal{I}(k+m)$ can occur at most one per an interval of length $1/(d^{k+m-1}-1)$, so there can be at most $|J|(d^{k+m-1}-1) < |J|d^{k+m}$ of them inside J. Noting, that $|J| < 2d^{1-k}$, we can write

$$\sum_{\text{short } I} |I|^\epsilon \leq \left(4d^{-2k}\right)^\epsilon |J|\, d^{m+k} = 4d^{m-k\epsilon}|J|^\epsilon \left(d^k|J|\right)^{1-\epsilon} \leq 8d^{m+1-k\epsilon}|J|^\epsilon .$$

Combining these observations and recalling that $k > k_0$, we obtain the estimate

$$\sum_{I \in \mathcal{I}(k+m),\, I \subset J} |I|^\epsilon \leq (m+1)\,|J|^\epsilon + 8d^{m+1-k\epsilon}|J|^\epsilon < 2m\,|J|^\epsilon .$$

When summed over all $J \in \mathcal{I}(k)$, it implies (4) and hence the Proposition. □

A different prove of this Proposition is given in [3].

4.2 From dynamical plane to parameter space

Only parameter ray such that kneading sequence of its angle is periodic can accumulate at the point $c \in \mathcal{M}_d$ with Julia set $z^d + c$ having indifferent cycle (see Section 5 of [16], Lemma 3.9 of

[29] and Theorem 14.4 of [15]). Since there are only countably many such kneading sequences, the corresponding set of angles has Hausdorff dimension zero by Proposition 1. See also [6] for a neat computation, explicitly showing that cardiod (set of parameters $c \in \mathcal{M}_2$ with $z^2 + c$ having an indifferent fixed point) has harmonic measure zero.

Now we can exclude Julia sets with indifferent or (super)attracting cycles (latter do not matter, since corresponding parameters are not on the boundary of $\mathcal{M}_d$ anyway), and Lemma 2.1 in [31] implies that if parameter ray R_θ accumulates in the fiber of point c, then dynamical ray R_θ for the Julia set $z^d + c$ also accumulates in the fiber of the critical value c. By the Beurling's theorem nearly every (i.e. except for logarithmic capacity zero and hence except for Hausdorff dimension zero – see [22], Thm. 9.19), parameter ray lands, so combining this observations with the Theorem 3, we arrive at the Theorem 4.

In the quadratic case one can give an alternative proof. C. McMullen shows in Theorem 8.4 of [20], that every infinitely renormalizable polynomial is infinitely simply renormalizable. Every simply renormalizable polynomial can be obtained by tuning, described in [7]. There is also a tuning procedure for angles, formally defined for their dyadic expansions, and described in [6]. If $z^2 + c'$ is obtained by tuning from $z^2 + c$, apart from countably many exceptions (some rays landing at Misiurewicz points) angles of parameter rays, landing at c' can be obtained by tuning from those of rays, landing at c. One can call an angle "renormalizable," if it can be obtained by tuning. Then except for countably many possibilities angles of all rays, landing at points c with infinitely renormalizable $z^2 + c$ are infinitely renormalizable. As A. Manning computed in [19], the set of infinitely renormalizable angles has zero Hausdorff dimension, hence so does the set of angles of parameter rays, landing at c with $z^2 + c$ infinitely renormalizable. Combining the observations above with Beurling theorem, we deduce that except for a set of angles of Hausdorff dimension zero all parameter rays land at points corresponding to finitely renormalizable quadratic polynomials without indifferent cycles. But J.-C. Yoccoz has shown (see [13] and [21]), that for those the Julia set is locally connected, and external rays with the same angles land at the point c both in the Julia set and on the boundary of the Mandelbrot set, so invoking the Theorem 3, again completes the proof.

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