CFT and SLE

and 2D statistical physics

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Recently much of the progress in understanding 2-dimensional critical phenomena resulted from

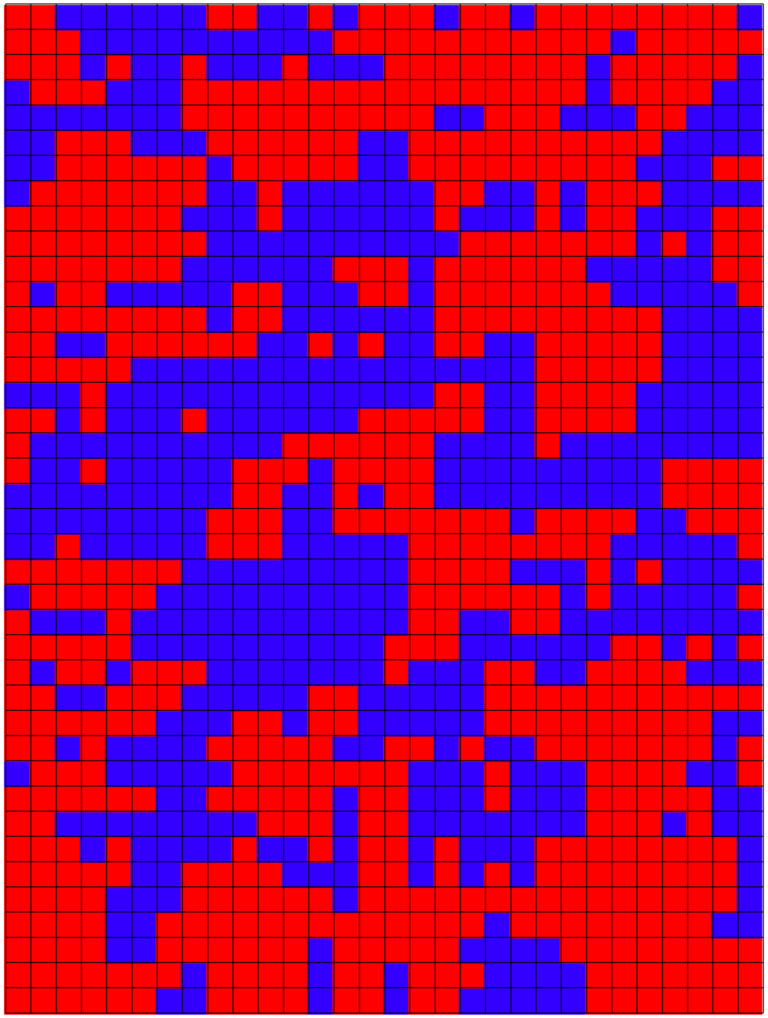
Conformal Field Theory (last 30 years)

Schramm-Loewner Evolution (last 15 years)

There was very fruitful interaction between mathematics and physics, algebraic and geometric arguments

We will try to describe some of it

An example: 2D Ising model



Squares of two colors, representing spins s=±1

Nearby spins want to be the same, parameter **x** :

Prob ≍ $x^{\#\{+-\text{neighbors}\}}$

≍ $\exp(-\beta\sum_{\text{neighbors}} s(u)s(v))$

[Peierls 1936]:

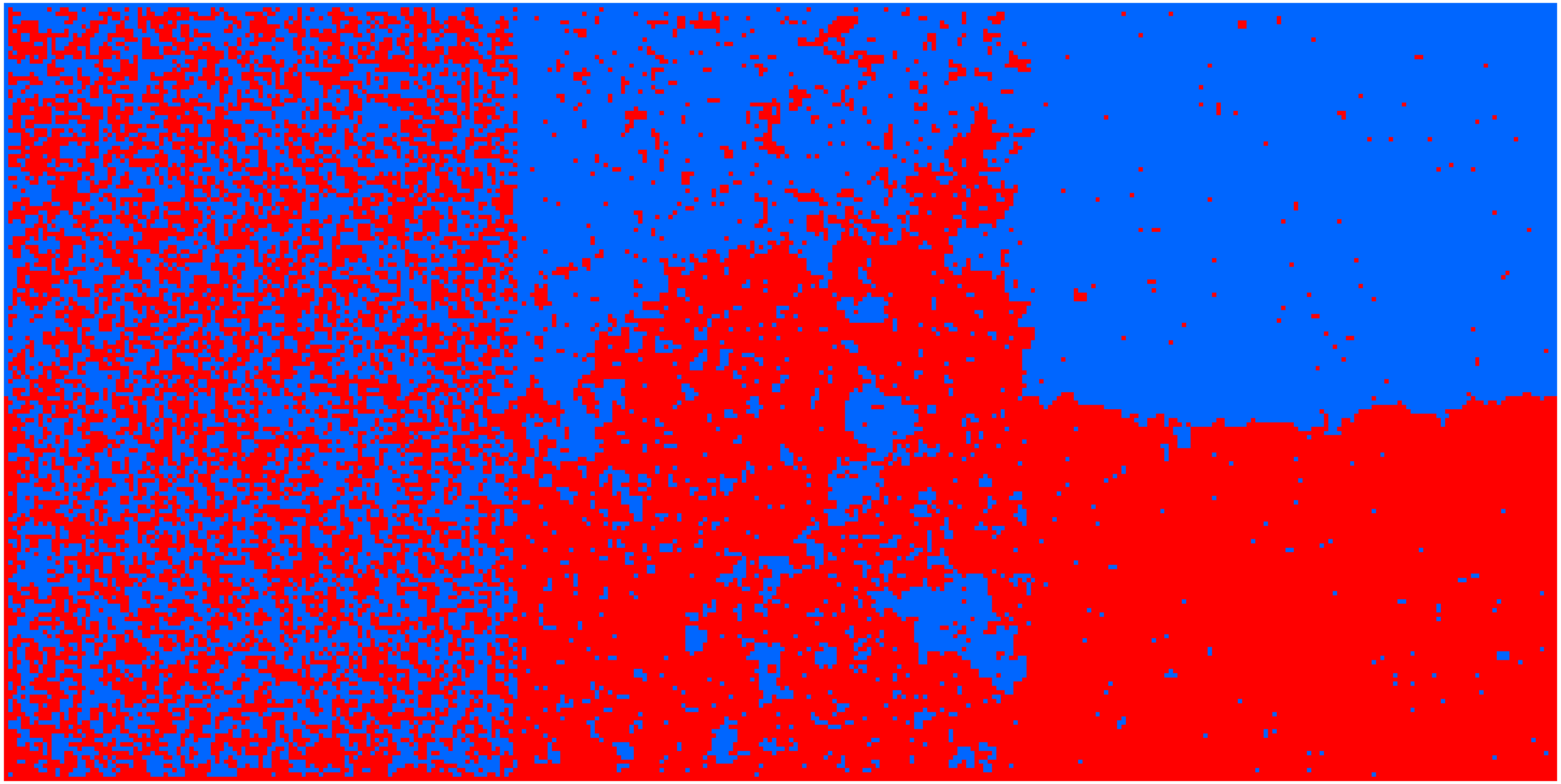
there is a phase transition

[Kramers-Wannier 1941]:

at $x_{crit} = 1/(1+\sqrt{2})$

Ising model: the phase transition

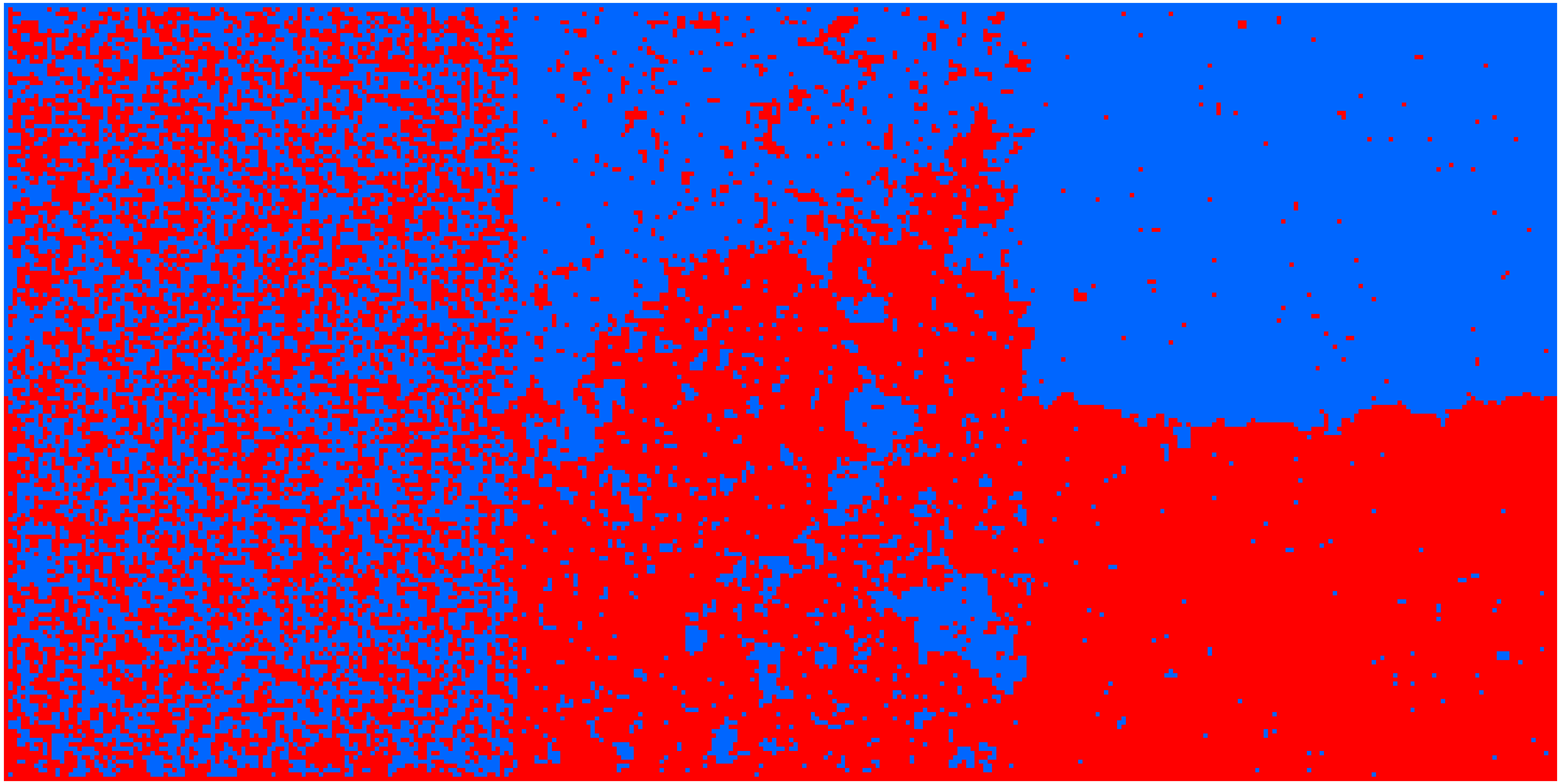
$x \approx 1$ $x = x_{crit}$ $x \approx 0$



Prob $\asymp$ $x^{\#\{+-\text{neighbors}\}}$

Ising model: the phase transition

$x>x_{crit}$ $x=x_{crit}$ $x<x_{crit}$



Prob $\asymp$ $x^{\#\{+-\text{neighbors}\}}$

Ising model is "exactly solvable"

Onsager, 1944: a famous calculation of the partition function (non-rigorous).

Many results followed, by different methods:

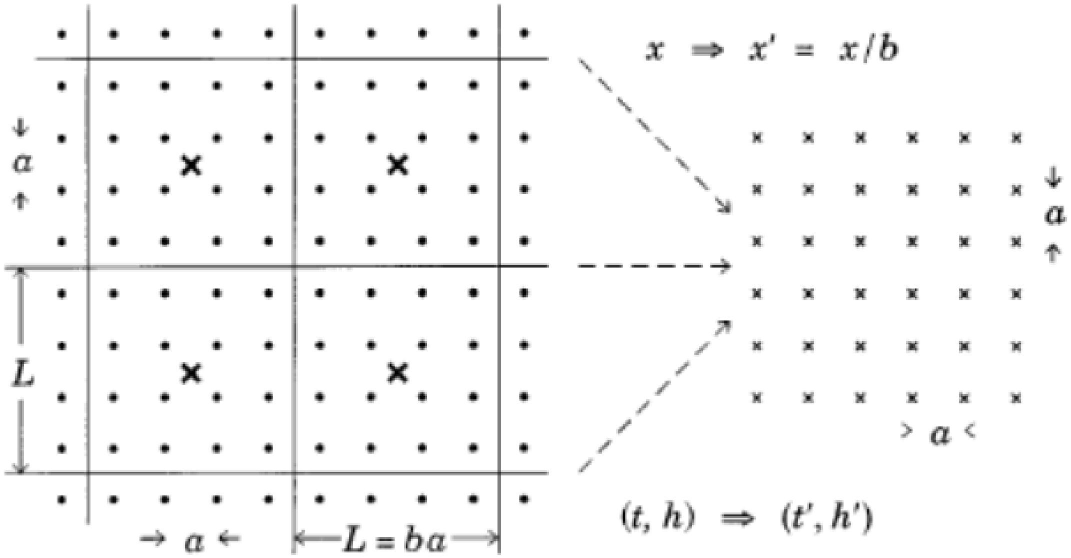
Kaufman, Onsager, Yang, Kac, Ward, Potts, Montroll, Hurst, Green, Kasteleyn, McCoy, Wu, Vdovichenko, Fisher, Baxter, …

- Only some results rigorous
- Limited applicability to other models

Renormalization Group

Petermann-Stueckelberg 1951, …
Kadanoff, Fisher, Wilson, 1963-1966, …

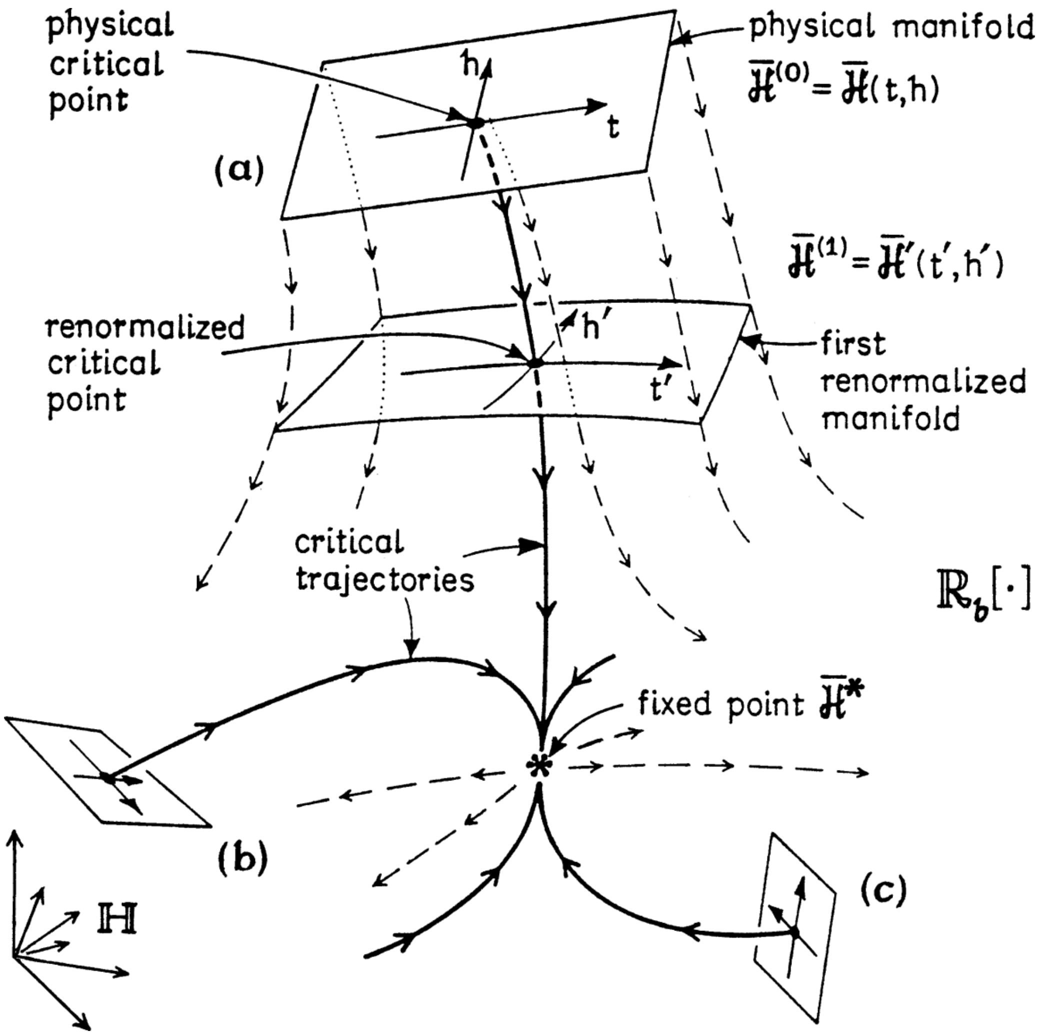
Block-spin renormalization ≈ rescaling



Conclusion:
At criticality the scaling limit is described by a massless field theory.
The **critical point** is **universal** and hence **translation, scale** and **rotation invariant**

Renormalization Group

From [Michael Fisher,1983]



A depiction of the space of Hamiltonians H showing initial or physical manifolds and the flows induced by repeated application of a discrete RG transformation R*b* with a spatial rescaling factor *b* (or induced by a corresponding continuous or differential RG). Critical trajectories are shown bold: they all terminate, in the region of H shown here, at a fixed point H*. The full space contains, in general, other nontrivial (and trivial) critical fixed points,...

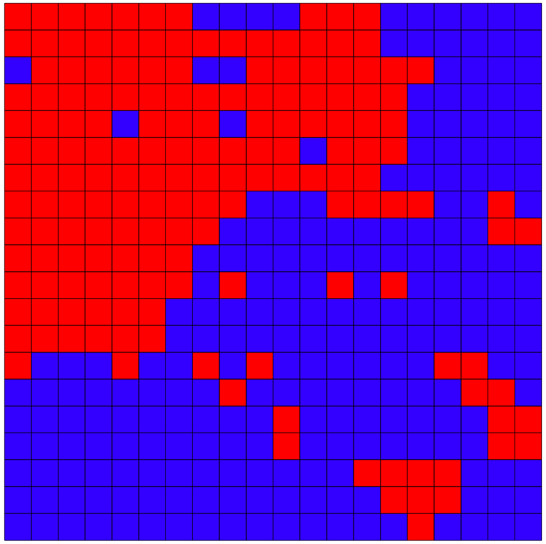
2D Conformal Field Theory

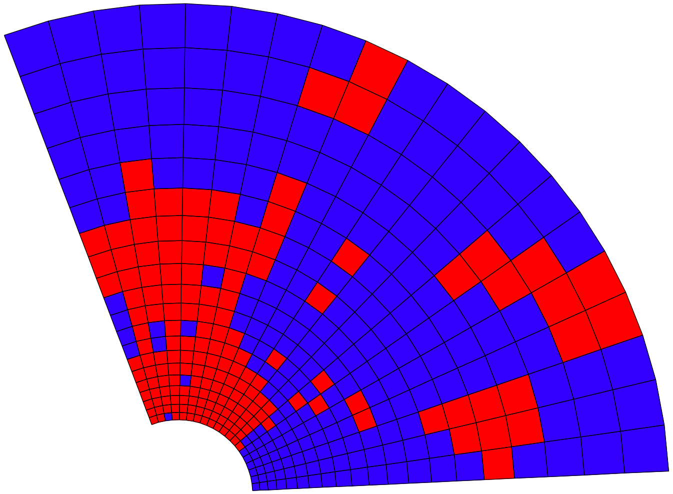
Conformal transformations
= those preserving angles
= analytic maps

Locally **translation** +
+ **rotation** + **rescaling**

So it is logical to conclude **conformal invariance**, but

- We must believe the RG
- Still there are counterexamples
- Still boundary conditions have to be addressed



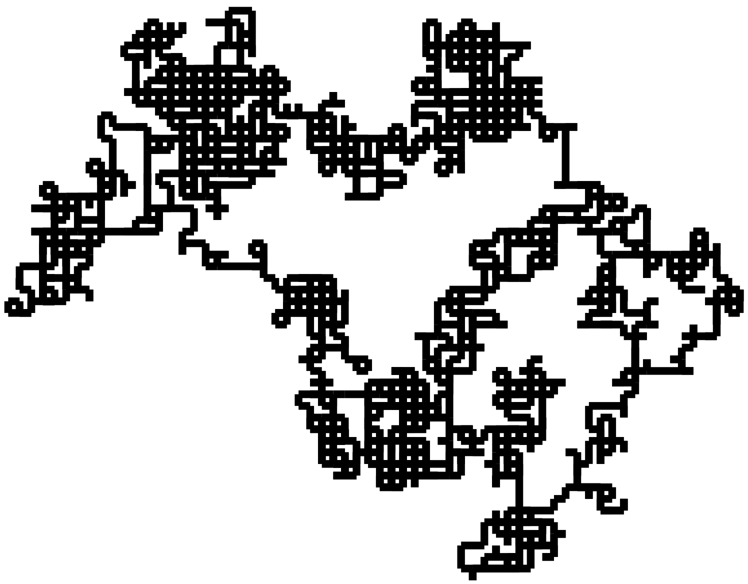


Conformal invariance

well-known example: 2D Brownian Motion
is the scaling limit of the Random Walk

Paul Lévy,1948: BM is **conformally invariant**

The trajectory is **preserved** (up to speed change) by **conformal maps.** **Not so in 3D!!!**



2D Conformal Field Theory

[Patashinskii-Pokrovskii; Kadanoff 1966]
scale, rotation and translation invariance

- allows to calculate two-point correlations

[Polyakov,1970] postulated inversion
(and hence **Möbius**) **invariance**

- allows to calculate three-point correlations

[Belavin, Polyakov, Zamolodchikov, 1984]
postulated **full conformal invariance**

- allows to do much more

[Cardy, 1984] worked out boundary fields,
applications to lattice models

2D Conformal Field Theory

Many more papers followed [...]

- Beautiful algebraic theory (Virasoro etc)
- Correlations satisfy ODEs, important role played by holomorphic correlations
- Spectacular predictions e.g.
  HDim (percolation cluster)= 91/48
- Geometric and analytical parts missing

Related methods

- [den Nijs, Nienhuis 1982] Coulomb gas
- [Knizhnik Polyakov Zamolodchikov; Duplantier] Quantum Gravity & RWs

More recently, since 1999

Two analytic and geometric approaches

1) Schramm-Loewner Evolution: a geometric description of the scaling limits at criticality
2) Discrete analyticity: a way to rigorously establish existence and conformal invariance of the scaling limit

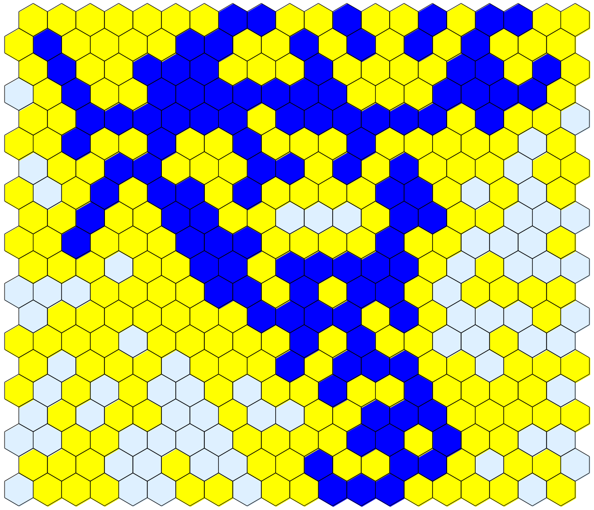
- **New physical approaches and results**
- **Rigorous proofs**
- **Cross-fertilization with CFT**

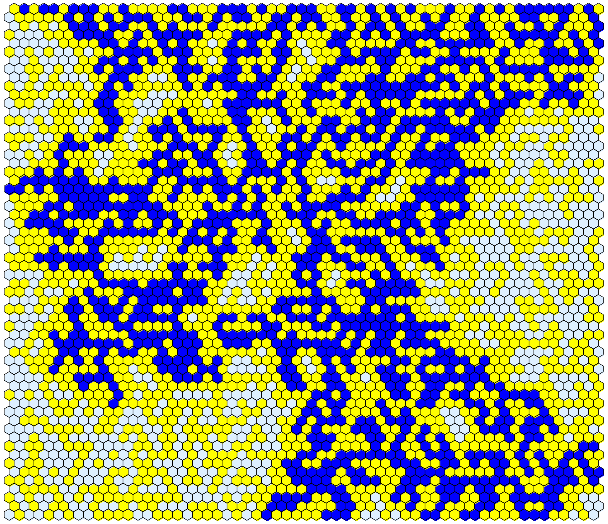
SLE prehistory

Robert Langlands spent much time looking for an analytic approach to CFT. With **Pouilot & Saint-Aubin,** BAMS'1994: study of crossing probabilities for percolation. They checked numerically

- **existence of the scaling limit,**
- **universality,**
- **conformal invariance** (suggested by Aizenman)

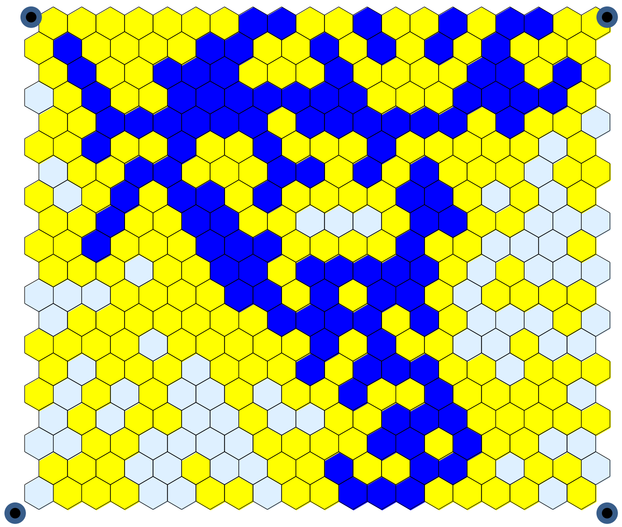
Very widely read!





Percolation: hexagons are coloured white or yellow independently with probability ½. Connected white cluster touching the upper side is coloured in blue.

CFT connection



Langlands, Pouilot , Saint-Aubin paper was very widely read and led to much research.

John Cardy in **1992** used CFT to deduce a formula for the limit of the crossing probability in terms of the **conformal modulus m** of the rectangle:

$$\mathbb{P}\,(\text{crossing}) = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})}\, m^{1/3}\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; m\right)$$

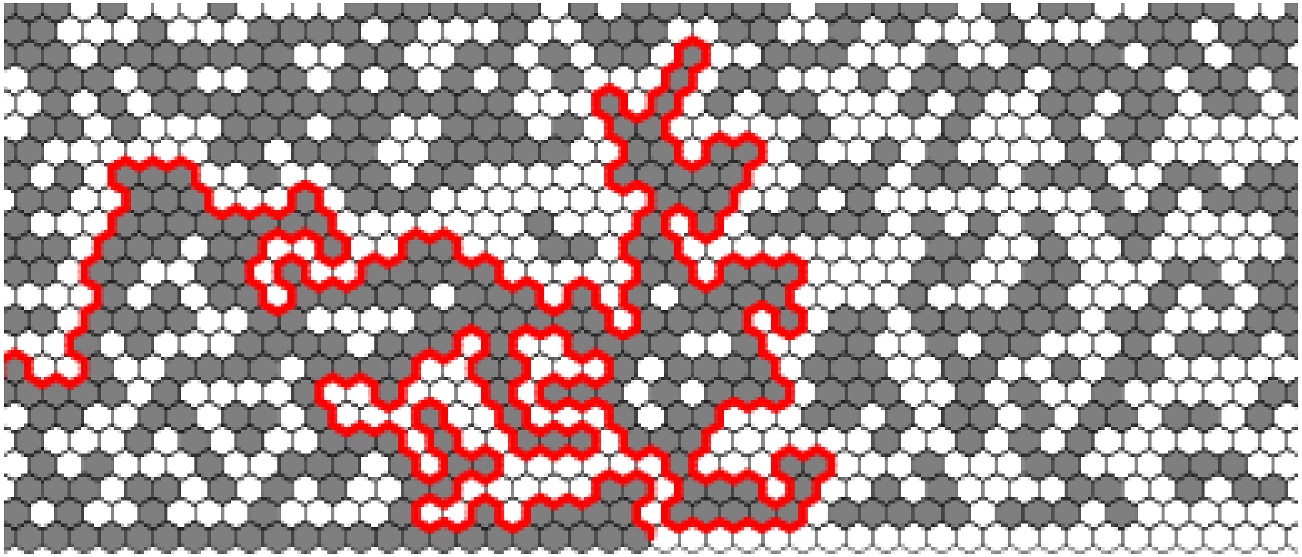
Lennart Carleson: the formula simplifies for equilateral triangles

Schramm-Loewner Evolution

A way to construct **random conformally invariant fractal curves**, introduced in 1999 by **Oded Schramm** (1961-2008), who decided to look at a more general object than crossing probabilities.

O. Schramm. *Scaling limits of loop-erased random walks and uniform spanning trees.* Israel J. Math., 118 (2000), 221-288; arxiv math/9904022

from Oded Schramm's talk 1999



In the figure, each of the hexagons is colored black with probability $1/2$, independently, except that the hexagons intersecting the positive real ray are all white, and the hexagons intersecting the negative real ray are all black. There is a boundary path β, passing through 0 and separating the black and the white connected components adjacent to 0. The curve β is a random path in the upper half-plane $\mathbb{H}$ connecting the boundary points 0 and ∞.

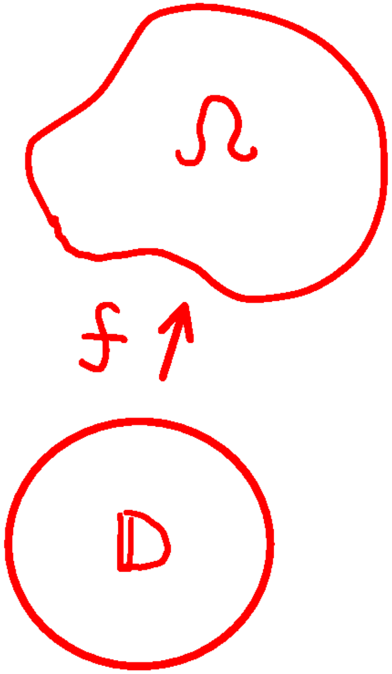
Loewner Evolution

- a tool to study variation of domains & maps in $\mathbb{C}$.
- introduced to attack Bieberbach's conjecture

K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I, *Math. Ann.* 89, 103-121 (1923).

- was instrumental in its proof

L. de Branges, A proof of the Bieberbach conjecture, *Acta. Math.* 154, 137-152 (1985).



Bieberbach's conjecture

de Branges' theorem

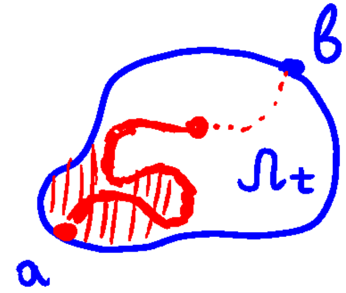
$f: \mathbb{D} \to \Omega$ a conformal map

$f(z) = \sum a_n z^n$.

Then $|a_n| \leq n|a_1|$, attained for $\Omega = \mathbb{C} \setminus \mathbb{R}_-$

Loewner Evolution

Deform domain by growing a slit from $a \in \partial\Omega$ to $b \in \Omega$ (or $b \in \partial\Omega$)

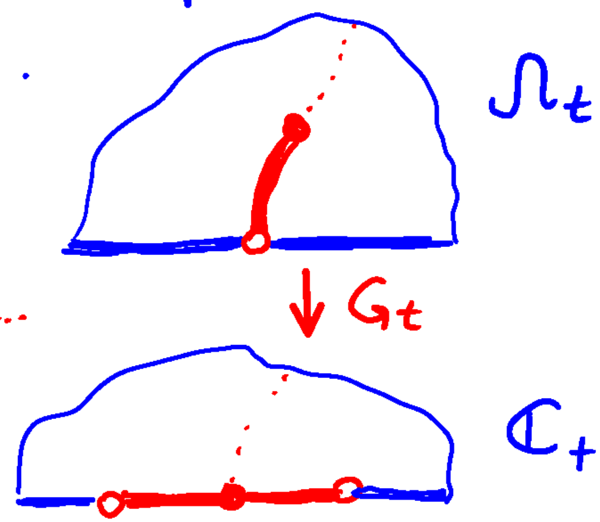


Map Ω to $\mathbb{C}_+$, so that $a \mapsto 0$, $b \mapsto \infty$.

Parametrize slit γ by time t.

Set $\Omega_t = \mathbb{C}_+ \setminus \gamma[0,t]$, component at ∞

$G_t : \Omega_t \to \mathbb{C}_+$ a conformal map with $\infty \mapsto \infty$, $G_t'(\infty) = 1$, $\gamma(t) \to 0$.



Expand at ∞:

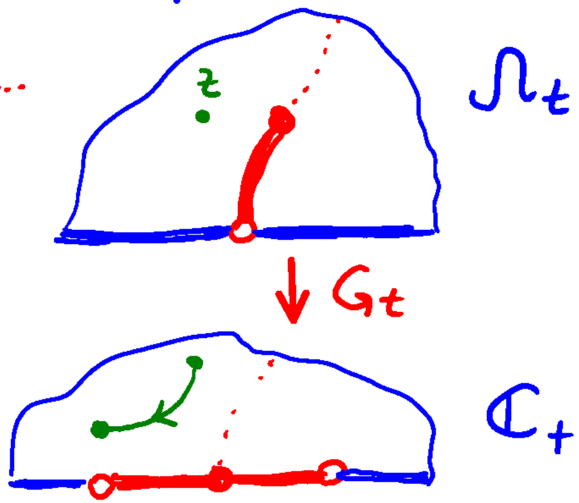
$$G_t(z) = z + a_0(t) + \frac{a_{-1}(t)}{z} + \frac{a_{-2}(t)}{z^2} + \ldots$$

Note: $G_t : \mathbb{R}S \Rightarrow a_k \in \mathbb{R}$

Loewner Evolution

$G_t : \Omega_t \to \mathbb{C}_+$ a conformal map

$$G_t(z) = z + a_0(t) + \frac{a_{-1}(t)}{z} + \frac{a_{-2}(t)}{z^2} + \dots$$



Note: $a_{-1}(t) = \mathrm{cap}_{\mathbb{C}_+}(\gamma[0,t])$

$\Rightarrow$ continuously increases $\Rightarrow$ can change time $a_{-1}(t) = 2t$

Denote $w(t) := -a_0(t)$.

Löwner equation $\quad \boxed{d_t\left(G_t(z) + w(t)\right) = \frac{2}{G_t(z)}}$

B.C. $G_0(z) = z$, $G_t(z) = z - w(t) + \frac{2t}{z} + \dots$ at ∞

gives a bijection {nice slits γ} $\leftrightarrow$ {continuous w}

- ODE for $G_t(z)$ involves $w(t)$ only!
- $d_t w(t)$ = "the turning speed"

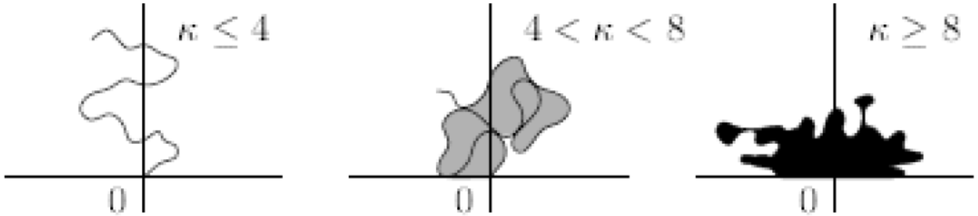
Schramm-Loewner Evolution

deterministic w $\leftrightarrow$ deterministic γ

random w $\leftrightarrow$ random γ $(\to \mu \in \mathrm{Prob}\{\text{curves}\})$

SLE(κ) is LE with $w(t) = \sqrt{\kappa}\, B_t$, $\kappa \in \mathbb{R}_+$

- γ a.s. a simple curve $0 \le \kappa \le 4$
  a self-touching curve $4 < \kappa < 8$
  a random Peano curve $8 \le \kappa$ [Rohde-Schramm]
- $\mathrm{HDim}(\gamma) = \min(1 + \frac{\kappa}{8}, 2)$ a.s. [Beffara]
- $\partial(\mathrm{SLE}(\kappa)) = \mathrm{SLE}(\frac{16}{\kappa})$, $\kappa > 4$ [Zhan], [Dubedat]

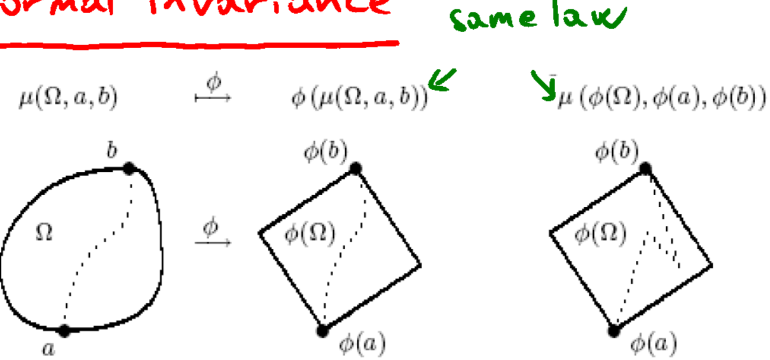


SLE computations = Itô calculus

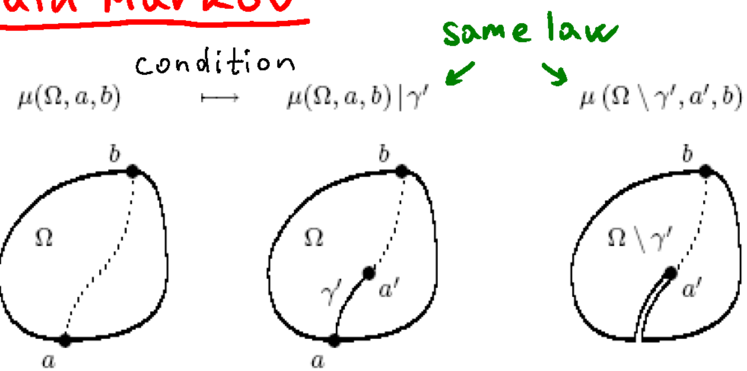
Relation to lattice models

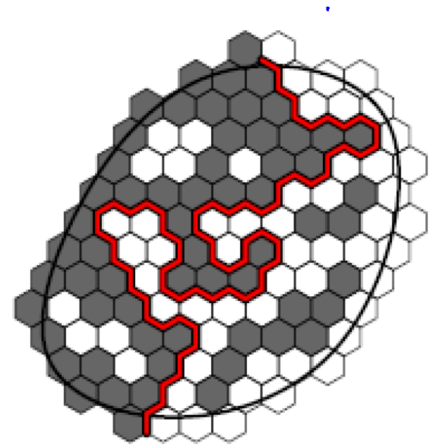
Schramm's principle Assume that an interface has a conformally invariant scaling limit.
Then it is SLE(κ) for some κ.

Conformal invariance

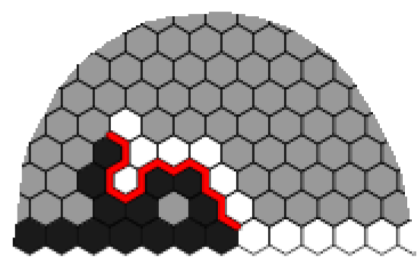


Domain Markov





← holds in the limit



↑

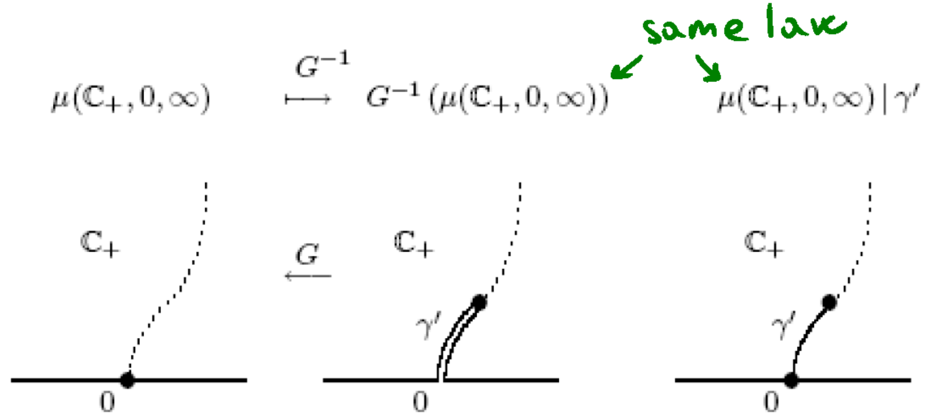
← holds on the lattice (nearest neighbour interaction)

Relation to lattice models

Conformal invariance + domain Markov $\Rightarrow$

Conformal Markov Property

$$G_{t+s} \mid G_t = G_t(G_s)$$



Expanding at ∞: $z - w(t+s) + \ldots \mid G_t =$

$$= (z - w(t) + \ldots) \circ (z - w(s) + \ldots) = z - (w(t) + w(s)) + \ldots$$

So $w(t+s) - w(t) \mid G_t = w(s)$, $w(t)$ a.s. continuous $\Big\} \Rightarrow$ $w(t) = \sqrt{\varkappa}\, B_t + \alpha t$, $\varkappa \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$

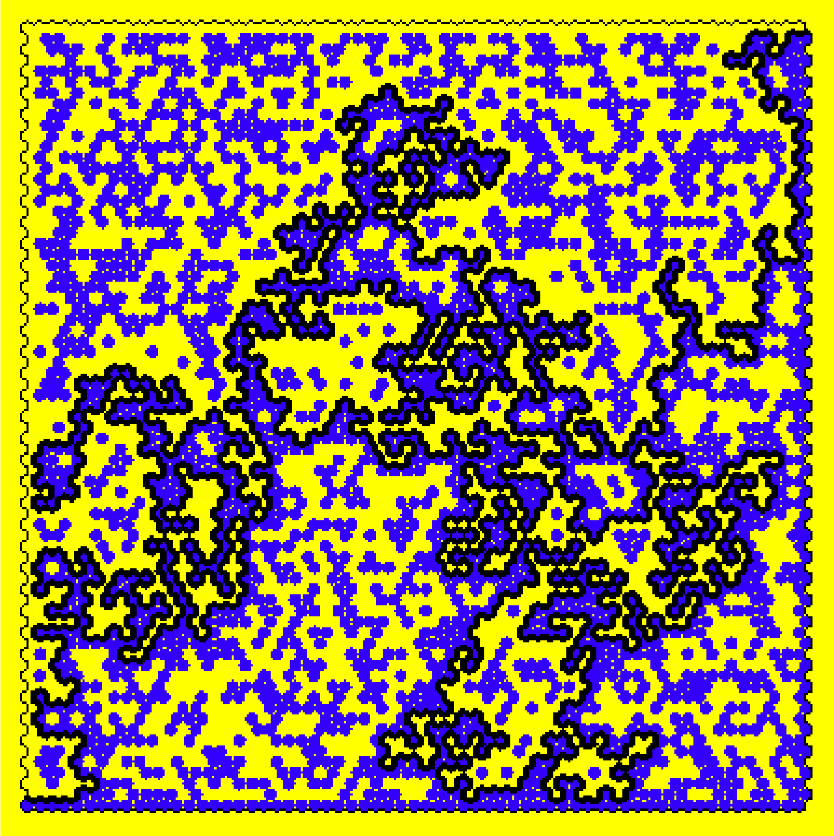
- $\alpha = 0$ by symmetry or scaling

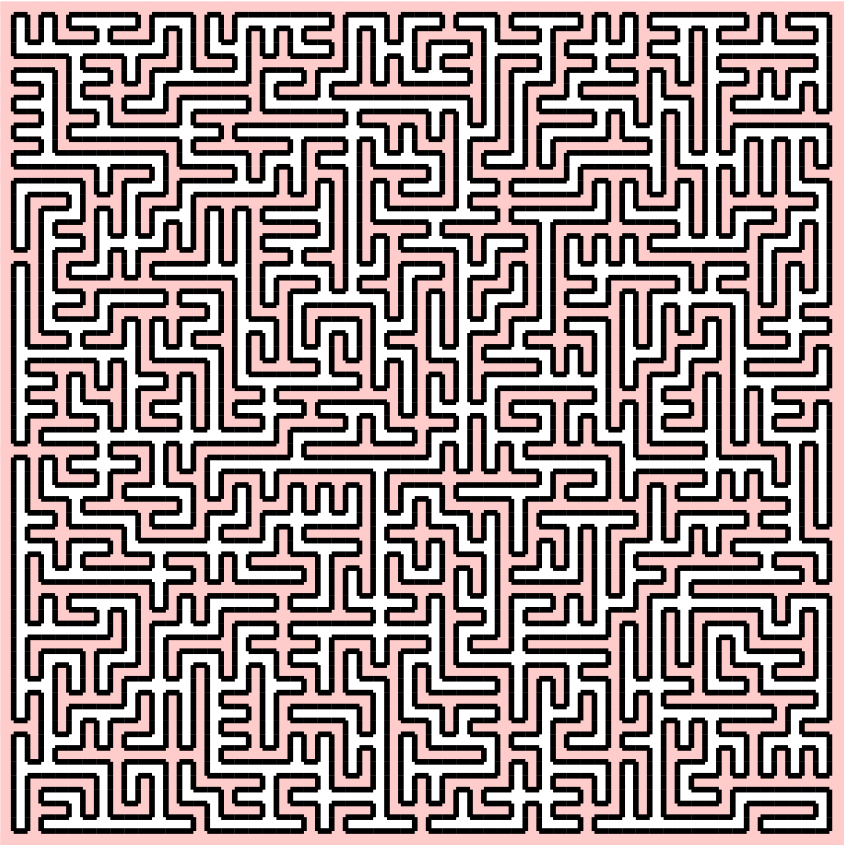
Even better: it is enough to find one conformally invariant observable

Relation to lattice models

Percolation→SLE(6) [Smirnov, 2001]

UST→SLE(8) [Lawler-Schramm-Werner, 2001]

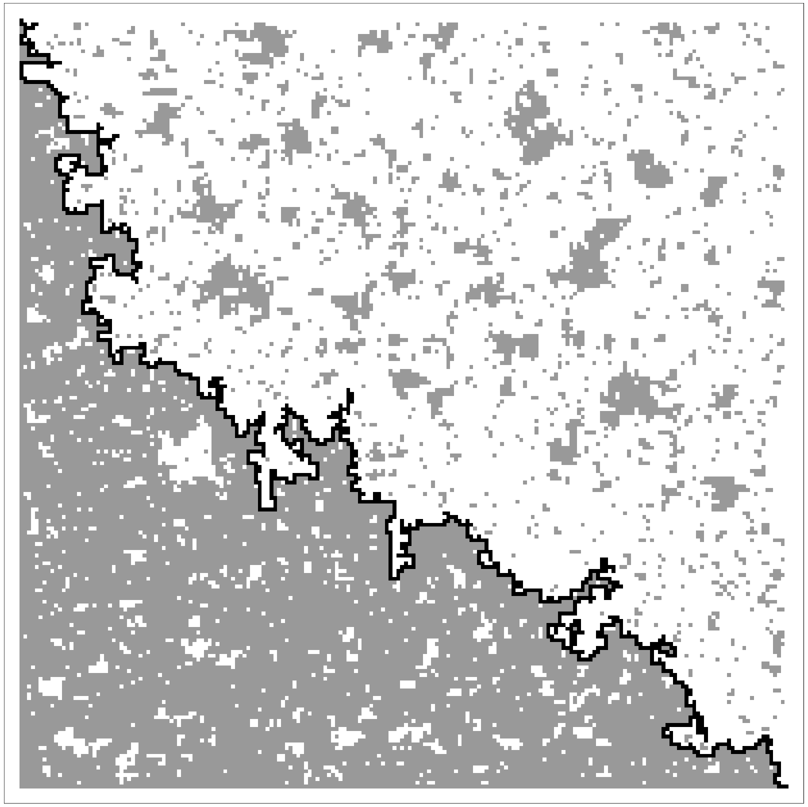




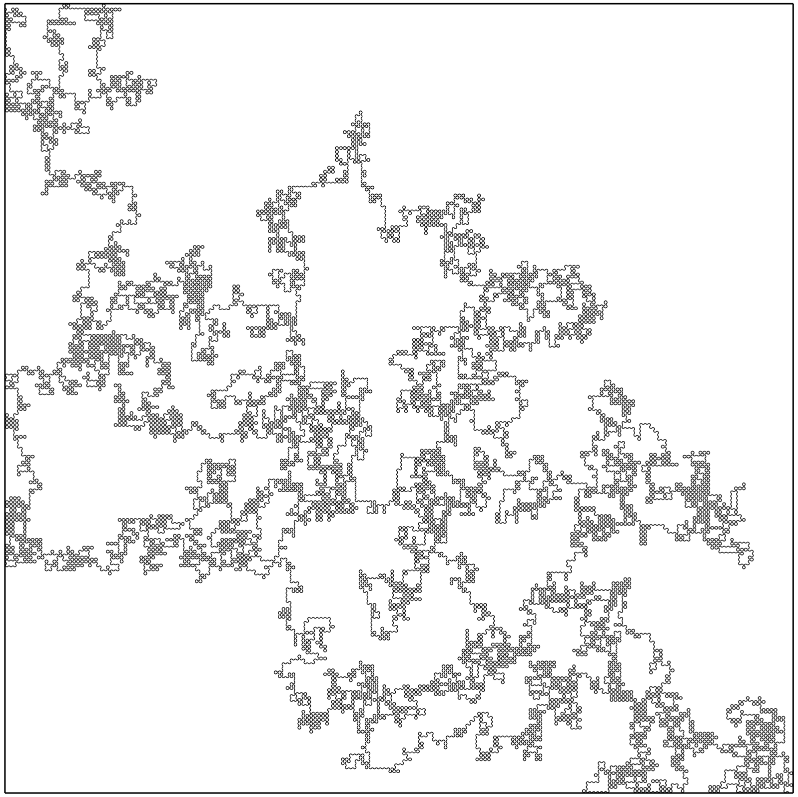
Hdim = 7/4

Relation to lattice models

[Chelkak, Smirnov 2008-10] Interfaces in critical spin-Ising and FK-Ising models on rhombic lattices converge to SLE(3) and SLE(16/3)



Hdim = 11/8



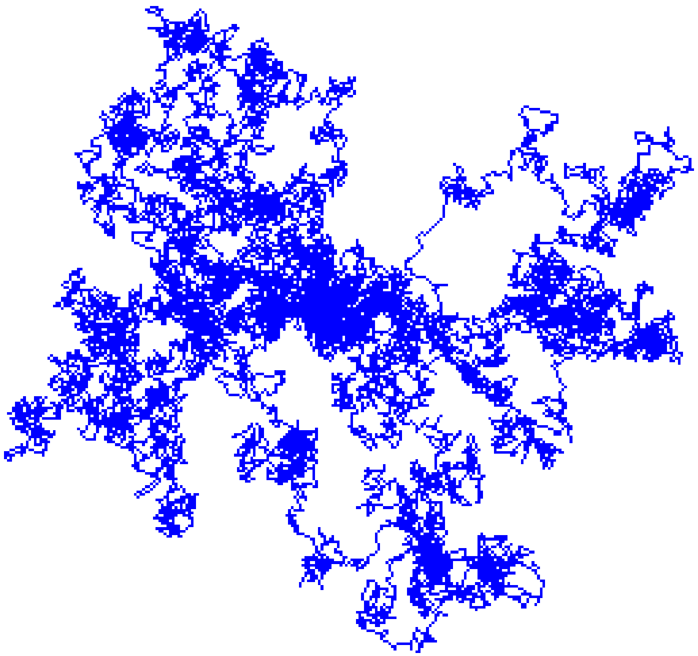
Hdim = 5/3

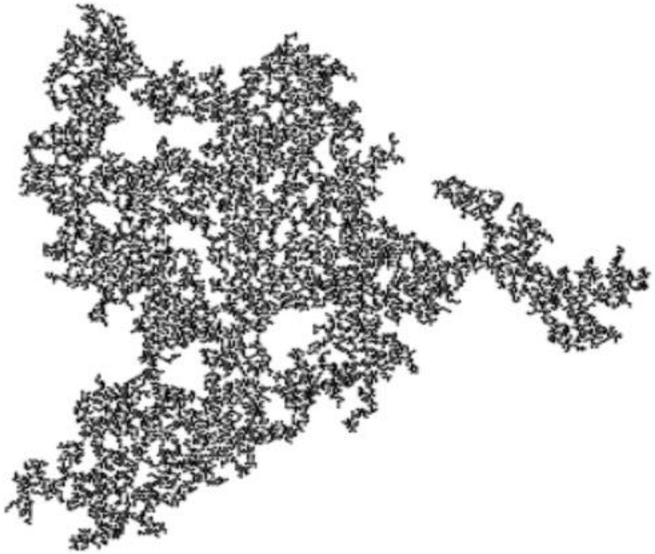
Relation to lattice models

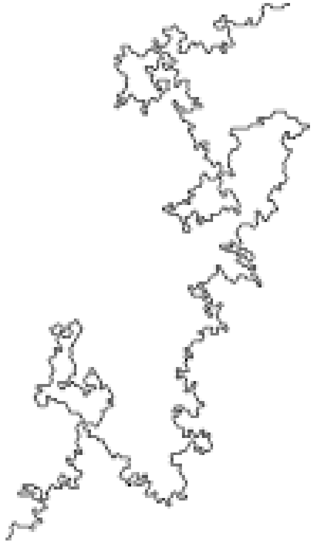
Lawler, Schramm, Werner; Smirnov

SLE(8/3) coincides with

- the boundary of the **2D Brownian motion**
- the **percolation cluster** boundary
- **(conjecturally)** the **self-avoiding walk ?**







Discrete analytic functions

New approach to 2D integrable models

- **Find an observable *F* (edge density, spin correlation, exit probability,. . .) which is discrete analytic and solves some BVP.**
- **Then in the scaling limit *F* converges to a holomorphic solution *f* of the same BVP.**

We conclude that

- ***F* has a conformally invariant scaling limit.**
- **Interfaces converge to Schramm's SLEs, allowing to calculate exponents.**
- ***F* is approximately equal to *f*, we infer some information even without SLE.**

Discrete analytic functions

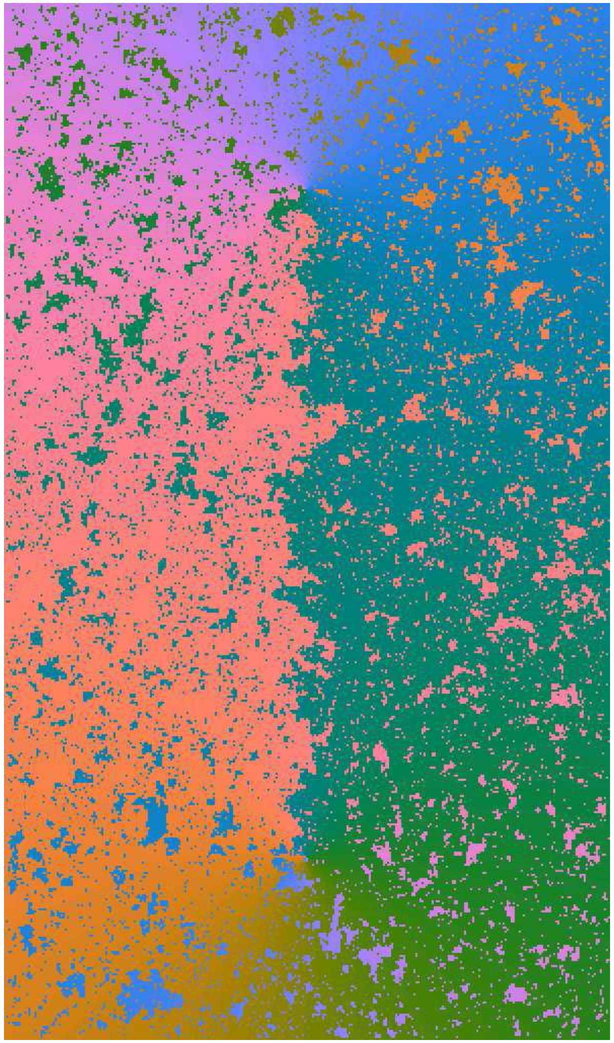
Several models were approached in this way:

- Random Walk – **[Courant, Friedrich & Lewy, 1928; ….]**
- Dimer model, UST – **[Kenyon, 1997-…]**
- Critical percolation – **[Smirnov, 2001]**
- Uniform Spanning Tree – **[Lawler, Schramm & Werner, 2003]**
- Random cluster model with q = 2 and Ising model at criticality – **[Smirnov; Chelkak & Smirnov 2006-2010]**

Most observables are CFT correlations!

Connection to SLE gives dimensions!

Energy field in the Ising model



Combination of two disorder operators is a discrete analytic Green's function solving Riemann-Hilbert BVP, then:

Theorem [Hongler - Smirnov]

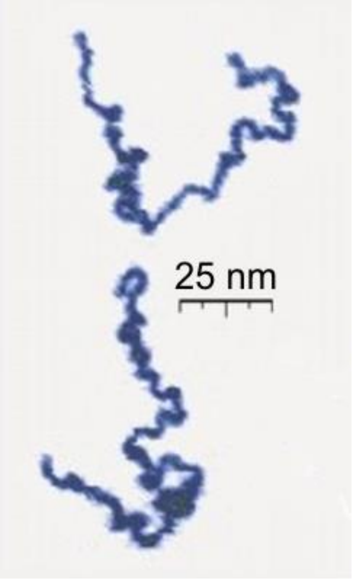
At β_c the correlation of neighboring spins satisfies (± depends on BC: + or free, ε is the lattice mesh, ρ is the hyperbolic metric element):

$$\mathbb{E}\, s(u)\, s(v) \;=\; \frac{1}{\sqrt{2}} \pm \frac{1}{\pi} \rho_\Omega(u)\, \varepsilon + O(\varepsilon^2)$$

Self-avoiding polymers

Paul Flory, 1948: Proposed to model a polymer molecule by a **self-avoiding walk** (= random walk without self-intersections)

- How many length ***n*** walks?
- What is a "typical" walk?
- What is its fractal dimension?



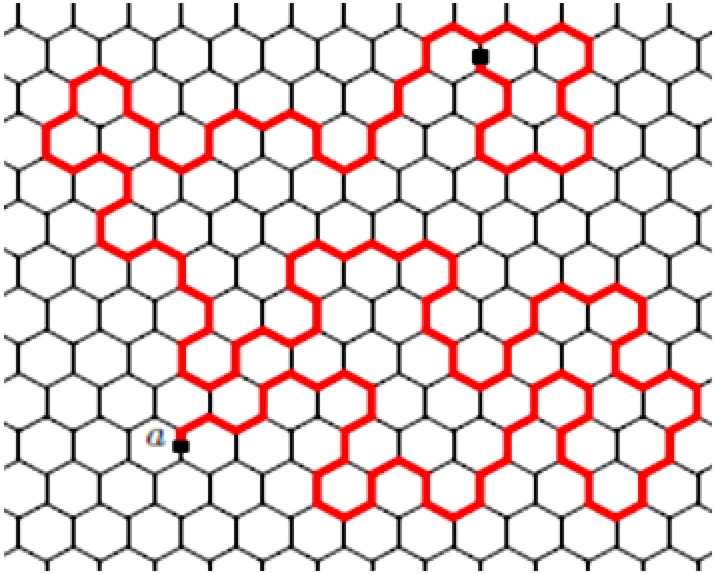
Flory: a fractal of dimension 4/3

- The argument is wrong…
- The answer is correct!

Physical explanation by **Nienhuis**, later by **Lawler, Schramm, Werner.**

Self-avoiding polymers

What is the number *C(n)* of length *n* walks?



Nienhuis predictions:

- $C(n) \approx \mu^n \cdot n^{11/32}$
- ***11/32*** is universal
- On hex lattice $\mu = \sqrt{2+\sqrt{2}}$

Theorem [Duminil-Copin & Smirnov, 2010]

On hexagonal lattice $\mu = x_c^{-1} = \sqrt{2+\sqrt{2}}$

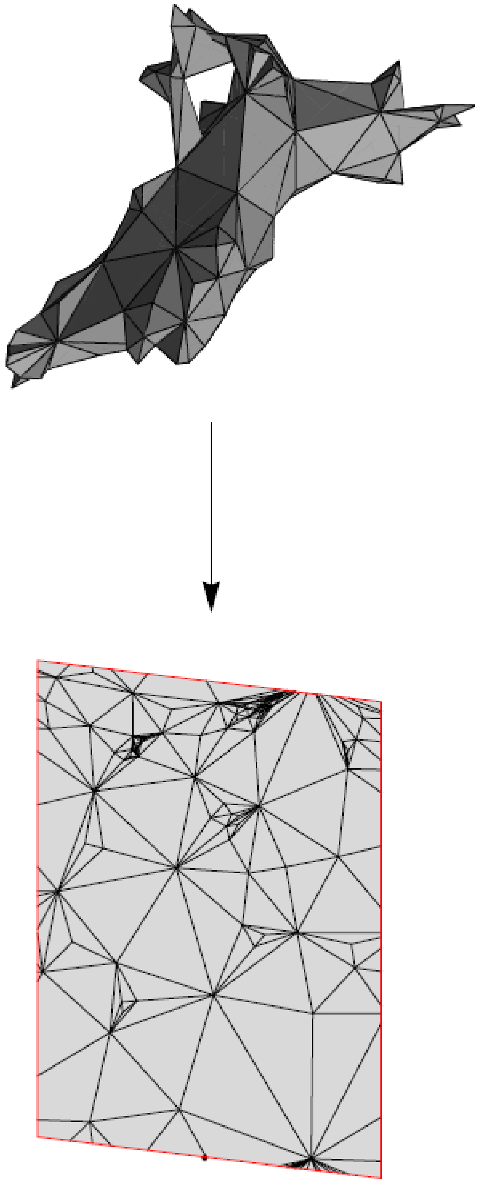
Idea: for $x=x_c$, $\lambda=\lambda_c$ discrete analyticity of

$$F(z) = \sum_{\text{self-avoiding walks } 0 \to z} \lambda^{\#\text{ turns}} x^{\text{length}}$$

Quantum gravity

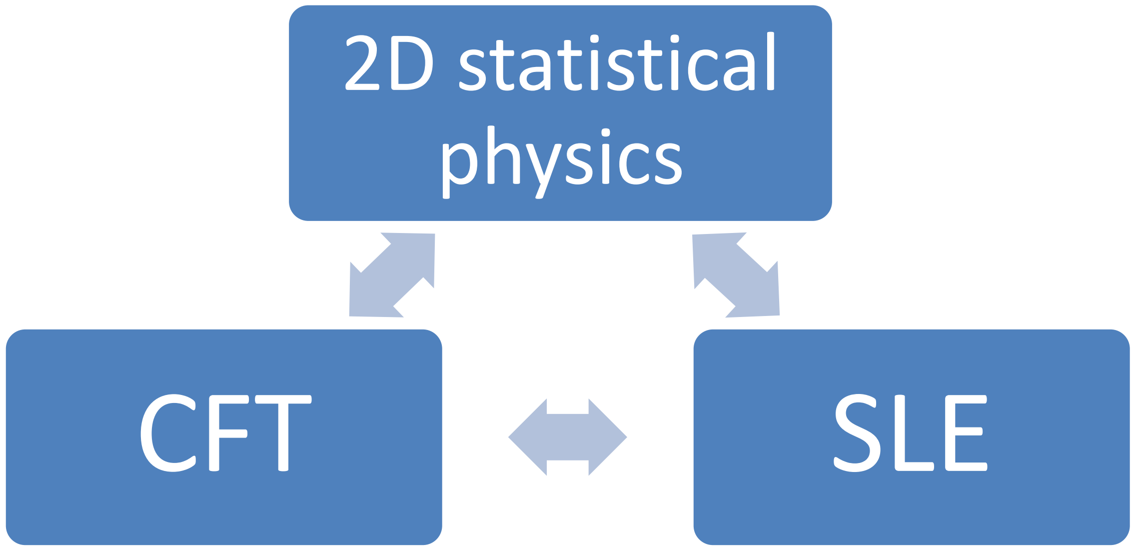
Miermont, Le Gall 2011:
Uniform random planar graph
(*taken as a metric space*)
has a universal scaling limit
(*a random metric space, topologically a plane*)

Duplantier-Sheffield, Sheffield, 2010:
Proposed relation to **SLE** and
Liouville Quantum Gravity
(*a random “metric” exp(γG)|dz|*)



from [Ambjorn-Barkeley-Budd]

Interactions



- Same objects studied from different angles
- Exchange of motivation and ideas
- Many new things, but many open questions: e.g. *SLE and CFT give different PDEs for correlations. Why solutions are the same?*

Goals for next N years

- **Prove conformal invariance for more models, establish universality**
- **Build rigorous renormalization theory**
- **Establish convergence of random planar graphs to LQG, prove LQG is a random metric**