

# Partition functions of the Ising model on some self-similar Schreier graphs

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**Abstract.** We study partition functions and thermodynamic limits for the Ising model on three families of finite graphs converging to infinite self-similar graphs. They are provided by three well-known groups realized as automorphism groups of regular rooted trees: the first Grigorchuk's group of intermediate growth; the iterated monodromy group of the complex polynomial  $z^2 - 1$  known as the "Basilica group"; and the Hanoi Towers group  $H^{(3)}$  closely related to the Sierpiński gasket.

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**Keywords.** Ising model, partition function, self-similar group, Schreier graph.

## 1. Introduction

### 1.1. The Ising model

The famous Ising model of ferromagnetism was introduced by W. Lenz in 1920, [11], and became the subject of the PhD thesis of his student E. Ising. It consists of discrete variables called spins arranged on the vertices of a finite graph  $Y$ . Each spin can take values  $\pm 1$  and only interacts with its nearest neighbours. Configuration of spins at two adjacent vertices  $i$  and  $j$  has energy  $J_{i,j} > 0$  if the spins have opposite values, and  $-J_{i,j}$  if the values are the same. Let  $|Vert(Y)| = N$ , and let  $\vec{\sigma} = (\sigma_1, \dots, \sigma_N)$  denote the configuration of spins, with  $\sigma_i \in \{\pm 1\}$ . The total energy of the system in configuration  $\vec{\sigma}$  is then

$$E(\vec{\sigma}) = - \sum_{i \sim j} J_{i,j} \sigma_i \sigma_j,$$

where we write  $i \sim j$  if the vertices  $i$  and  $j$  are adjacent in  $Y$ .

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The probability of a particular configuration at temperature  $T$  is given by

$$\mathbb{P}(\vec{\sigma}) = \frac{1}{Z} \exp(-\beta E(\vec{\sigma})),$$

where  $\beta$  is the "inverse temperature" conventionally defined as  $\beta \equiv 1/(k_B T)$ , and  $k_B$  denotes the Boltzmann constant.

As usual in statistical physics, the normalizing constant that makes the distribution above a probability measure is called the *partition function*:

$$Z = \sum_{\vec{\sigma}} \exp(-\beta E(\vec{\sigma})).$$

One can rewrite this formula by using  $\exp(K\sigma_i\sigma_j) = \cosh(K) + \sigma_i\sigma_j \sinh(K)$ , so as to get the so-called "high temperature expansion" :

$$\begin{aligned} Z &= \prod_{i \sim j} \cosh(\beta J_{i,j}) \sum_{\vec{\sigma}} \left( 1 + \sum_{i \sim j} \sigma_i \sigma_j \tanh(\beta J_{i,j}) \right. \\ &+ \sum_{\substack{i \sim j \\ l \sim m}} (\sigma_i \sigma_j)(\sigma_l \sigma_m) \tanh(\beta J_{i,j}) \tanh(\beta J_{l,m}) + \dots \left. \right). \end{aligned}$$

After changing the order of summation, observe that the non-vanishing terms in  $Z$  are exactly those with an even number of occurrences of each  $\sigma_i$ . We can interpret this by saying that non-vanishing terms in this expression are in bijection with *closed polygons* of  $Y$ , i.e., subgraphs in which every vertex has even degree. Consequently we can rewrite  $Z$  as

$$Z = \left( \prod_{i \sim j} \cosh(\beta J_{i,j}) \right) \cdot 2^N \sum_{X \text{ closed polygon of } Y} \prod_{(i,j) \in \text{Edges}(X)} \tanh(\beta J_{i,j}), \quad (1)$$

where in the RHS we have the *generating series of closed polygons* of  $Y$  with weighted edges, the weight of an edge  $(i, j)$  being  $\tanh(\beta J_{i,j})$ .

In the case of constant  $J$ , the above expression specializes to

$$Z = (\cosh(\beta J))^{|Edges(Y)|} \cdot 2^N \Gamma^{cl}(\tanh(\beta J))$$

with  $\Gamma^{cl}(z) = \sum_{n=0}^{\infty} A_n^{cl} z^n$ , where  $A_n^{cl}$  is the number of closed polygons with  $n$  edges in  $Y$ . (In particular, the total number of closed polygons is given by  $\Gamma^{cl}(1)$ .)

From the physics viewpoint it is interesting to study the model when the system (i.e. the number of vertices in the graph) grows. One way to express this mathematically is to consider growing sequences of finite graphs converging to an infinite graph. If the limit

$$\lim_{n \rightarrow \infty} \frac{\log(Z_n)}{|Vert(Y_n)|}$$

for a sequence of finite graphs  $Y_n$  with partition functions  $Z_n$  exists, it is called the *thermodynamic limit*.

In the thermodynamic limit, at some critical temperature, a phase transition can occur between ordered and disordered phase in the behaviour of the model. Existence of a phase transition depends on the graph. In his thesis in 1925, Ising studied the case of one-dimensional Euclidean lattice; computed the partition functions and the thermodynamic limit and showed that there is no phase transition [10]. The Ising model in  $\mathbb{Z}^d$  with  $d \geq 2$  undergoes a phase transition. This was first established for  $d = 2$  by R. Peierls. At high temperature,  $T > T_C$ , the clusters of vertices with equal spins grow similarly for two different types of spin, whereas for  $T < T_C$  the densities of the types of spin are different and the system “chooses” one of them.

The infinite graphs that will be studied in this paper all have *finite order  $\mathcal{R}$  of ramification*, i.e., for any connected bounded part  $\mathcal{X}$  of the graph there exists a set  $\mathcal{A}$  of at most  $\mathcal{R}$  vertices such that any infinite self-avoiding path in the graph that begins in  $\mathcal{X}$  necessarily goes through  $\mathcal{A}$ . Finite order of ramification ensures that the critical temperature is  $T=0$ , and there is no phase transition in the Ising model (see [6]).

Typically, an infinite lattice is viewed as the limit of an exhaustive sequence of finite subgraphs. This is a simple example of the so-called *pointed Hausdorff-Gromov convergence* (see Proposition 1.3 below). Another typical case of this convergence is that of covering graph sequences. In this paper we will be studying the Ising model on families of finite graphs coming from the theory of *self-similar groups* (see Definition 1.2 below), and their infinite limits. Any finitely generated group of automorphisms of a regular rooted tree provides us with a sequence of finite graphs describing the action of the group on the levels of the tree. When the action is self-similar the sequence converges in the above sense to infinite graphs describing the action of the group on the boundary of the tree. The graphs that we study here are determined by group actions, and so their edges are labeled naturally by the generators of the acting group. Different weights on the edges lead to weighted partition functions, with  $J_{i,j}$  depending on the label of the edge  $(i, j)$ .

## 1.2. Groups of automorphisms of rooted regular trees

Let  $T$  be the infinite regular rooted tree of degree  $q$ , i.e., the rooted tree in which each vertex has  $q$  children. Each vertex of the  $n$ -th level of the tree can be regarded as a word of length  $n$  in the alphabet  $X = \{0, 1, \dots, q-1\}$ . Moreover, one can identify the set  $X^\omega$  of infinite words in  $X$  with the set  $\partial T$  of infinite geodesic rays starting at the root of  $T$ . Now let  $G < \text{Aut}(T)$  be a group acting on  $T$  by automorphisms, generated by a finite symmetric set of generators  $S$ . Suppose moreover that the action is transitive on each level of the tree.

**Definition 1.1.** *The  $n$ -th Schreier graph  $\Sigma_n$  of the action of  $G$  on  $T$ , with respect to the generating set  $S$ , is a graph whose vertex set coincides with the set of vertices of the  $n$ -th level of the tree, and two vertices  $u, v$  are adjacent if and only if there exists  $s \in S$  such that  $s(u) = v$ . If this is the case, the edge joining  $u$  and  $v$  is*

labeled by  $s$ . For any infinite ray  $\xi \in \partial T$ , the orbital Schreier graph  $\Sigma_\xi$  has vertices  $G \cdot \xi$  and edges determined by the action of generators, as above.

The vertices of  $\Sigma_n$  are labeled by words of length  $n$  in  $X$  and the edges are labeled by elements of  $S$ . The Schreier graph is thus a regular graph of degree  $d = |S|$  with  $q^n$  vertices, and it is connected since the action of  $G$  is level-transitive.

**Definition 1.2.** *A finitely generated group  $G < \text{Aut}(T)$  is self-similar if, for all  $g \in G, x \in X$ , there exist  $h \in G, y \in X$  such that*

$$g(xw) = yh(w),$$

for all finite words  $w$  in the alphabet  $X$ .

Self-similarity implies that  $G$  can be embedded into the wreath product  $\text{Sym}(q) \wr G$ , where  $\text{Sym}(q)$  denotes the symmetric group on  $q$  elements, so that any automorphism  $g \in G$  can be represented as

$$g = \tau(g_0, \dots, g_{q-1}),$$

where  $\tau \in \text{Sym}(q)$  describes the action of  $g$  on the first level of  $T$  and  $g_i \in G, i = 0, \dots, q-1$  is the restriction of  $g$  on the full subtree of  $T$  rooted at the vertex  $i$  of the first level of  $T$  (observe that any such subtree is isomorphic to  $T$ ). Hence, if  $x \in X$  and  $w$  is a finite word in  $X$ , we have

$$g(xw) = \tau(x)g_x(w).$$

It is not difficult to see that the orbital Schreier graphs of a self-similar group are infinite and that the finite Schreier graphs  $\{\Sigma_n\}_{n=1}^\infty$  form a sequence of graph coverings (see [12] and references therein for more information about this interesting class of groups, also known as *automata groups*.)

Take now an infinite ray  $\xi \in X^\omega$  and denote by  $\xi_n$  the  $n$ -th prefix of the word  $\xi$ . Then the sequence of rooted graphs  $\{(\Sigma_n, \xi_n)\}$  converges to the infinite rooted graph  $(\Sigma_\xi, \xi)$  in the space of rooted graphs, in the following sense.

**Proposition 1.3.** ([9], Chapter 3.) *Let  $\mathcal{X}$  be the space of connected graphs having a distinguished vertex called the root;  $\mathcal{X}$  can be endowed with the following metric: given two rooted graphs  $(Y_1, v_1)$  and  $(Y_2, v_2)$ ,*

$$\text{Dist}((Y_1, v_1), (Y_2, v_2)) := \inf \left\{ \frac{1}{r+1}; B_{Y_1}(v_1, r) \text{ is isomorphic to } B_{Y_2}(v_2, r) \right\}$$

where  $B_Y(v, r)$  is the ball of radius  $r$  in  $Y$  centered in  $v$ . Under the assumption of uniformly bounded degrees,  $\mathcal{X}$  endowed with the metric  $\text{Dist}$  is a compact space.

### 1.3. Plan of the paper

Our aim in this paper is to study the Ising model on the Schreier graphs of three key examples of self-similar groups:

- the first Grigorchuk's group of intermediate (i.e., strictly between polynomial and exponential) growth (see [7] for a detailed account and further references);

- the “Basilica” group that can be described as the iterated monodromy group of the complex polynomial  $z^2 - 1$  (see [12] connections of self-similar groups to complex dynamics);

- and the Hanoi Towers group  $H^{(3)}$  whose action on the ternary tree models the famous Hanoi Towers game on three pegs, see [8].

It is known [1] that the infinite Schreier graphs associated with these groups (and, more generally, with all groups generated by *bounded automata*) have finite order of ramification. Hence the Ising model on these graphs exhibits no phase transition.

We first compute the partition functions and prove existence of thermodynamic limit for the model where interactions between vertices are constant: in Section 2 we treat the Grigorchuk’s group and the Basilica group, and in Section 3 the Hanoi Towers group  $H^{(3)}$  and its close relative the Sierpiński gasket are considered.

In Section 4, we study weighted partition functions for all the graphs previously considered, and we find the distribution of the number of occurrences of a fixed weight in a random configuration. The relation between the Schreier graphs of  $H^{(3)}$  and the Sierpiński gasket is also discussed from the viewpoint of Fisher’s theorem establishing a correspondence between the Ising model on the Sierpiński gasket and the dimers model on the Schreier graphs of  $H^{(3)}$ .

**1.4. Acknowledgments**

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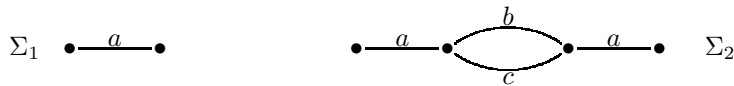
**2. Partition functions and thermodynamic limit for the Grigorchuk’s group and for the Basilica group**

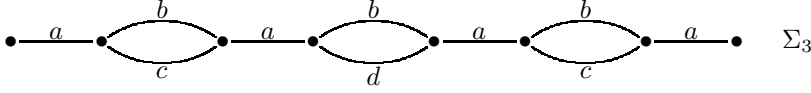
**2.1. Grigorchuk’s group**

This group admits the following easy description as a self-similar subgroup of automorphisms of the binary tree. It is generated by the elements

$$a = \epsilon(id, id), \quad b = e(a, c), \quad c = e(a, d), \quad d = e(id, b),$$

where  $e$  and  $\epsilon$  are respectively the trivial and the non-trivial permutations in  $Sym(2)$ . These recursive formulae allow easily to construct finite Schreier graphs for the action of the group on the binary tree. Here are three first graphs in the sequence, with loops erased.





In general, the Schreier graph  $\Sigma_n$  has the same linear shape, with  $2^{n-1}$  simple edges, all labeled by  $a$ , and  $2^{n-1} - 1$  cycles of length 2. It is therefore very easy to compute the generating function of closed polygons of  $\Sigma_n$ , for each  $n$ .

**Theorem 2.1.** *The generating function of closed polygons for the  $n$ -th Schreier graph of the Grigorchuk group is  $\Gamma_n^{cl}(z) = (1+z^2)^{2^{n-1}-1}$ . In particular, the number of all closed polygons in  $\Sigma_n$  is  $2^{2^{n-1}-1}$ .*

The partition function of the Ising model is given by

$$Z_n = \cosh(\beta J)^{3 \cdot 2^{n-1} - 2} \cdot 2^{2^n} \cdot (1 + \tanh^2(\beta J))^{2^{n-1} - 1}$$

and the thermodynamic limit exists and satisfies:

$$\lim_{n \rightarrow \infty} \frac{\log(Z_n)}{2^n} = \frac{3}{2} \log(\cosh(\beta J)) + \log 2 + \frac{1}{2} \log(1 + \tanh^2(\beta J)).$$

*Proof.* It is clear that a closed polygon in  $\Sigma_n$  is the union of 2-cycles. So we can easily compute the number  $A_{k,n}^{cl}$  of closed polygons with  $k$  edges in  $\Sigma_n$ , for all  $k = 0, 1, \dots, 2^n - 2$ . For  $k$  odd, one has  $A_{k,n}^{cl} = 0$ . For  $k$  even, we have to choose  $\frac{k}{2}$  cycles of length 2 to get a closed polygon with  $k$  edges, which implies

$$A_{k,n}^{cl} = \binom{2^{n-1} - 1}{\frac{k}{2}}.$$

So the generating function of closed polygons for  $\Sigma_n$  is given by

$$\Gamma_n^{cl}(z) = \sum_{k=0}^{2^{n-1}-1} \binom{2^{n-1}-1}{k} z^{2k} = (1+z^2)^{2^{n-1}-1}.$$

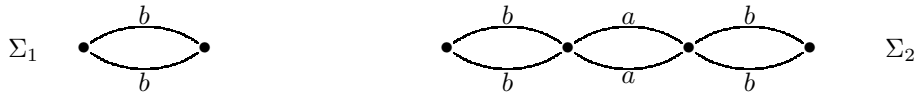
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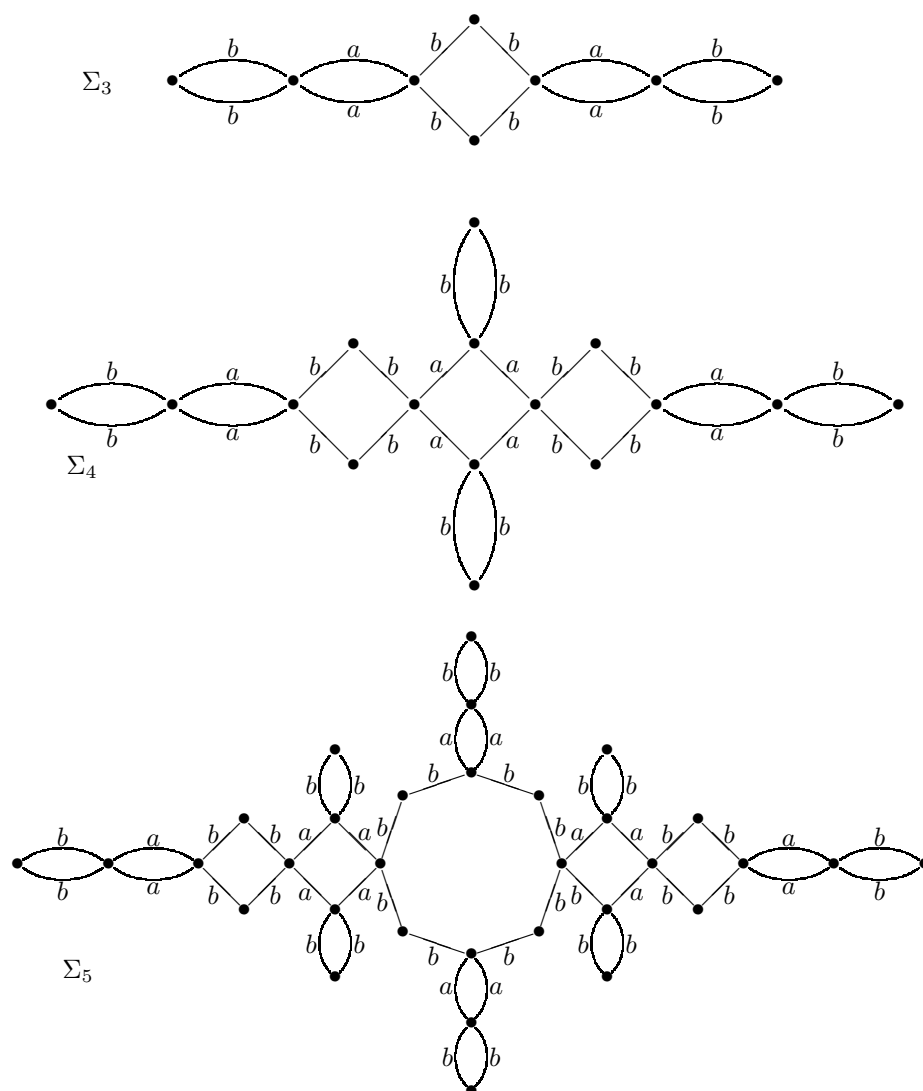
## 2.2. The Basilica group

The Basilica group is a self-similar group of automorphisms of the binary tree generated by the elements

$$a = e(b, id), \quad b = \epsilon(a, id).$$

The following pictures of graphs  $\Sigma_n$  for  $n = 1, 2, 3, 4, 5$  with loops erased give an idea of how finite Schreier graphs of the Basilica group look like. See [4] for a comprehensive analysis of finite and infinite Schreier graphs of this group. Note also that  $\{\Sigma_n\}_{n=1}^{\infty}$  is an approximating sequence for the Julia set of the polynomial  $z^2 - 1$ , the famous ‘‘Basilica’’ fractal (see [12]).





In general, it follows from the recursive definition of the generators, that each  $\Sigma_n$  is a cactus, i.e., a union of cycles (in this example all of them are of length power of 2) arranged in a tree-like way. The maximal length of a cycle in  $\Sigma_n$  is  $2^{\lceil \frac{n}{2} \rceil}$ . Denote by  $a_j^i$  the number of cycles of length  $j$  labeled by  $a$  in  $\Sigma_i$  and analogously denote by  $b_j^i$  the number of cycles of length  $j$  labeled by  $b$  in  $\Sigma_i$ .

**Proposition 2.2.** *For any  $n \geq 4$  consider the Schreier graph  $\Sigma_n$  of the Basilica group.*

For each  $k \geq 1$ , the number of cycles of length  $2^k$  labeled by  $a$  is

$$a_{2^k}^n = \begin{cases} 2^{n-2k-1} & \text{for } 1 \leq k \leq \frac{n-1}{2} - 1, \\ 2 & \text{for } k = \lfloor \frac{n}{2} \rfloor \end{cases}, \quad \text{for } n \text{ odd,}$$

$$a_{2^k}^n = \begin{cases} 2^{n-2k-1} & \text{for } 1 \leq k \leq \frac{n}{2} - 1, \\ 1 & \text{for } k = \frac{n}{2} \end{cases}, \quad \text{for } n \text{ even}$$

and the number of cycles of length  $2^k$  labeled by  $b$  is

$$b_{2^k}^n = \begin{cases} 2^{n-2k} & \text{for } 1 \leq k \leq \frac{n-1}{2} - 1 \\ 2 & \text{for } k = \lfloor \frac{n}{2} \rfloor \\ 1 & \text{for } k = \lceil \frac{n}{2} \rceil \end{cases}, \quad \text{for } n \text{ odd,}$$

$$b_{2^k}^n = \begin{cases} 2^{n-2k} & \text{for } 1 \leq k \leq \frac{n}{2} - 1, \\ 2 & \text{for } k = \frac{n}{2} \end{cases}, \quad \text{for } n \text{ even.}$$

*Proof.* The recursive formulae for the generators imply that, for each  $n \geq 3$ , one has

$$a_2^n = b_2^{n-1} \quad \text{and} \quad b_2^n = a_1^{n-1} = 2^{n-2}$$

and in general  $a_{2^k}^n = a_2^{n-2(k-1)}$  and  $b_{2^k}^n = b_2^{n-2(k-1)}$ . In particular, for each  $n \geq 4$ , the number of 2-cycles labeled by  $a$  is  $2^{n-3}$  and the number of 2-cycles labeled by  $b$  is  $2^{n-2}$ . More generally, the number of cycles of length  $2^k$  is given by

$$a_{2^k}^n = 2^{n-2k-1}, \quad b_{2^k}^n = 2^{n-2k},$$

where the last equality is true if  $n - 2k + 2 \geq 4$ , i.e. for  $k \leq \frac{n}{2} - 1$ . Finally, for  $n$  odd, there is only one cycle of length  $2^{\lceil \frac{n}{2} \rceil}$  labeled by  $b$  and four cycles of length  $2^{\lfloor \frac{n}{2} \rfloor}$ , two of them labeled by  $a$  and two labeled by  $b$ ; for  $n$  even, there are three cycles of length  $2^{\frac{n}{2}}$ , two of them labeled by  $b$  and one labeled by  $a$ .  $\square$

**Corollary 2.3.** *For each  $n \geq 4$ , the number of cycles labeled by  $a$  in the Schreier graph  $\Sigma_n$  of the Basilica group is*

$$\begin{cases} \frac{2^{n-1}+2}{3} & \text{for } n \text{ odd,} \\ \frac{2^{n-1}+1}{3} & \text{for } n \text{ even.} \end{cases}$$

and the number of  $b$ -cycles in  $\Sigma_n$  is

$$\begin{cases} \frac{2^n+1}{3} & \text{for } n \text{ odd,} \\ \frac{2^n+2}{3} & \text{for } n \text{ even.} \end{cases}$$

The total number of cycles of length  $\geq 2$  is  $2^{n-1} + 1$  and the total number of edges, without loops, is  $3 \cdot 2^{n-1}$ .

The computations above lead to the following formula for the partition function of the Ising model on the Schreier graphs  $\Sigma_n$  associated with the action of the Basilica group.



**Theorem 2.4.** *The partition function of the Ising model on the  $n$ -th Schreier graph  $\Sigma_n$  of the Basilica group is*

$$Z_n = 2^{2^n} \cdot \cosh(\beta J)^{3 \cdot 2^{n-1}} \cdot \Gamma_n^{cl}(\tanh(\beta J)),$$

where  $\Gamma_n^{cl}(z)$  is the generating function of closed polygons for  $\Sigma_n$  given by

$$\Gamma_n^{cl}(z) = \prod_{k=1}^{\frac{n-1}{2}-1} (1+z^{2^k})^{3 \cdot 2^{n-2k-1}} \cdot \left(1+z^{2^{\frac{n-1}{2}}}\right)^4 \cdot \left(1+z^{2^{\frac{n+1}{2}}}\right),$$

for  $n \geq 5$  odd and

$$\Gamma_n^{cl}(z) = \prod_{k=1}^{\frac{n}{2}-1} (1+z^{2^k})^{3 \cdot 2^{n-2k-1}} \cdot \left(1+z^{2^{\frac{n}{2}}}\right)^3,$$

for  $n \geq 4$  even. Moreover,  $\Gamma_1^{cl} = 1+z^2$ ,  $\Gamma_2^{cl} = (1+z^2)^3$  and  $\Gamma_3^{cl} = (1+z^2)^4(1+z^4)$ .

*Proof.* Recall that  $Z_n = 2^{|\text{Vert}(\Sigma_n)|} \cosh(\beta J)^{|\text{Edges}(\Sigma_n)|} \cdot \Gamma_n^{cl}(\tanh(\beta J))$ , where  $\Gamma_n^{cl}(z)$  is the generating function of closed polygons in  $\Sigma_n$ . In our case we have  $|\text{Edges}(\Sigma_n)| = 3 \cdot 2^{n-1}$  and  $|\text{Vert}(\Sigma_n)| = 2^n$ .

The formulae for  $\Gamma_n^{cl}(z)$  with  $n = 1, 2, 3$  can be directly verified. For  $n \geq 4$ , we can use Proposition 2.2. Since the length of each cycle of  $\Sigma_n$  is even, it is clear that the coefficient  $A_{k,n}^{cl}$  is zero for every odd  $k$ . The coefficient  $A_{k,n}^{cl}$  is nonzero for every even  $k$  such that  $0 \leq k \leq 3 \cdot 2^{n-1}$ . In fact,  $3 \cdot 2^{n-1}$  is the total number of edges of  $\Sigma_n$  ( $2^n$  labeled by  $b$  and  $2^{n-1}$  labeled by  $a$ ). By taking the exact number of cycles of length  $2^i$  in  $\Sigma_n$ , we get the assertion.  $\square$

**Theorem 2.5.** *The thermodynamic limit  $\lim_{n \rightarrow \infty} \frac{\log(Z_n)}{|\text{Vert}(\Sigma_n)|}$  exists.*

*Proof.* Since  $|\text{Edges}(\Sigma_n)| = 3 \cdot 2^{n-1}$  and  $|\text{Vert}(\Sigma_n)| = 2^n$ , the limit reduces to (choosing, for example,  $n$  even)

$$\log(2) + \frac{3}{2} \log(\cosh(\beta J)) + \lim_{n \rightarrow \infty} \frac{\log(\Gamma_n^{cl}(z))}{2^n},$$

where  $z = \tanh(\beta J)$  takes values between 0 and 1. Now

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log(\Gamma_n^{cl}(z))}{2^n} &= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^{\frac{n}{2}-1} 3 \cdot 2^{n-2k-1} \log(1+z^{2^k}) + 3 \log(1+z^{2^{\frac{n}{2}}})}{2^n} \\ &= \frac{3}{2} \sum_{k=1}^{\infty} \frac{\log(1+z^{2^k})}{4^k} + \lim_{n \rightarrow \infty} \frac{3 \log(1+z^{2^{\frac{n}{2}}})}{2^n} \\ &\leq \frac{3}{2} \sum_{k=1}^{\infty} \frac{\log(2)}{4^k} < \infty, \end{aligned}$$

giving the assertion.  $\square$

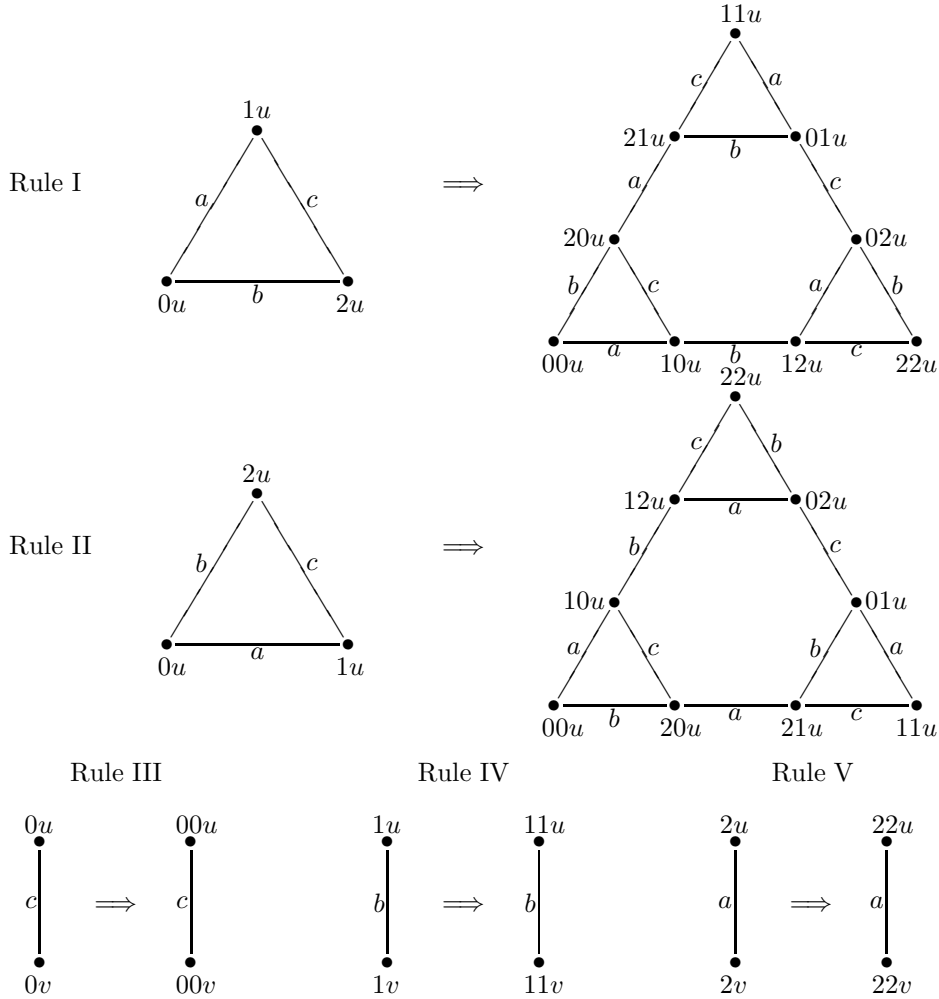
### 3. Partition functions and thermodynamic limits for the Hanoi Towers group $H^{(3)}$ and for the Sierpiński gasket

#### 3.1. Hanoi Towers group $H^{(3)}$

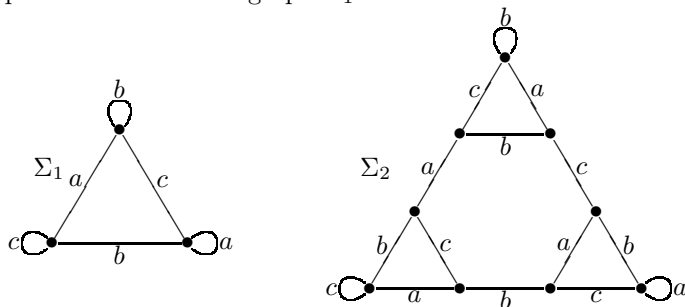
The Hanoi Towers group  $H^{(3)}$  is generated by three automorphisms of the ternary rooted tree admitting the following self-similar presentation [8]:

$$a = (01)(id, id, a) \quad b = (02)(id, b, id) \quad c = (12)(c, id, id),$$

where  $(01)$ ,  $(02)$  and  $(12)$  are transpositions in  $Sym(3)$ . The associated Schreier graphs are self-similar in the sense of [13], that is, each  $\Sigma_{n+1}$  contains three copies of  $\Sigma_n$  glued together by three edges. These graphs can be recursively constructed via the following substitutional rules [8]:

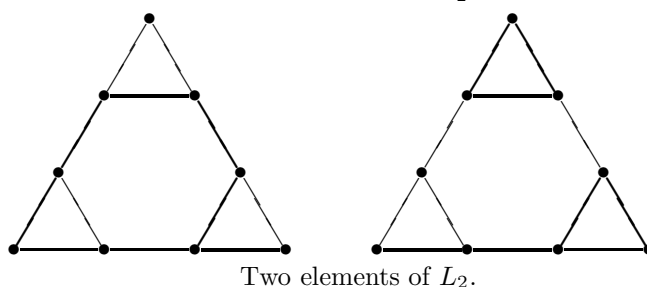
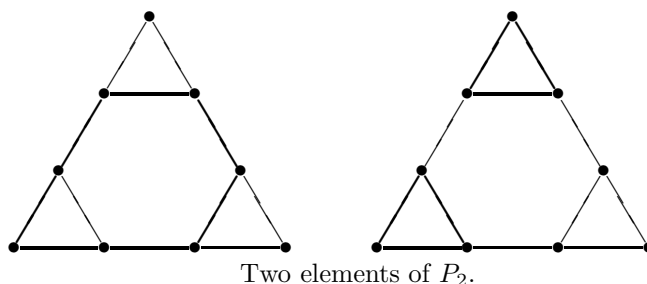


The starting point is the Schreier graph  $\Sigma_1$  of the first level.



**Remark 3.1.** Observe that, for each  $n \geq 1$ , the graph  $\Sigma_n$  has three loops, at the vertices  $0^n, 1^n$  and  $2^n$ , labeled by  $c, b$  and  $a$ , respectively. Moreover, these are the only loops in  $\Sigma_n$ . The Ising model will be studied on  $\Sigma_n$  considered without loops.

Let us now proceed to the computation of closed polygons in  $\Sigma_n$ . Denote by  $P_n$  the set of closed polygons in  $\Sigma_n$ , and by  $L_n$  the set of all subgraphs of  $\Sigma_n$  consisting of self-avoiding paths joining the left-most vertex to the right-most vertex in  $\Sigma_n$ , together with closed polygons having no common edge with the path.



Each closed polygon in  $\Sigma_n$  can be obtained in the following way: either it is a union of closed polygons living in the three copies  $\Sigma_{n-1}$  or it contains the three special edges joining the three subgraphs isomorphic to  $\Sigma_{n-1}$ . The subgraphs of the first type can be identified with the elements of the set  $P_{n-1}^3$ , whereas the other ones are obtained by joining three elements in  $L_{n-1}$ , each one belonging to one of the

three copies of  $\Sigma_{n-1}$ , so that they can be identified with elements of the set  $L_{n-1}^3$ . This gives

$$P_n = P_{n-1}^3 \coprod L_{n-1}^3. \quad (2)$$

On the other hand, each element in  $L_n$  can be described in the following way: if it contains a path that does not reach the up-most triangle isomorphic to  $\Sigma_{n-1}$ , it can be regarded as an element in  $L_{n-1}^2 \times P_{n-1}$ ; if it contains a path which goes through all three copies of  $\Sigma_{n-1}$ , then it is in  $L_{n-1}^3$ . This gives

$$L_n = (L_{n-1}^2 \times P_{n-1}) \coprod L_{n-1}^3, \quad (3)$$

from which we deduce

**Proposition 3.2.** *For each  $n \geq 1$  the number  $|P_n|$  of closed polygons in the Schreier graph  $\Sigma_n$  of  $H^{(3)}$  is  $2^{\frac{3^n-1}{2}}$ .*

We are now ready to compute the generating series for closed polygons and the partition function of the Ising model on Schreier graphs of  $H^{(3)}$ . Denote by  $\Gamma_n^{cl}(z)$  the generating function of the set of subgraphs in  $P_n$  and by  $\Upsilon_n(z)$  the generating function of the set of subgraphs in  $L_n$ . The equation (2) gives

$$\Gamma_n^{cl}(z) = (\Gamma_{n-1}^{cl}(z))^3 + z^3 \Upsilon_{n-1}^3(z). \quad (4)$$

The factor  $z^3$  in (4) is explained by the fact that each term in  $\Upsilon_n^3(z)$  corresponds to a set of edges that becomes a closed polygon after adding the three special edges connecting the three copies of  $\Sigma_{n-1}$ . We have consequently that the second summand is the generating function for the closed polygons containing the three special edges. Analogously, from (3) we have

$$\Upsilon_n(z) = z \Upsilon_{n-1}^2(z) \Gamma_{n-1}^{cl}(z) + z^2 \Upsilon_{n-1}^3(z). \quad (5)$$

**Theorem 3.3.** *For each  $n \geq 1$ , the partition function of the Ising model on the Schreier graph  $\Sigma_n$  of the group  $H^{(3)}$  is*

$$Z_n = 2^{3^n} \cdot \cosh(\beta J)^{\frac{3^{n+1}-3}{2}} \cdot \Gamma_n^{cl}(\tanh(\beta J)),$$

with

$$\Gamma_n^{cl}(z) = z^{3^n} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1),$$

where  $\psi_1(z) = \frac{z+1}{z}$  and  $\psi_k(z) = \psi_{k-1}^2(z) - 3\psi_{k-1}(z) + 4$ , for each  $k \geq 2$ .

*Proof.* Recall that  $Z_n = 2^{|Vert(\Sigma_n)|} \cosh(\beta J)^{|Edges(\Sigma_n)|} \cdot \Gamma_n^{cl}(\tanh(\beta J))$ , where  $\Gamma_n^{cl}(z)$  is the generating function of closed polygons in  $\Sigma_n$ . In our case we have  $|Edges(\Sigma_n)| = \frac{3^{n+1}-3}{2}$  and  $|Vert(\Sigma_n)| = 3^n$ .

We know that the generating functions  $\Gamma_n^{cl}(z)$  and  $\Upsilon_n(z)$  satisfy equations (4) and (5), and the initial conditions can be easily computed as:

$$\Gamma_1^{cl}(z) = 1 + z^3 \quad \Upsilon_1(z) = z^2 + z.$$

We now show by induction on  $n$  that the solutions of the system of equations (4) and (5) are

$$\begin{cases} \Gamma_n^{cl}(z) = z^{3^n} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1) \\ \Upsilon_n(z) = z^{3^{n-1}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z). \end{cases}$$

For  $n = 1$ , we get  $\Gamma_1^{cl}(z) = z^3 \psi_1(z)(\psi_2(z) - 1) = z^3 + 1$  and  $\Upsilon_1(z) = z^2 \psi_1(z) = z^2 + z$  and so the claim is true. Now suppose that the assertion is true for  $n$  and let us show that it is true for  $n + 1$ . One gets:

$$\begin{aligned} \Gamma_{n+1}^{cl}(z) &= \left( z^{3^n} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1) \right)^3 + z^3 \left( z^{3^{n-1}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \right)^3 \\ &= z^{3^{n+1}} \prod_{k=1}^n \psi_k^{3^{n-k+1}}(z) (\psi_{n+1}^3(z) - 3\psi_{n+1}^2(z) + 3\psi_{n+1}(z)) \\ &= z^{3^{n+1}} \prod_{k=1}^{n+1} \psi_k^{3^{n-k+1}}(z) (\psi_{n+2}(z) - 1) \end{aligned}$$

and

$$\begin{aligned} \Upsilon_{n+1}(z) &= z \left( z^{3^{n-1}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \right)^2 \left( z^{3^n} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1) \right) \\ &\quad + z^2 \left( z^{3^{n-1}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \right)^3 \\ &= z^{3^{n+1}-1} \prod_{k=1}^n \psi_k^{3^{n-k+1}}(z) \cdot \psi_{n+1}(z) \\ &= z^{3^{n+1}-1} \prod_{k=1}^{n+1} \psi_k^{3^{n-k+1}}(z). \end{aligned}$$

□

**Theorem 3.4.** *The thermodynamic limit  $\lim_{n \rightarrow \infty} \frac{\log(Z_n)}{|\text{Vert}(\Sigma_n)|}$  exists.*

*Proof.* Since  $|\text{Edges}(\Sigma_n)| = \frac{3^{n+1}-3}{2}$  and  $|\text{Vert}(\Sigma_n)| = 3^n$ , the limit reduces to

$$\log(2) + \frac{3}{2} \log(\cosh(\beta J)) + \lim_{n \rightarrow \infty} \frac{\log(\Gamma_n^{cl}(z))}{3^n},$$

where  $\tanh(\beta J)$  takes values between 0 and 1. It is straightforward to show, by induction, that  $\psi_k(z) = \frac{\varphi_k(z)}{z^{2^{k-1}}}$ , for every  $k \geq 1$ , where  $\varphi_k(z)$  is a polynomial of degree  $2^{k-1}$  in  $z$ . Hence, the limit  $\lim_{n \rightarrow \infty} \frac{\log(\Gamma_n^{cl}(z))}{3^n}$  becomes

$$\lim_{n \rightarrow \infty} \frac{\log \left( \prod_{k=1}^n \varphi_k^{3^{n-k}}(z) \cdot (\varphi_{n+1}(z) - z^{2^n}) \right)}{3^n} =$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\log(\varphi_k(z))}{3^k} + \lim_{n \rightarrow \infty} \frac{\log(\varphi_{n+1}(z) - z^{2^n})}{3^n}.$$

Let us show that the series  $\sum_{k=1}^{\infty} \frac{\log(\varphi_k(z))}{3^k}$  converges absolutely, and that  $\lim_{n \rightarrow \infty} \frac{\log(\varphi_{n+1}(z) - z^{2^n})}{3^n} = 0$ . It is not difficult to show by induction that

$$2z^{2^{k-1}} \leq \varphi_k(z) \leq 2^{2^{k-1}}$$

for each  $k \geq 2$  and  $z \in [0, 1]$ , so that

$$|\log(\varphi_k(z))| \leq \max\{2^{k-1}|\log(z)|, (2^k - 1)|\log(2)|\}.$$

Note that  $1 \leq \varphi_1(z) \leq 2$  for each  $0 \leq z \leq 1$ . Moreover, one can directly verify that  $\varphi_2(z)$  has a minimum at  $c_2 = 1/4$  and  $\varphi_2'(z) < 0$  for each  $z \in (0, c_2)$ . Let us call  $c_k$  the point where  $\varphi_k(z)$  has the first minimum. One can prove by induction that  $\varphi_{k+1}'(z) < 0$  for each  $z \in (0, c_k]$  and so  $c_k < c_{k+1}$  for every  $k \geq 2$ . In particular,  $\varphi_k(z)$  satisfies

$$-2^k \log(2) \leq \log(\varphi_k(z)) \leq (2^k - 1) \log(2),$$

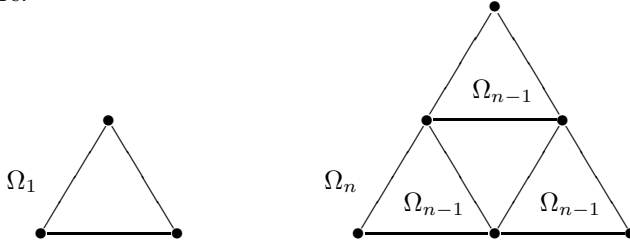
that gives  $|\log(\varphi_k(z))| \leq 2^k \log(2)$  for each  $k \geq 2$ . So we can conclude that

$$\sum_{k=1}^{\infty} \frac{|\log(\varphi_k(z))|}{3^k} \leq \frac{\log(2)}{3} + \sum_{k=2}^{\infty} \frac{2^k \log(2)}{3^k} < \infty.$$

Moreover  $\lim_{n \rightarrow \infty} \frac{|\log(\varphi_{n+1}(z) - z^{2^n})|}{3^n} \leq \lim_{n \rightarrow \infty} \frac{2^n \log(2)}{3^n} = 0$ .  $\square$

### 3.2. The Sierpiński gasket

In this section we use the high temperature expansion and counting of closed polygons in order to compute the partition function for the Ising model on a sequence of graphs  $\{\Omega_n\}_{n \geq 1}$  converging to the Sierpiński gasket. The graphs  $\Omega_n$  are close relatives the Schreier graphs  $\Sigma_n$  of the group  $H^{(3)}$  considered above. More precisely, one can obtain  $\Omega_n$  from  $\Sigma_n$  by contracting the edges between copies of  $\Sigma_{n-1}$  in  $\Sigma_n$ . The graphs  $\Omega_n$  are also self-similar in the sense of [13], as can be seen in the picture.



Similarly to the case of  $H^{(3)}$  above, define sets  $P_n$  and  $L_n$ . The same recursive rules hold, and the total number of closed polygons is again  $2^{\frac{3^n-1}{2}}$ , since the initial conditions are the same.

Let  $\Gamma_n^{cl}(z)$  denote the generating function of the subgraphs in  $P_n$  and let

$\Upsilon_n(z)$  denote the generating function of the subgraphs in  $L_n$ . From relations (2) and (3) we deduce the following formulae:

$$\Gamma_n^{cl}(z) = \left(\Gamma_{n-1}^{cl}(z)\right)^3 + \Upsilon_{n-1}^3(z). \quad (6)$$

and

$$\Upsilon_n(z) = \Upsilon_{n-1}^2(z)\Gamma_{n-1}^{cl}(z) + \Upsilon_{n-1}^3(z). \quad (7)$$

Note that in (6) and (7) there are no factors  $z, z^2, z^3$  occurring in (4) and (5), because the special edges connecting elementary triangles have been contracted in  $\Omega_n$ .

**Theorem 3.5.** *For each  $n \geq 1$ , the partition function of the Ising model on the  $n$ -th Sierpiński graph  $\Omega_n$  is*

$$Z_n = 2^{\frac{3^n+3}{2}} \cdot \cosh(\beta J)^{3^n} \cdot \Gamma_n^{cl}(\tanh(\beta J)),$$

with

$$\Gamma_n^{cl}(z) = z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1),$$

where  $\psi_1(z) = \frac{z+1}{z^{1/2}}$ ,  $\psi_2(z) = \frac{z^2+1}{z}$  and  $\psi_k(z) = \psi_{k-1}^2(z) - 3\psi_{k-1}(z) + 4$ , for each  $k \geq 3$ .

*Proof.* Again we shall use the expression  $Z_n = 2^{|Vert(\Omega_n)|} \cosh(\beta J)^{|Edges(\Omega_n)|} \cdot \Gamma_n^{cl}(\tanh(\beta J))$ , where  $\Gamma_n^{cl}(z)$  is the generating function of closed polygons in  $\Omega_n$ . In our case we have  $|Edges(\Omega_n)| = 3^n$  and  $|Vert(\Omega_n)| = \frac{3^n+3}{2}$ .

We know that the generating functions  $\Gamma_n^{cl}(z)$  and  $\Upsilon_n(z)$  satisfy the equations (6) and (7), with the initial conditions

$$\Gamma_1^{cl}(z) = 1 + z^3 \quad \Upsilon_1(z) = z^2 + z.$$

Let us show by induction on  $n$  that the solutions of the system of equations (6) and (7) are

$$\begin{cases} \Gamma_n^{cl}(z) = z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1) \\ \Upsilon_n(z) = z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z). \end{cases}$$

For  $n = 1$ , we get  $\Gamma_1^{cl}(z) = z^{\frac{3}{2}}\psi_1(z)(\psi_2(z) - 1) = z^3 + 1$  and  $\Upsilon_1(z) = z^{\frac{3}{2}}\psi_1(z) = z^2 + z$  and so the claim is true. Now suppose that the assertion is true for  $n$  and let us show that it is true for  $n + 1$ . One gets:

$$\begin{aligned} \Gamma_{n+1}^{cl}(z) &= \left( z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1) \right)^3 + \left( z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \right)^3 \\ &= z^{\frac{3^{n+1}}{2}} \prod_{k=1}^n \psi_k^{3^{n-k+1}}(z) (\psi_{n+1}^3(z) - 3\psi_{n+1}^2(z) + 3\psi_{n+1}(z)) \\ &= z^{\frac{3^{n+1}}{2}} \prod_{k=1}^{n+1} \psi_k^{3^{n-k+1}}(z) \cdot (\psi_{n+2}(z) - 1) \end{aligned}$$

and

$$\begin{aligned}
\Upsilon_{n+1}(z) &= \left( z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \right)^2 \left( z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \cdot (\psi_{n+1}(z) - 1) \right) \\
&+ \left( z^{\frac{3^n}{2}} \prod_{k=1}^n \psi_k^{3^{n-k}}(z) \right)^3 \\
&= z^{\frac{3^{n+1}}{2}} \prod_{k=1}^n \psi_k^{3^{n-k+1}}(z) \cdot \psi_{n+1}(z) \\
&= z^{\frac{3^{n+1}}{2}} \prod_{k=1}^{n+1} \psi_k^{3^{n-k+1}}(z).
\end{aligned}$$

□

**Remark 3.6.** The existence of the thermodynamic limit can be shown in exactly the same way as for the Schreier graphs of the Hanoi Towers group.

### 3.3. Renormalization approach

Expressions for the partition function of the Ising model on the Sierpiński gasket are well known to physicists. A renormalization equation for it can be found for example in [6] (see also references therein), and a more detailed analysis is given in [2]. Using renormalization, Burioni et al [2] give the following recursion for the partition function of the Ising model on the graphs  $\Omega_n$ ,  $n \geq 1$ :

$$Z_{n+1}(y) = Z_n(f(y))[c(y)]^{3^{n-1}}, \quad (8)$$

where  $y = \exp(\beta J)$ ;  $f(y)$  is a substitution defined by

$$y \rightarrow f(y) = \left( \frac{y^8 - y^4 + 4}{y^4 + 3} \right)^{1/4};$$

and

$$c(y) = \frac{y^4 + 1}{y^3} [(y^4 + 3)^3 (y^8 - y^4 + 4)]^{1/4};$$

with

$$Z_1(y) = 2y^3 + 6y^{-1}.$$

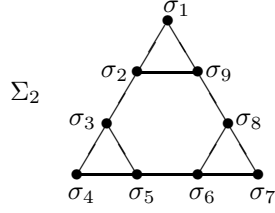
A similar computation can be performed in the case of the Schreier graph  $\Sigma_n$  of the group  $H^{(3)}$ , where the self-similarity of the graph allows to compare the partition function  $Z_1$  of the first level with the partition function of level 2, where the sum is taken only over the internal spins  $\sigma_2, \sigma_3, \sigma_5, \sigma_6, \sigma_8, \sigma_9$  (see figure below). The resulting recurrence is the same as (8), but with

$$y \rightarrow f(y) = \left( \frac{y^8 - 2y^6 + 2y^4 + 2y^2 + 1}{2(y^4 + 1)} \right)^{1/4}$$



and

$$c(y) = \frac{(y^4 - y^2 + 2)(y^2 + 1)^3}{y^6} (8(y^4 + 1)^3 (y^8 - 2y^6 + 2y^4 + 2y^2 + 1))^{1/4} .$$



**Remark 3.7.** The above recursions for the partition functions for  $\Omega_n$ 's and  $\Sigma_n$ 's can be deduced from our Theorems 3.3 and 3.5 by rewriting the formulae in the variable  $y = \exp(\beta J)$  and substituting  $z = \tanh(\beta J) = (y^2 - 1)/(y^2 + 1)$ .

#### 4. Statistics on weighted closed polygons

This Section is devoted to the study of weighted generating functions of closed polygons, i.e., we allow the edges of the graph to have different weights  $\tanh(\beta J_{i,j})$ , as in RHS of (1). We also take into account the fact that the graphs we consider are Schreier graphs of some self-similar group  $G$  with respect to a certain generating set  $S$ , and their edges are therefore labeled by these generators. It is thus natural to allow the situations where the energy between two neighbouring spins takes a finite number of possible values encoded by the generators  $S$ . Logarithmic derivatives of the weighted generating function with respect to  $s \in S$  give us the mean density of  $s$ -edges in a random configuration. We can further find the variance and show that the limiting distribution is normal.

##### 4.1. The Schreier graphs of the Grigorchuk's group

Recall from 2.1 that the simple edges in  $\Sigma_n$  are always labeled by  $a$ . Moreover, the 2-cycles can be labeled by the couples of labels  $(b, c)$ ,  $(b, d)$  and  $(c, d)$ . We want to compute the weighted generating function of closed polygons, with respect to the weights given by the labels  $a, b, c, d$ . Let us set, for each  $n \geq 1$ :

$$X_n = |\{2\text{-cycles with labels } b, c\}| \quad Y_n = |\{2\text{-cycles with labels } b, d\}|$$

$$W_n = |\{2\text{-cycles with labels } c, d\}|$$

One can easily check by using self-similar formulae for the generators, that the following equations hold:

$$\begin{cases} X_n = W_{n-1} + 2^{n-2} \\ Y_n = X_{n-1} \\ W_n = Y_{n-1}. \end{cases}$$

In particular, one gets

$$\begin{cases} X_n = X_{n-3} + 2^{n-2} \\ Y_n = X_{n-1} \\ W_n = X_{n-2}, \end{cases}$$

with initial conditions  $X_1 = 0$ ,  $X_2 = 1$  and  $X_3 = 2$ . One gets the following values:

$$X_n = \begin{cases} \frac{2^{n+1}-2}{7} & \text{if } n \equiv 0(3) \\ \frac{2^{n+1}-4}{7} & \text{if } n \equiv 1(3) \\ \frac{2^{n+1}-1}{7} & \text{if } n \equiv 2(3) \end{cases} \quad Y_n = \begin{cases} \frac{2^n-1}{7} & \text{if } n \equiv 0(3) \\ \frac{2^n-2}{7} & \text{if } n \equiv 1(3) \\ \frac{2^n-4}{7} & \text{if } n \equiv 2(3) \end{cases}$$

$$W_n = \begin{cases} \frac{2^{n-1}-4}{7} & \text{if } n \equiv 0(3) \\ \frac{2^{n-1}-1}{7} & \text{if } n \equiv 1(3) \\ \frac{2^{n-1}-2}{7} & \text{if } n \equiv 2(3) \end{cases},$$

and, consequently,

**Theorem 4.1.** *For each  $n \geq 1$ , the weighted generating function of closed polygons in  $\Sigma_n$  is*

$$\begin{cases} \Gamma_n^{cl}(a, b, c, d) = (1+bc)^{\frac{2^{n+1}-2}{7}} (1+bd)^{\frac{2^n-1}{7}} (1+cd)^{\frac{2^{n-1}-4}{7}} & \text{if } n \equiv 0(3) \\ \Gamma_n^{cl}(a, b, c, d) = (1+bc)^{\frac{2^{n+1}-4}{7}} (1+bd)^{\frac{2^n-2}{7}} (1+cd)^{\frac{2^{n-1}-1}{7}} & \text{if } n \equiv 1(3) \\ \Gamma_n^{cl}(a, b, c, d) = (1+bc)^{\frac{2^{n+1}-1}{7}} (1+bd)^{\frac{2^n-4}{7}} (1+cd)^{\frac{2^{n-1}-2}{7}} & \text{if } n \equiv 2(3). \end{cases}$$

**Proposition 4.2.** *Let  $w_n$  be the number of edges labeled  $w$  in a random closed polygon in  $\Sigma_n$ , where  $w = a, b, c, d$ . Denote by  $\mu_{n,w}$  and  $\sigma_{n,w}^2$  the mean and the variance of  $w_n$ . Then,*

- for each  $n \geq 1$ ,  $a_n = 0$ ;
- The means and the variances of the random variables  $b_n, c_n, d_n$  are given in the following table:

	$n \equiv 0(3)$	$n \equiv 1(3)$	$n \equiv 2(3)$
$\mu_{n,b}$	$\frac{3}{14}(2^n - 1)$	$\frac{3}{7}(2^{n-1} - 1)$	$\frac{3 \cdot 2^n - 5}{14}$
$\sigma_{n,b}^2$	$\frac{3}{28}(2^n - 1)$	$\frac{3}{14}(2^{n-1} - 1)$	$\frac{3 \cdot 2^n - 5}{28}$
$\mu_{n,c}$	$\frac{5 \cdot 2^{n-2} - 3}{7}$	$\frac{5}{14}(2^{n-1} - 1)$	$\frac{5 \cdot 2^{n-1} - 3}{14}$
$\sigma_{n,c}^2$	$\frac{5 \cdot 2^{n-2} - 3}{14}$	$\frac{5}{28}(2^{n-1} - 1)$	$\frac{5 \cdot 2^{n-1} - 3}{28}$
$\mu_{n,d}$	$\frac{3 \cdot 2^{n-1} - 5}{14}$	$\frac{3}{14}(2^{n-1} - 1)$	$\frac{3}{7}(2^{n-2} - 1)$
$\sigma_{n,d}^2$	$\frac{3 \cdot 2^{n-1} - 5}{28}$	$\frac{3}{28}(2^{n-1} - 1)$	$\frac{3}{14}(2^{n-2} - 1)$

- the random variables  $b_n, c_n, d_n$  are asymptotically normal, as  $n \rightarrow \infty$ .

*Proof.* It is clear that an edge labeled by  $a$  never belongs to a closed polygon of  $\Sigma_n$ , so that  $a_n = 0$ . Let us choose, for instance,  $n \equiv 0(3)$ . Putting

$$\Gamma_n^{cl}(b) := \Gamma_n^{cl}(1, b, 1, 1) = 2^{\frac{2^{n+1}-2}{7}} (1+b)^{\frac{3 \cdot 2^n - 3}{7}},$$

we can obtain the mean  $\mu_{n,b}$  and the variance  $\sigma_{n,b}^2$  for  $b_n$  by studying the derivatives of the function  $\log(\Gamma_n^{cl}(b))$ . One gets

$$\mu_{n,b} = \frac{3}{14}(2^n - 1) \quad \sigma_{n,b}^2 = \frac{3}{28}(2^n - 1).$$

Similar computations can be done for  $c$  and  $d$ .

In all cases above we can find explicitly the moment generating function of the corresponding normalized random variables: a direct computation of its limit for  $n \rightarrow \infty$  shows that the asymptotic distribution is normal.  $\square$

#### 4.2. The Schreier graphs of the Basilica group

We also compute the weighted generating function of closed polygons for the Basilica group, with respect to the weights given by the labels  $a$  and  $b$  on the edges of its Schreier graph  $\Sigma_n$ . We use here the computations from Proposition 2.2.

**Theorem 4.3.** *The weighted generating function of closed polygons in the Schreier graph  $\Sigma_n$  of the Basilica group is*

$$\begin{aligned} \Gamma_n^{cl}(a, b) &= \prod_{k=1}^{\frac{n-1}{2}-1} (1 + a^{2^k})^{2^{n-2k-1}} \prod_{k=1}^{\frac{n-1}{2}-1} (1 + b^{2^k})^{2^{n-2k}} \\ &\cdot \left(1 + a^{2^{\frac{n-1}{2}}}\right)^2 \left(1 + b^{2^{\frac{n-1}{2}}}\right)^2 \left(1 + b^{2^{\frac{n+1}{2}}}\right) \end{aligned}$$

for  $n \geq 5$  odd and

$$\Gamma_n^{cl}(a, b) = \prod_{k=1}^{\frac{n}{2}-1} (1 + a^{2^k})^{2^{n-2k-1}} \prod_{k=1}^{\frac{n}{2}-1} (1 + b^{2^k})^{2^{n-2k}} (1 + a^{2^{\frac{n}{2}}}) (1 + b^{2^{\frac{n}{2}}})^2$$

for  $n \geq 4$  even.

**Proposition 4.4.** *The means and the variances of the densities  $a_n$  and  $b_n$  are given in the following table:*

	$n \geq 5$ odd	$n \geq 4$ even
$\mu_{n,a}$	$2^{n-2}$	$2^{n-2}$
$\sigma_{n,a}^2$	$(n+1)2^{n-4}$	$(n+2)2^{n-4}$
$\mu_{n,b}$	$2^{n-1}$	$2^{n-1}$
$\sigma_{n,b}^2$	$(n+3)2^{n-3}$	$(n-2)2^{n-3}$

#### 4.3. The Schreier graphs of $H^{(3)}$

Let us denote by  $\Upsilon_n^{lr}(a, b, c)$  the weighted generating function of the subgraphs that belong to the set  $P_n$ , defined in Subsection 3.1 (the exponent  $lr$  stands for left-right, as self-avoiding paths in  $P_n$  join the left-most and the right-most vertices of  $\Sigma_n$ .) Analogously, we define  $\Upsilon_n^{lu}(a, b, c)$  and  $\Upsilon_n^{ru}(a, b, c)$ , where the exponents  $lu$  and  $ru$  stand for left-up and right-up, respectively. By using the self-similar

expressions for the generators given in Subsection 3.1, we find that these functions satisfy the following system of equations (we omit the arguments  $a, b, c$ ):

$$\begin{cases} \Gamma_{n+1}^{cl} = (\Gamma_n^{cl})^3 + abc\Upsilon_n^{lr}\Upsilon_n^{lu}\Upsilon_n^{ru} \\ \Upsilon_{n+1}^{lu} = a\Upsilon_n^{lr}\Upsilon_n^{ru}\Gamma_n^{cl} + bc(\Upsilon_n^{lu})^3 \\ \Upsilon_{n+1}^{ru} = c\Upsilon_n^{lu}\Upsilon_n^{lr}\Gamma_n^{cl} + ab(\Upsilon_n^{ru})^3 \\ \Upsilon_{n+1}^{lr} = b\Upsilon_n^{lu}\Upsilon_n^{ru}\Gamma_n^{cl} + ac(\Upsilon_n^{lr})^3 \end{cases} \quad (9)$$

with the initial conditions  $\Gamma_1^{cl}(a, b, c) = 1 + abc$ ,  $\Upsilon_1^{lr}(a, b, c) = ac + b$ ,  $\Upsilon_1^{lu}(a, b, c) = a + bc$ ,  $\Upsilon_1^{ru}(a, b, c) = c + ab$ .

**Proposition 4.5.** *The mean and the variance for  $w_n$ , with  $w = a, b, c$ , are:*

$$\mu_{n,w} = \frac{3^n - 1}{4} \quad \sigma_{n,w}^2 = \frac{3^n - 1}{8}.$$

*The random variables  $w_n$  with  $w = a, b, c$  are asymptotically normal.*

*Proof.* If we put  $a = b = 1$ , the system (9) reduces to

$$\begin{cases} \Gamma_{n+1}^{cl} = (\Gamma_n^{cl})^3 + c\Upsilon_n^{lr}\Upsilon_n^{lu}\Upsilon_n^{ru} \\ \Upsilon_{n+1}^{lu} = \Upsilon_n^{lr}\Upsilon_n^{ru}\Gamma_n^{cl} + c(\Upsilon_n^{lu})^3 \\ \Upsilon_{n+1}^{ru} = c\Upsilon_n^{lu}\Upsilon_n^{lr}\Gamma_n^{cl} + (\Upsilon_n^{ru})^3 \\ \Upsilon_{n+1}^{lr} = \Upsilon_n^{lu}\Upsilon_n^{ru}\Gamma_n^{cl} + c(\Upsilon_n^{lr})^3 \end{cases} \quad (10)$$

with the initial conditions  $\Gamma_1^{cl}(1, 1, c) = \Upsilon_1^{lr}(1, 1, c) = \Upsilon_1^{lu}(1, 1, c) = \Upsilon_1^{ru}(1, 1, c) = 1 + c$ .

One can prove, by induction on  $n$ , that the solutions of the system (10) are

$$\Gamma_n^{cl}(1, 1, c) = \Upsilon_n^{lr}(1, 1, c) = \Upsilon_n^{lu}(1, 1, c) = \Upsilon_n^{ru}(1, 1, c) = (1 + c)^{\frac{3^n - 1}{2}} \quad \text{for each } n.$$

By studying the derivatives of the function  $\log(\Gamma_n^{cl}(1, 1, c))$  with respect to  $c$ , one gets:

$$\mu_{n,c} = \frac{3^n - 1}{4} \quad \sigma_{n,c}^2 = \frac{3^n - 1}{8}.$$

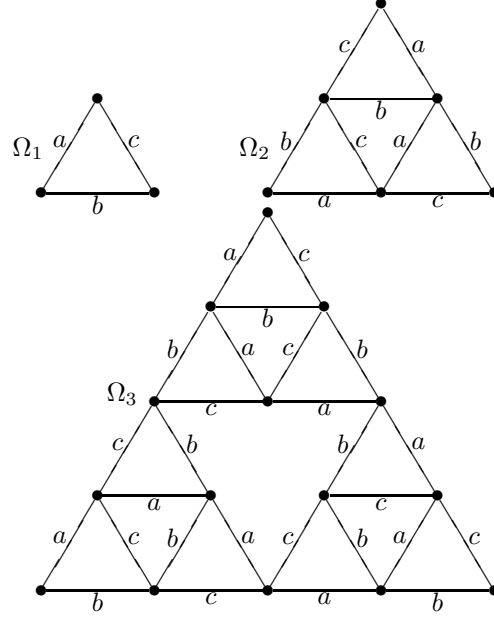
Symmetry of the labeling of the graph ensures that the same values arise for the random variables  $a_n, b_n$ .  $\square$

#### 4.4. The Sierpiński graphs

The Sierpiński graphs  $\Omega_n$  being not regular, they cannot be realized as Schreier graphs of any group. There exist however a number of natural, geometric labelings of edges of  $\Omega_n$  by letters  $a, b, c$  (see [3]). Here we will be interested in one particular labeling that is obtained by considering the labeled Schreier graph  $\Sigma_n$  of the Hanoi Towers group and then by contracting the edges connecting copies of  $\Sigma_{n-1}$  in  $\Sigma_n$ ; and so we call this the "Schreier" labeling of  $\Omega_n$ .

**Remark 4.6.** The "Schreier" labeling on  $\Omega_n$  can be constructed recursively, as follows. Start with the graph  $\Omega_1$  in the picture below; then, for each  $n \geq 2$ , the graph  $\Omega_n$  is defined as the union of three copies of  $\Omega_{n-1}$ . For each one of the

out-most (corner) vertices of  $\Omega_n$ , the corresponding copy of  $\Omega_{n-1}$  is reflected with respect to the bisector of the corresponding angle.



Let  $\Upsilon_n^{lr}(a, b, c)$ ,  $\Upsilon_n^{lu}(a, b, c)$  and  $\Upsilon_n^{ru}(a, b, c)$  be defined as for the Schreier graphs  $\Sigma_n$  of the Hanoi Towers group in the previous subsection. Then one can easily check that these functions satisfy the following system of equations:

$$\begin{cases} \Gamma_{n+1}^{cl} = (\Gamma_n^{cl})^3 + \Upsilon_n^{lr} \Upsilon_n^{lu} \Upsilon_n^{ru} \\ \Upsilon_{n+1}^{lu} = \Upsilon_n^{lr} \Upsilon_n^{ru} \Gamma_n^{cl} + (\Upsilon_n^{lu})^3 \\ \Upsilon_{n+1}^{ru} = \Upsilon_n^{lu} \Upsilon_n^{lr} \Gamma_n^{cl} + (\Upsilon_n^{ru})^3 \\ \Upsilon_{n+1}^{lr} = \Upsilon_n^{lu} \Upsilon_n^{ru} \Gamma_n^{cl} + (\Upsilon_n^{lr})^3 \end{cases}$$

with the initial conditions  $\Gamma_1^{cl}(a, b, c) = 1 + abc$ ,  $\Upsilon_1^{lr}(a, b, c) = ac + b$ ,  $\Upsilon_1^{lu}(a, b, c) = a + bc$ ,  $\Upsilon_1^{ru}(a, b, c) = c + ab$ .

Proceeding as in Subsection 4.3, we find:

$$\Gamma_n^{cl}(1, 1, c) = \Upsilon_n^{lr}(1, 1, c) = \Upsilon_n^{lu}(1, 1, c) = \Upsilon_n^{ru}(1, 1, c) = 2^{\frac{3^n - 1}{2} - 1} (1 + c)^{3^{n-1}},$$

which implies the following

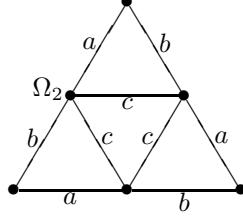
**Proposition 4.7.** *The mean and the variance for the random variable  $w_n$ , with  $w = a, b, c$ , for  $\Omega_n$  with the "Schreier" labeling are:*

$$\mu_{n,w} = \frac{3^{n-1}}{2} \quad \sigma_{n,w}^2 = \frac{3^{n-1}}{4}.$$

*The random variables  $a_n, b_n, c_n$  are asymptotically normal.*

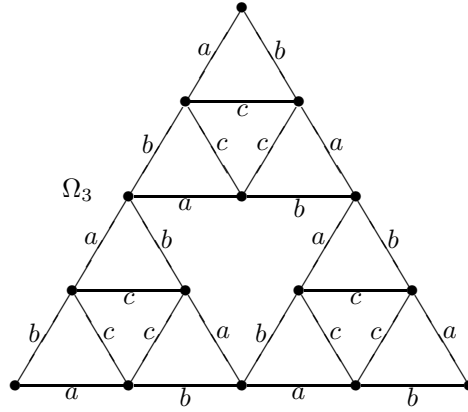
It is interesting to compare these computations with those for a different labeling of  $\Omega_n$ , that we call the "rotation-invariant" labeling of Sierpiński graphs, defined recursively as follows. (Compare the construction to the recursive description of the "Schreier labeling" in Remark 4.6.)

Let  $\Omega_2$  be the weighted graph in the following picture.



Then define, for each  $n \geq 3$ ,  $\Omega_n$  as the union of three copies of  $\Omega_{n-1}$ , rotated by  $k\pi/3$  with  $k = 0, 1, 2$ .

For  $n = 3$ , one gets the following graph.



It turns out that the weighted generating function of closed polygons is easier to compute for  $\Omega_n$  with the "rotation-invariant" labeling, than with the "Schreier" labeling. More precisely, we have the following

**Theorem 4.8.** *For each  $n \geq 2$ , the weighted generating function of closed polygons for the graph  $\Omega_n$  with the "rotation-invariant" labeling is*

$$\Gamma_n^{cl}(a, b, c) = ((a+bc)(b+ac))^{\frac{7 \cdot 3^{n-2}}{4}} \psi_1^{3^{n-2}}(a, b, c) \prod_{k=2}^n \psi_k^{3^{n-k}}(a, b, c) \cdot (\psi_{n+1}(a, b, c) - 1)$$

$$\text{where } \psi_1(a, b, c) = \frac{1+c}{((a+bc)(b+ac))^{\frac{1}{4}}}, \quad \psi_2(a, b, c) = \frac{1+ab}{((a+bc)(b+ac))^{\frac{1}{2}}},$$

$$\psi_3(a, b, c) = \frac{a^2b^2c^2 - a^2b^2c + a^2b^2 + 4abc + c^2 - c + 1 + a^2c + b^2c}{(a+bc)(b+ac)} \text{ and, for each } k \geq 4,$$

$$\psi_k(a, b, c) = \psi_{k-1}^2(a, b, c) - 3\psi_{k-1}(a, b, c) + 4.$$

*Proof.* Consider the graph  $\Omega_n$ . For each  $n \geq 2$ , define the sets  $P_n$  and  $L_n$  as in Subsection 3.1 and let  $\Gamma_n^{cl}(a, b, c)$  and  $\Upsilon_n(a, b, c)$  be the associated weighted

generating functions. By using the symmetry of the labeling, one can check that these functions satisfy the following equations

$$\begin{cases} \Gamma_n^{cl}(a, b, c) = (\Gamma_{n-1}^{cl}(a, b, c))^3 + \Upsilon_{n-1}^3(a, b, c) \\ \Upsilon_n(a, b, c) = \Upsilon_{n-1}^3(a, b, c) + \Upsilon_{n-1}^2(a, b, c)\Gamma_{n-1}^{cl}(a, b, c). \end{cases} \quad (11)$$

with the initial conditions

$$\begin{cases} \Gamma_2^{cl} = (1+c)(1+ab)(a^2b^2c^2 - a^2b^2c + a^2b^2 + 4abc - abc^2 - ab + c^2 - c + 1) \\ \Upsilon_2 = (1+c)(1+ab)(a+bc)(b+ac). \end{cases}$$

As in the proof of Theorem 3.5, one shows by induction on  $n$  that the solutions of the system are

$$\begin{cases} \Gamma_n^{cl} = ((a+bc)(b+ac))^{\frac{7 \cdot 3^{n-2}}{4}} \psi_1^{3^{n-2}}(a, b, c) \prod_{k=2}^n \psi_k^{3^{n-k}}(a, b, c) \cdot (\psi_{n+1}(a, b, c) - 1) \\ \Upsilon_n = ((a+bc)(b+ac))^{\frac{7 \cdot 3^{n-2}}{4}} \psi_1^{3^{n-2}}(a, b, c) \prod_{k=2}^n \psi_k^{3^{n-k}}(a, b, c) \end{cases}$$

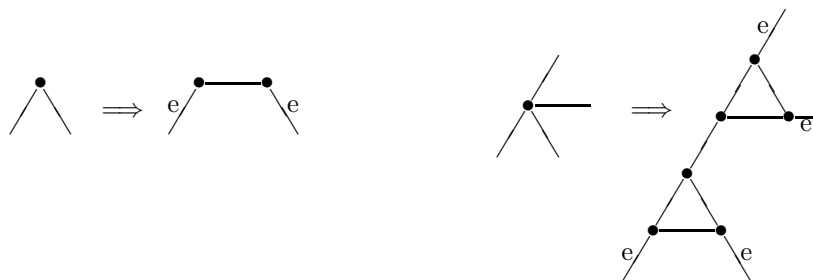
□

**Remark 4.9.** Although the labels  $a$  and  $b$  are not symmetric to the label  $c$  in the "rotation-invariant" labeling, computations show that the functions  $\Gamma_n^{cl}(a, 1, 1)$ ,  $\Gamma_n^{cl}(1, b, 1)$  and  $\Gamma_n^{cl}(1, 1, c)$  are the same in this case as in the case of the "Schreier" labeling. It follows that the values of the mean and the variance of the random variables  $a_n, b_n, c_n$  remain the same as in the "Schreier" labeling, see Proposition 4.7.

#### 4.5. Correspondences via Fisher's Theorem

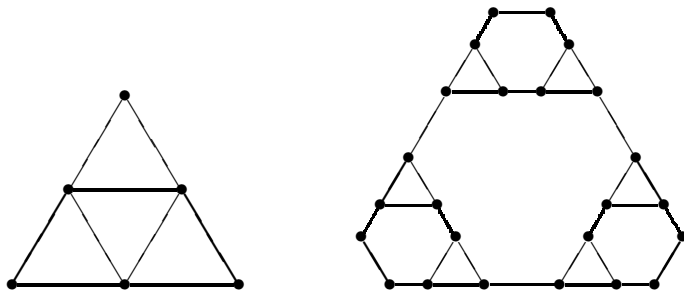
In [5] M. Fisher proposed a method of computation for the partition function of the Ising model on a (finite) planar lattice  $Y$  by relating it to the partition function of the dimers model (with certain weights) on another planar lattice  $Y^\Delta$  constructed from  $Y$ . (The latter partition function can then be found by computing the corresponding Pfaffian given by Kasteleyn's theorem.) This method uses the expression (1) for the partition function in terms of the generating function of closed polygons in  $Y$ . The new lattice  $Y^\Delta$  is constructed in such a way that Ising polygon configurations on  $Y$  are in one-to-one correspondence with dimer configurations on  $Y^\Delta$ . In order to have equality of generating functions however, the edges of  $Y^\Delta$  should be weighted in such a way that the edges coming from  $Y$  have the same weight  $\tanh(\beta J_{i,j})$  as in the RHS of (1), and other edges have weight 1.

Applying Fisher's construction to Sierpiński graphs, one concludes easily that if  $Y = \Omega_n$  for some  $n \geq 1$ , then  $Y^\Delta = \tilde{\Sigma}_{n+1}$ , the  $(n+1)$ -st Schreier graph of the Hanoi Towers group  $H^{(3)}$  with three corner vertices deleted. Note that the corner vertices are the only vertices in  $\Sigma_n$  with loops attached to them, and so it is anyway natural to forget about them when counting dimer coverings. The construction consists in applying to  $Y$  the following substitutions, where edges labeled by  $e$  in  $Y^\Delta$  are in bijection with edges in  $Y$ , and should be assigned weight  $\tanh(\beta J_{i,j})$ . Other edges should be assigned weight 1.



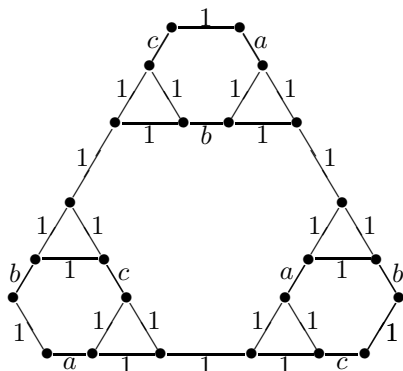
The correspondence between closed polygons in  $Y$  and dimer coverings of  $Y^\Delta$  is as follows: if an edge in  $Y$  belongs to a closed polygon, then the corresponding  $e$ -edge in  $Y^\Delta$  does not belong to the dimer covering of  $Y^\Delta$  associated with that closed polygon, and vice versa.

The following pictures give an example of a closed polygon in  $\Omega_2$  and of the associated dimer covering of  $\tilde{\Sigma}_3$ :



If, for a certain  $n \geq 1$ , the Sierpiński graph  $\Omega_n$  is considered with the "Schreier" labeling, then the labeling of the graph  $\tilde{\Sigma}_{n+1}$  given by Fisher's construction will be a restriction of the usual Schreier labeling of  $\Sigma_{n+1}$ . More precisely, only the edges that connect copies of  $\Sigma_{n-1}$  but not copies of  $\Sigma_n$  will be labeled (other edges have weight 1), and the labels are the same as in the standard labeling of  $\Sigma_{n+1}$  as a Schreier graph of the group  $H^{(3)}$ . The following picture represents  $\tilde{\Sigma}_3$  as  $Y^\Delta$  with  $Y = \Omega_2$  with the "Schreier" labeling.





**Remark 4.10.** One can wonder what Fisher’s construction gives for  $\{\Sigma_n\}_{n \geq 1}$ . It turns out that if  $Y = \Sigma_n$ , the  $n$ -th Schreier graph of  $H^{(3)}$ , then  $Y^\Delta = \tilde{\Sigma}_{n+1}$ , the same as for  $Y = \Omega_n$ , the  $n$ -th Sierpiński graph .

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