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Mathematical Problem Solving – current findings and needs in this field of research

OVERVIEW

(1) WHY / WHAT IS PROBLEM SOLVING

(2) CLARIFICATION OF TERMS

(3) INSIGHTS INTO RESEARCH: HEURISTICS

(4) INSIGHTS INTO RESEARCH: LEARNING VIA PROBLEM SOLVING

(5) INSIGHTS INTO RESEARCH: TESTING PROBLEM SOLVING

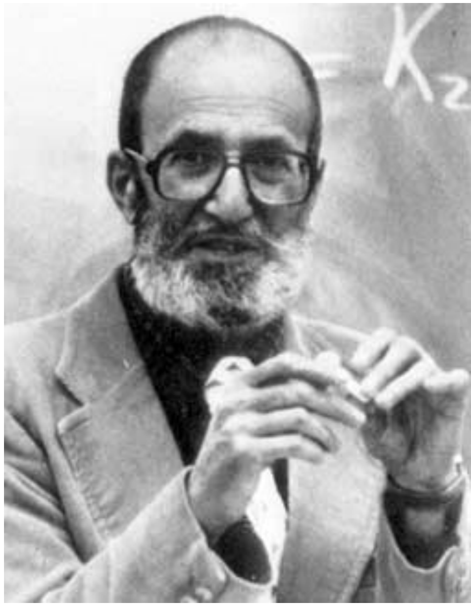
(6) CONCLUSIONS

WHY PROBLEM SOLVING?

What does mathematics *really* consist of? Axioms […]? Theorems […]? Proofs […]? Concepts […]? Definitions […]? Theories […]? Formulas […]? Methods […]?

Mathematics could surely not exist without these ingredients; […] nevertheless […] none of them is at the heart of the subject, […]

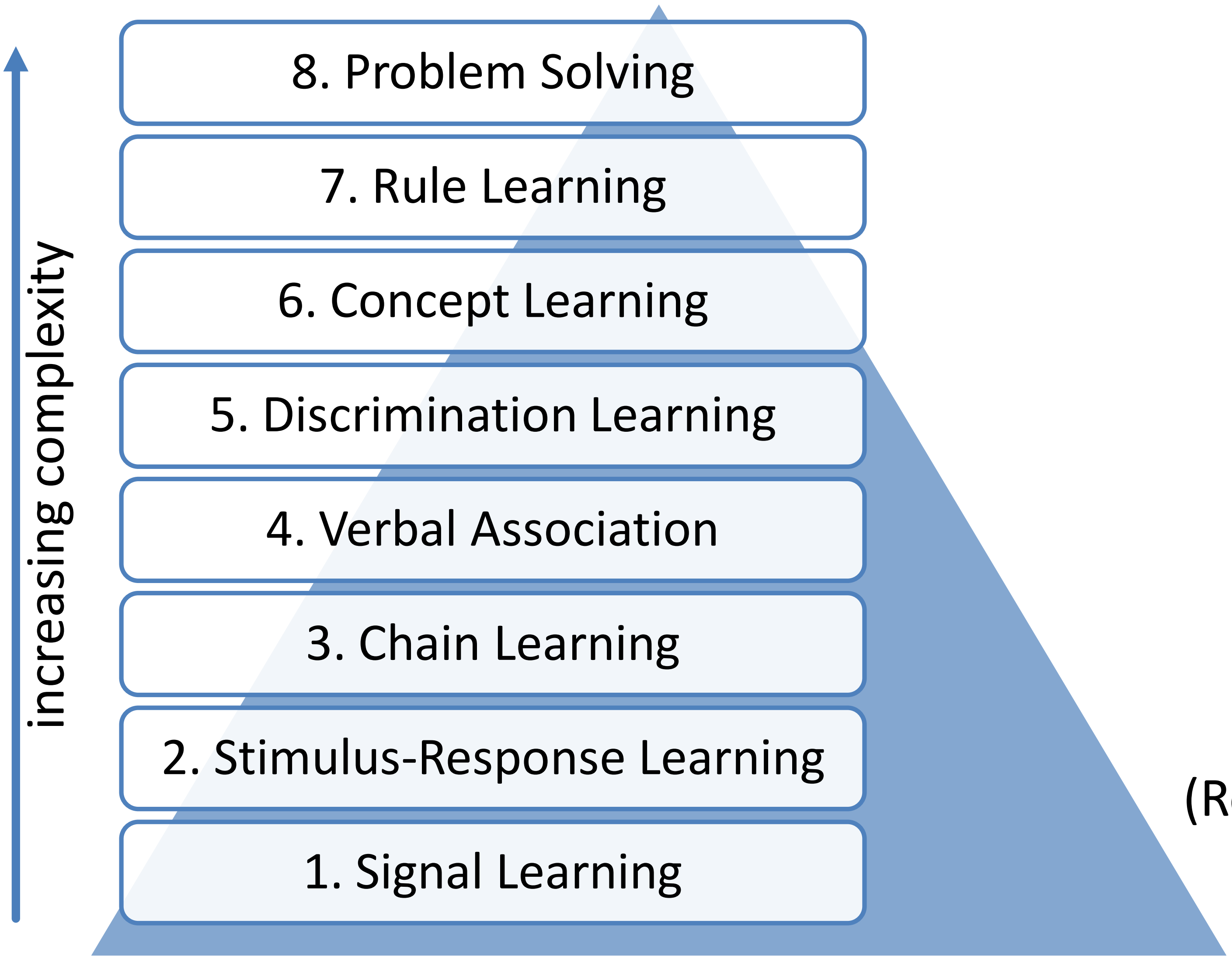
the mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics *really* consists of is problems and solutions.

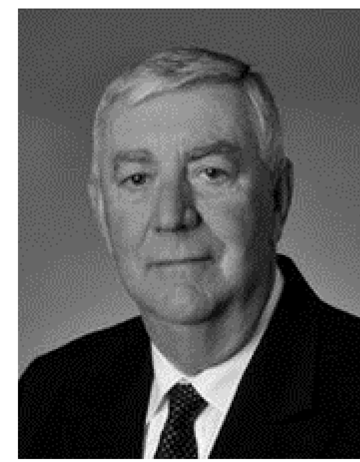


(Paul Halmos)

Halmos (1980, p. 519)

Learning Hierarchy





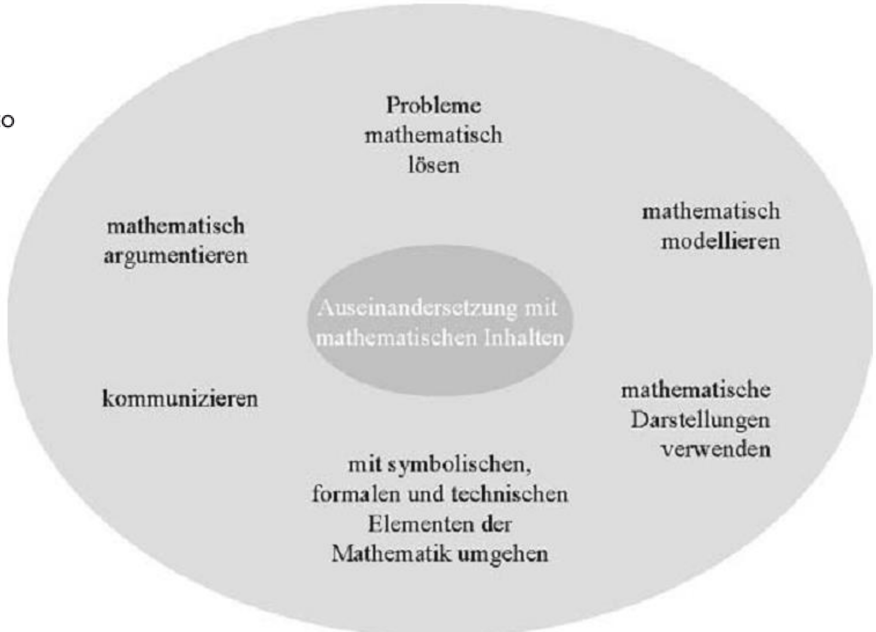
(Robert M. Gagné)

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1 Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and

Common Core State Standards for Mathematics (NCTM, 2010)

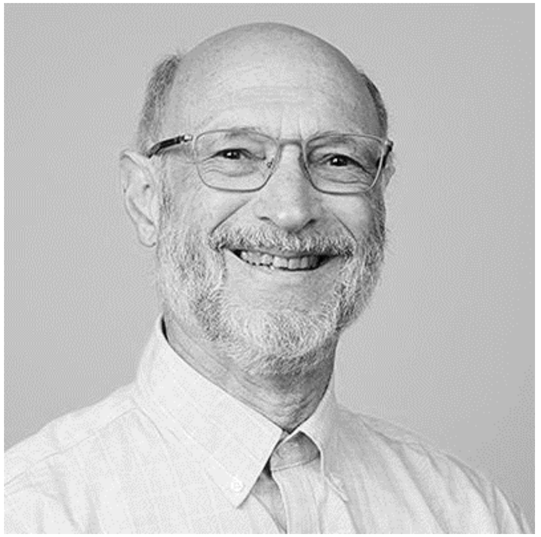


Bildungsstandards (KMK, 2003)

WHAT IS PROBLEM SOLVING?

Let me begin with a definition. For any student, a mathematical problem is a task (a) in which the student is interested and engaged and for which he wishes to obtain a resolution, and (b) for which the student does not have a readily accessible mathematical means by which to achieve that resolution.

[…] Second, it implies that tasks are not ‘problems’ in and of themselves; […]. Third, most of the textbook and homework ‘problems’ assigned to students are not problems according to this definition, but exercises. […]

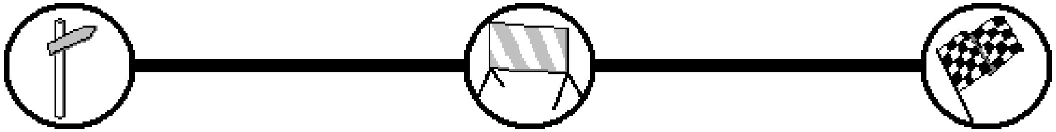


(Alan H. Schoenfeld)

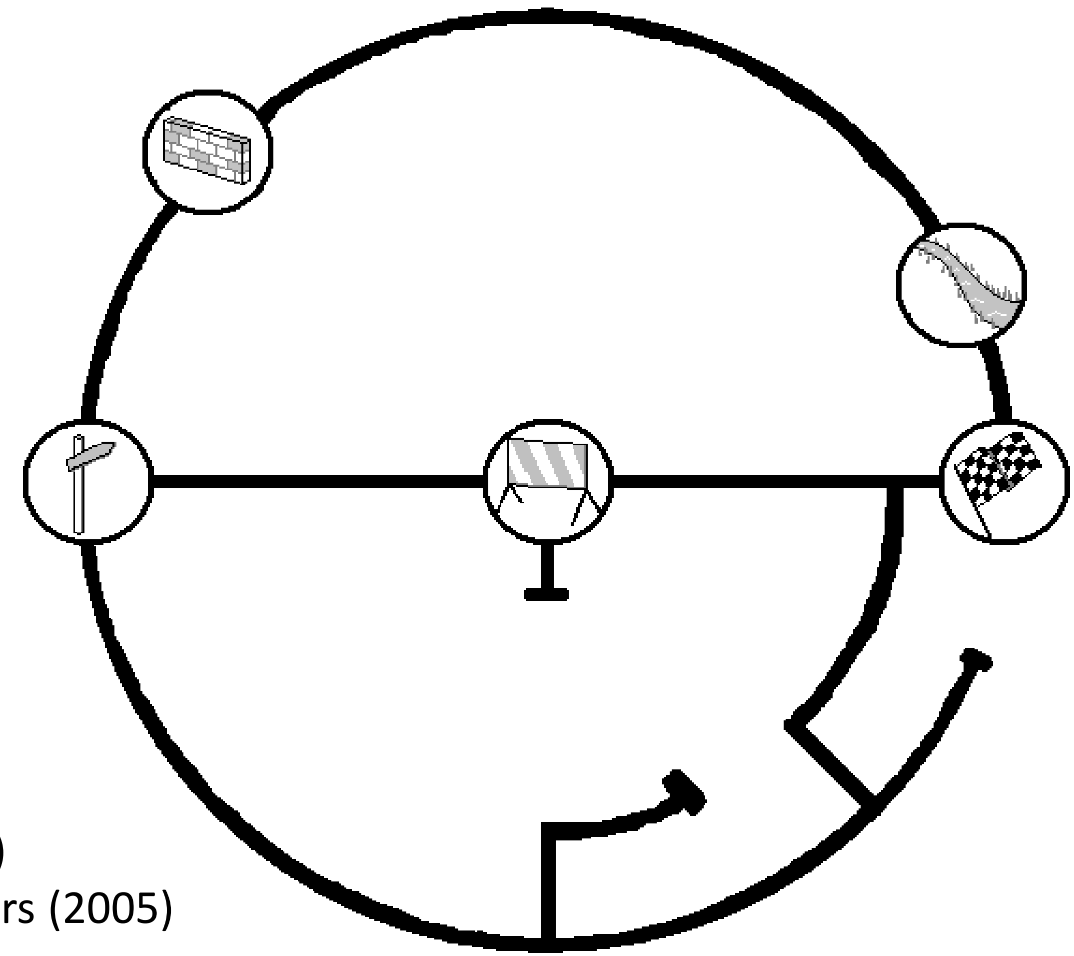
Schoenfeld (1989, p. 87 f.)

CLARIFICATION OF TERMS

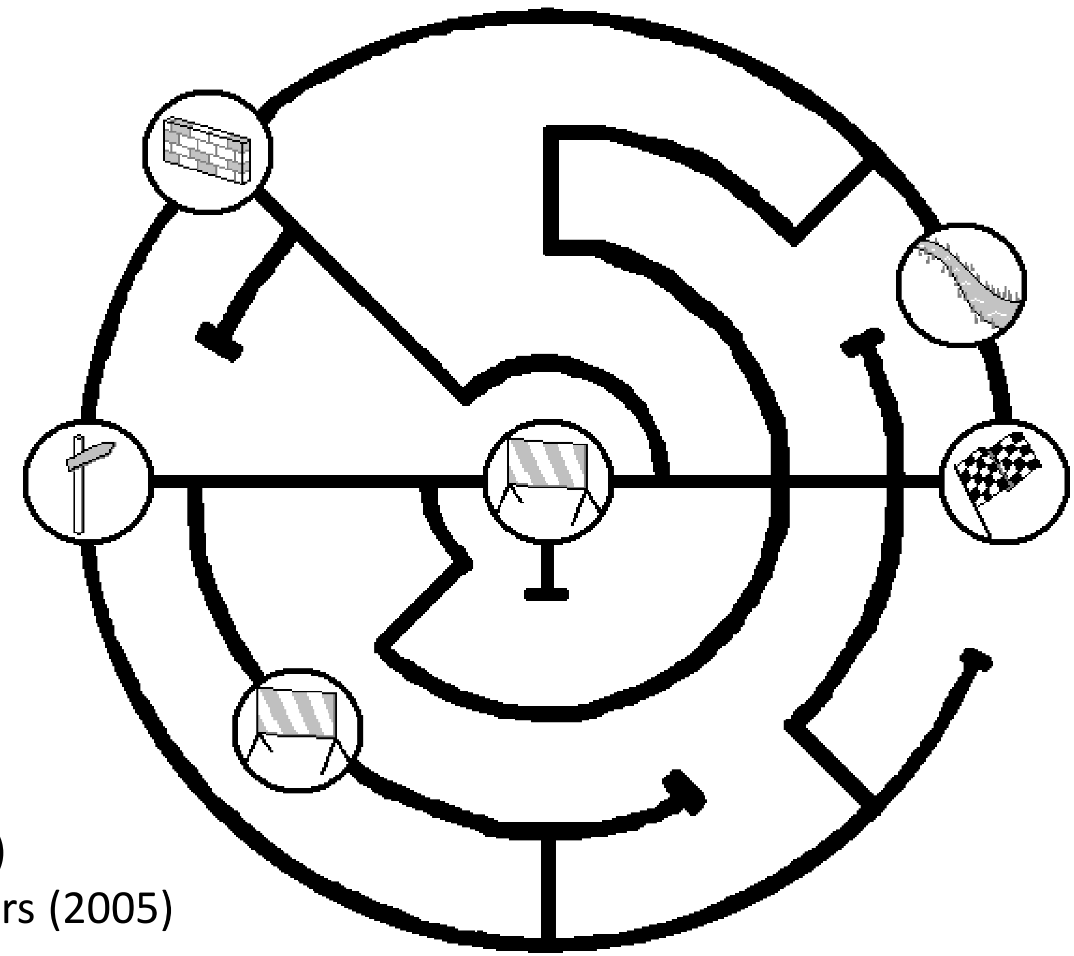
Duncker (1935)
Dörner (1979)
Pólya (1980)
Schoenfeld (1985)



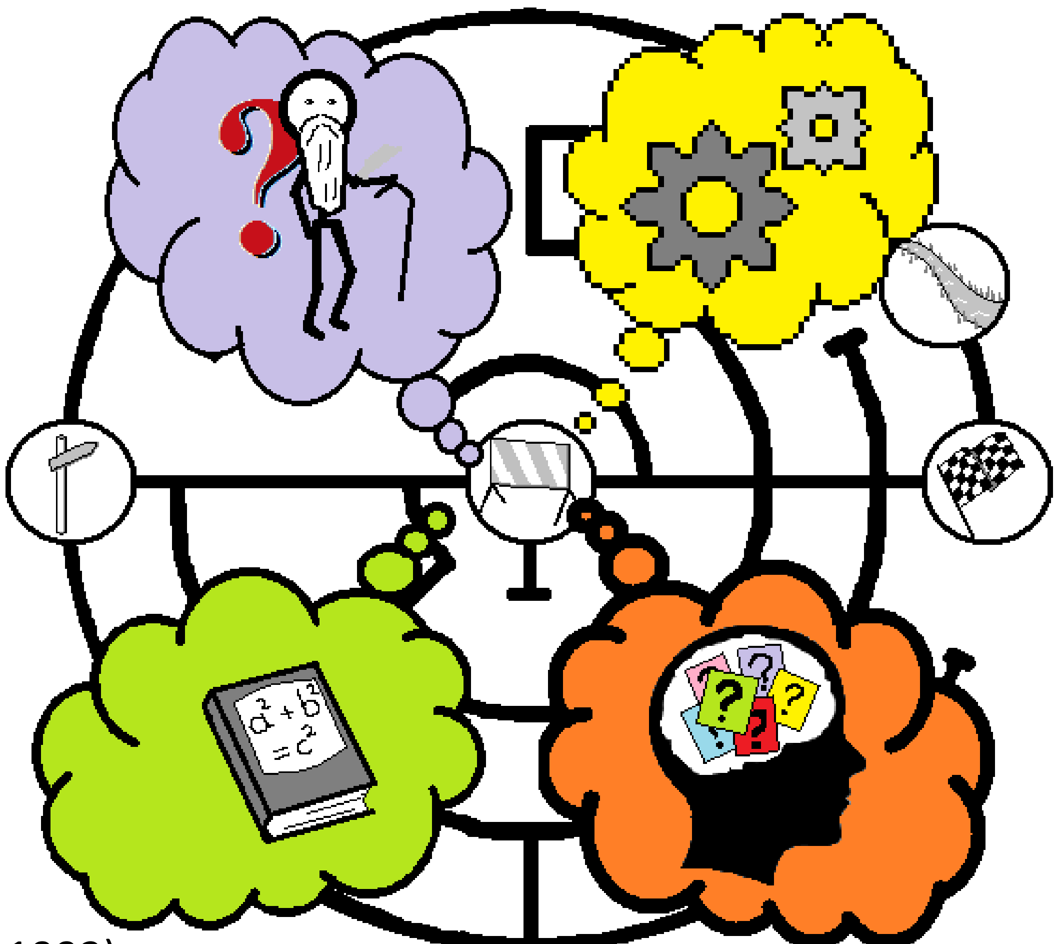
Duncker (1935)
Dörner (1979)
Pólya (1980)
Schoenfeld (1985)



Klix (1971)
Pehkonen (2004)
Büchter & Leuders (2005)

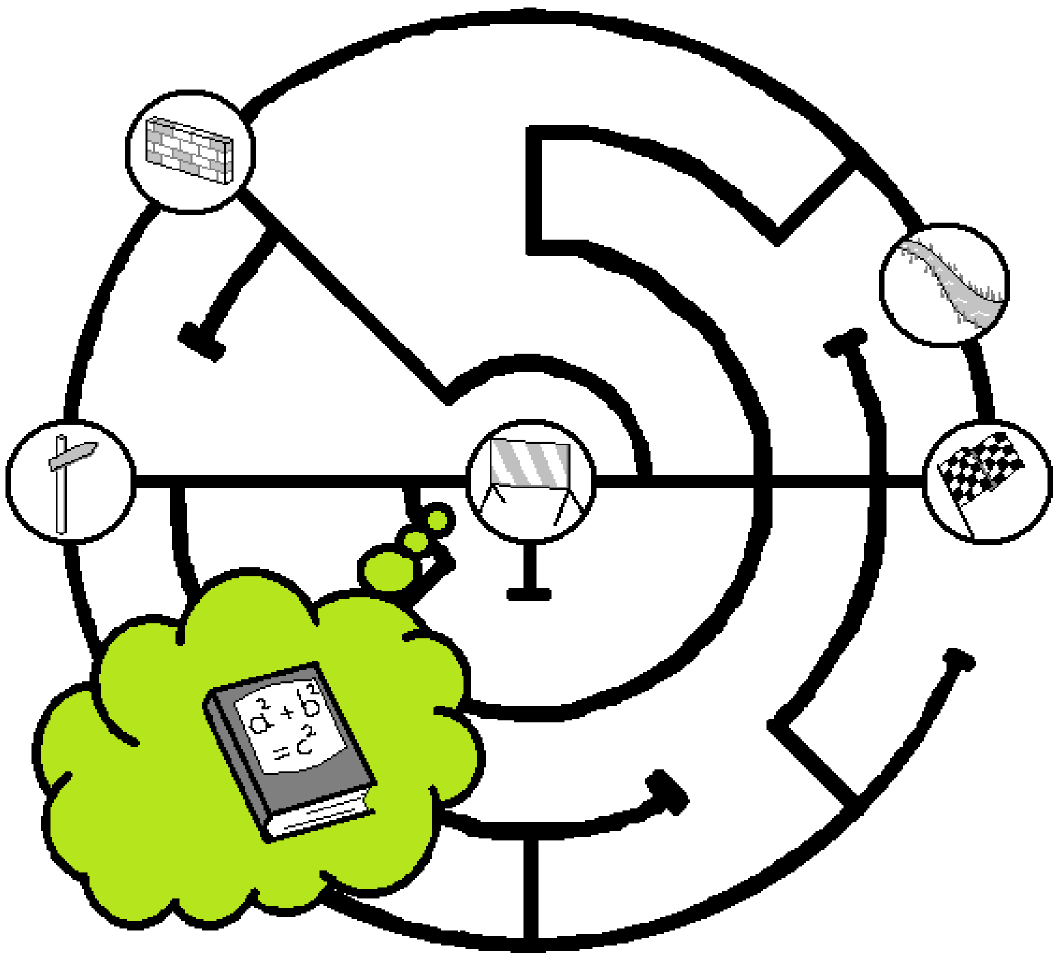


Klix (1971)
Pehkonen (2004)
Büchter & Leuders (2005)



Flavell (1976)
Mason, Burton &
Stacey (1982)
Brown (1984)
Winter (1989)
Bruder (2000)
Schoenfeld (1985; 1992)
Kilpatrick (1976)

Resources



Resources

Aufgabe 3 ☆ ☆ ☆ ☆ ★

A plant grows 20% in the 17th calendar week and 10% in the 18th calendar week.
By what percentage has the plant grown in the last two weeks?

a) 30 % b) 31 % c) 32 % d) 35 % e) 15 %

Resources

Aufgabe 3 ☆☆☆☆★

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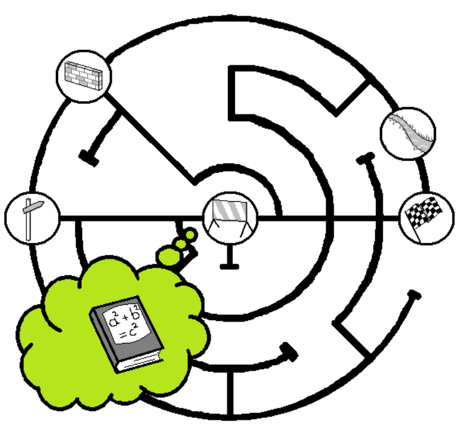
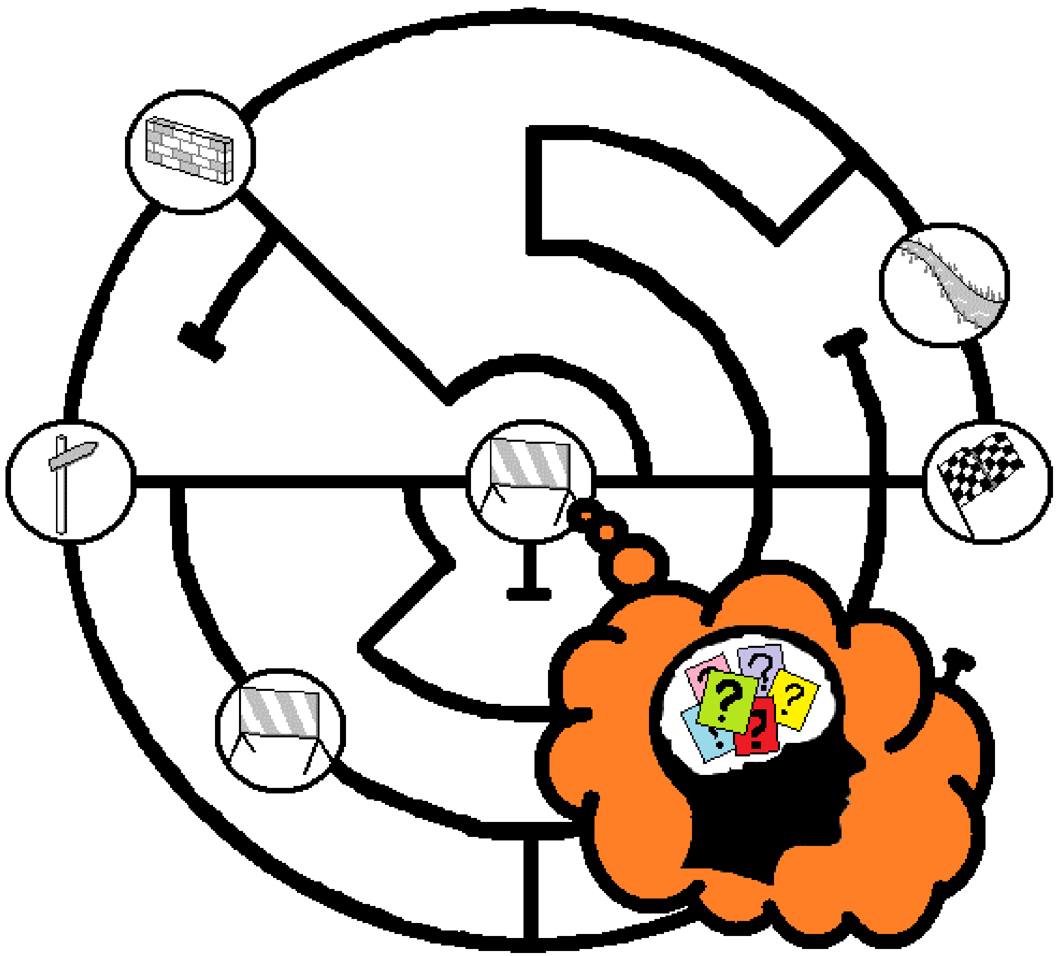
By what percentage has the plant grown in the last two weeks?

a) 30 %   b) 31 %   c) 32 %   d) 35 %   e) 15 %

Hilfsmaßeinheit	
Einheitenname	Prozent
Einheitenzeichen	%
Formelzeichen	p
Typ	Quotient
Definition	$1\,\% = 1 \cdot 10^{-2} = 0{,}01$
Benannt nach	italienisch *per cento*, „von Hundert“
Siehe auch: Promille, ppm, ppb	



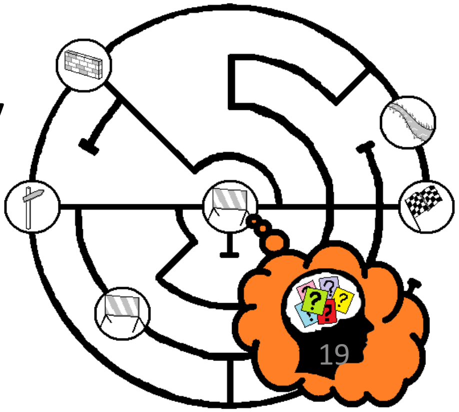
Beliefs



What are Beliefs?

- ***Beliefs*** are (stable, but also situation-related) characteristics of persons who indirectly influence action.
- They serve as a "lens" and filter our perception.
- Similar constructs are *motivation*, *affects*, and *attitudes*. These are often unconscious and can refer to mathematics in general or to problem solving in particular.
- Beliefs (or belief systems) exist for many aspects.

(Schoenfeld 1992; Thompson 1992; Philipp 2007)



Examples for beliefs of students

Math problems have just the right solution.

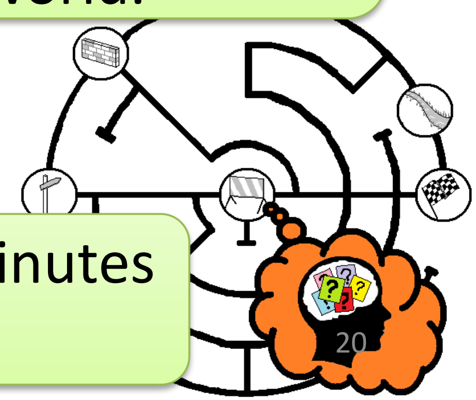
Normal students can't understand math. You learn the rules by heart and apply them.

Math problems have just the right solution. Usually the rule last introduced by the math teacher.

School mathematics has nothing to do with the real world.

It takes a maximum of 5 minutes to solve a math problem.

Schoenfeld (1992)



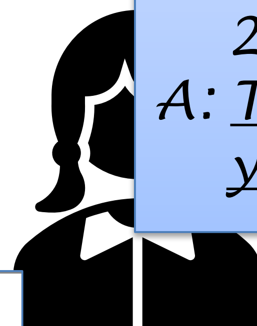
A school trip is being planned: How many buses with 20 seats each are needed for 310 students?

R: 310 : 20 = 15.5
A: One needs 15.5 buses.

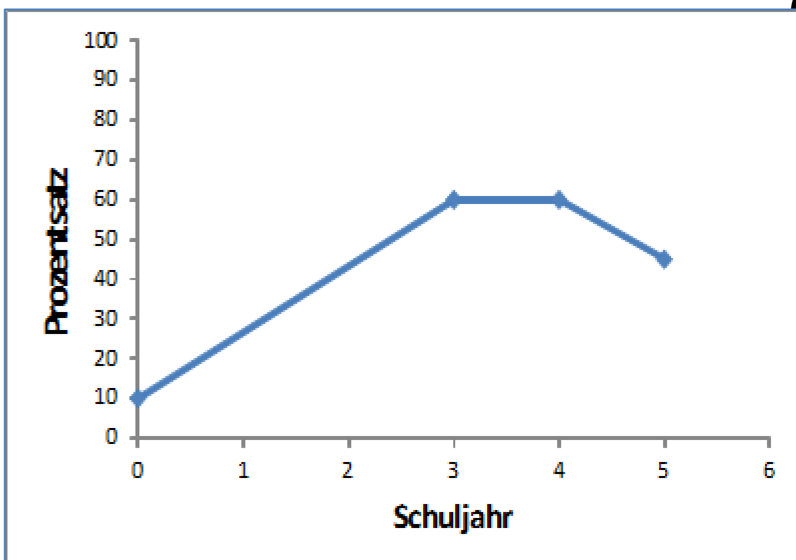
r be

A captain carries 26 sheep and 10 goats on his ship.
How old is the captain?

R: 26 · 10?
26 + 10 = 36
A: The captain is 36 years old.

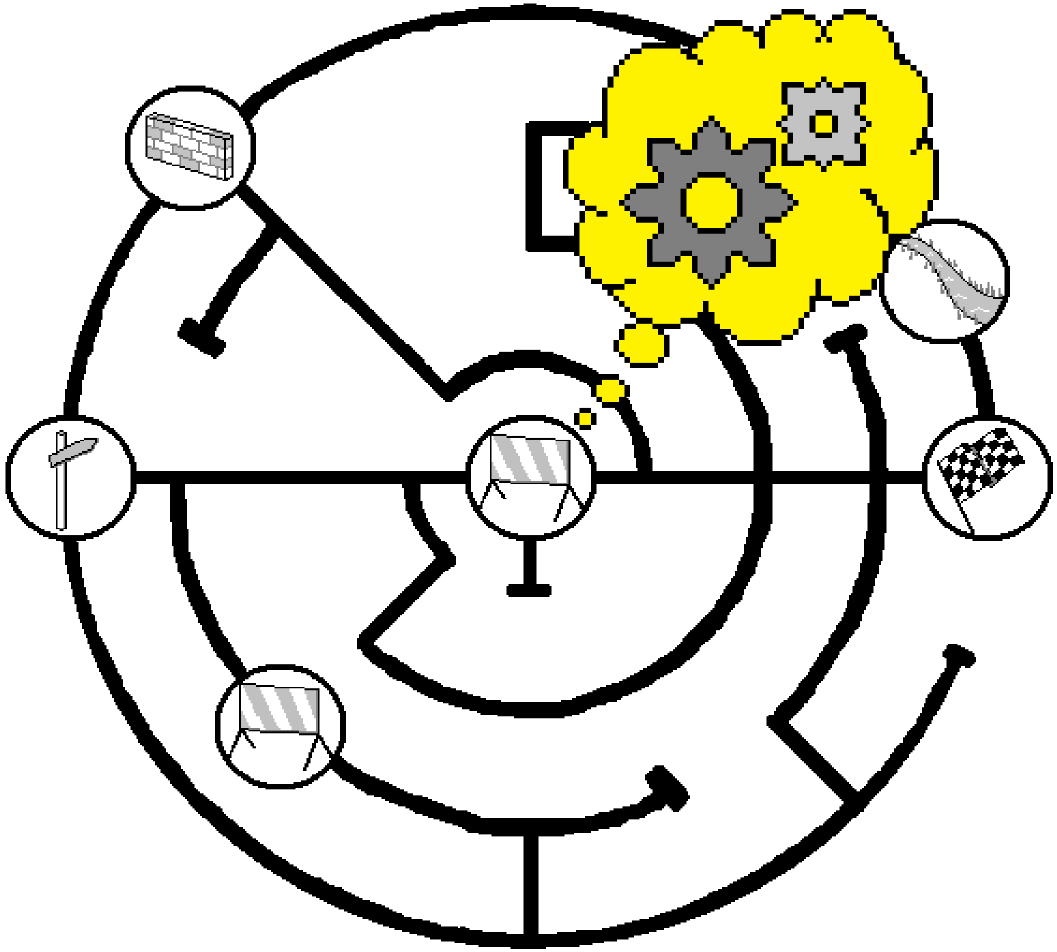


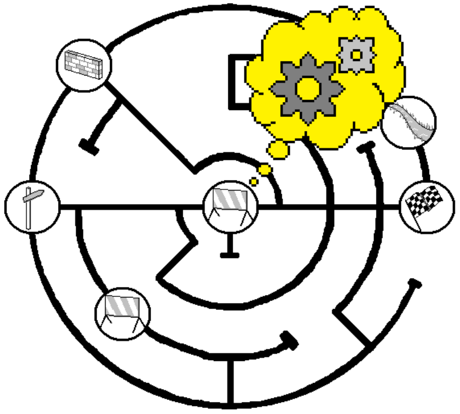
School mathematics has nothing to do with the real world.



Study on the “Captain's task” in Belgium from kindergarten to grade 5. The share of “solutions” by addition is given.
(Verschaffel & DeCorte, 1997)

Heuristics, Problem-Solving Strategies





Heuristics

Problem-solving strategies

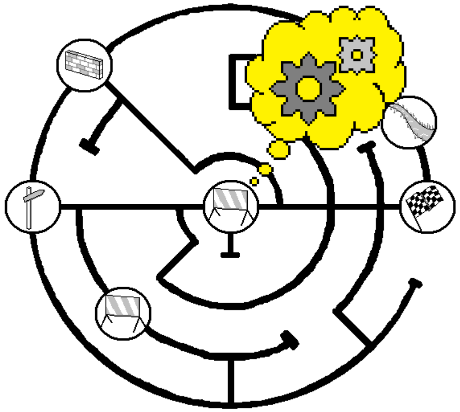
- *„the ancient, ill-defined discipline called 'heuristics'“* (McClintock 1979, S. 174)

The definitions of different authors sometimes differ considerably, e.g. with regard to

- algorithmic procedures
- metacognition

Pólya (1949); Kilpatrick (1969); Kantowski (1974); Schoenfeld (1985); Bruder (2000)

Heuristics

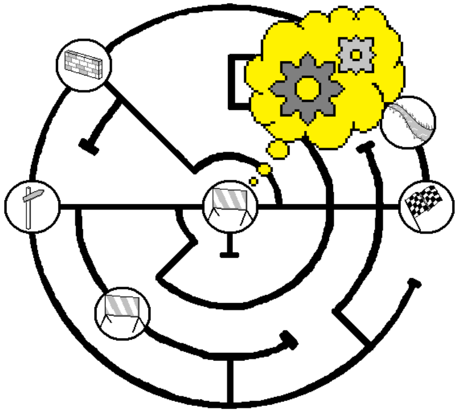


Problem-solving strategies

Working definition:

Heuristics is a collective term for devices, methods, or (cognitive) tools, often based on experience. They are used under the assumption of being helpful when solving a problem (but do not guarantee a solution). […] Though their nature is cognitive, the application and evaluation of heuristics is operated by *metacognition*.

Rott (2014)

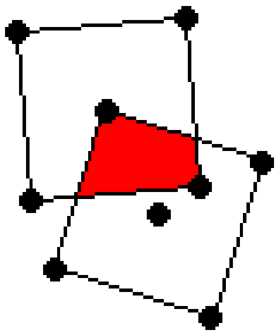


Examples of Heuristics

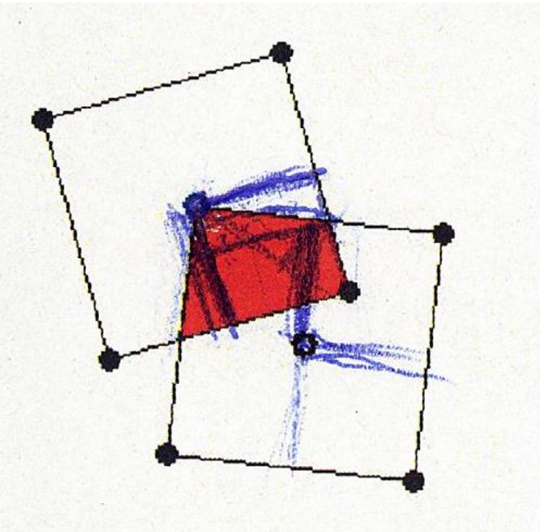
Beverage Coasters

The two pictured squares depict coasters. They are placed so, that the corner of one coaster lies in the center of the other.

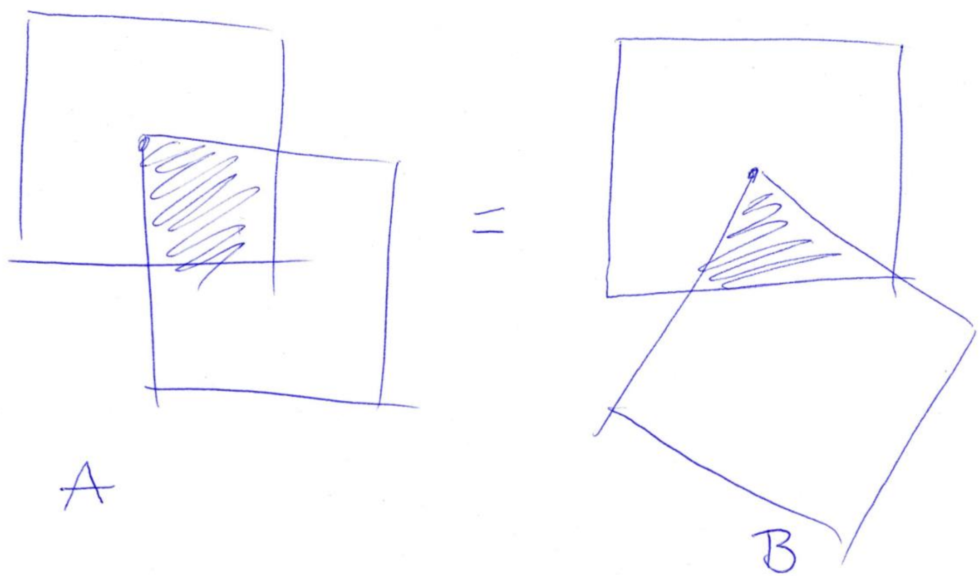
Examine the size of the area covered by **both** coasters.



Examples of Heuristics

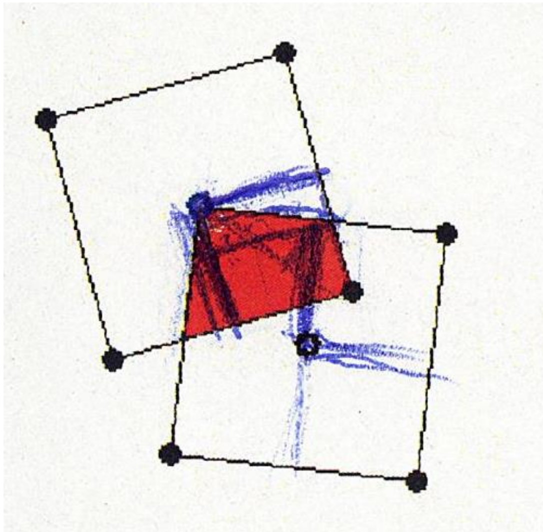


Student‘s solution by **Lucy** (5. grade)



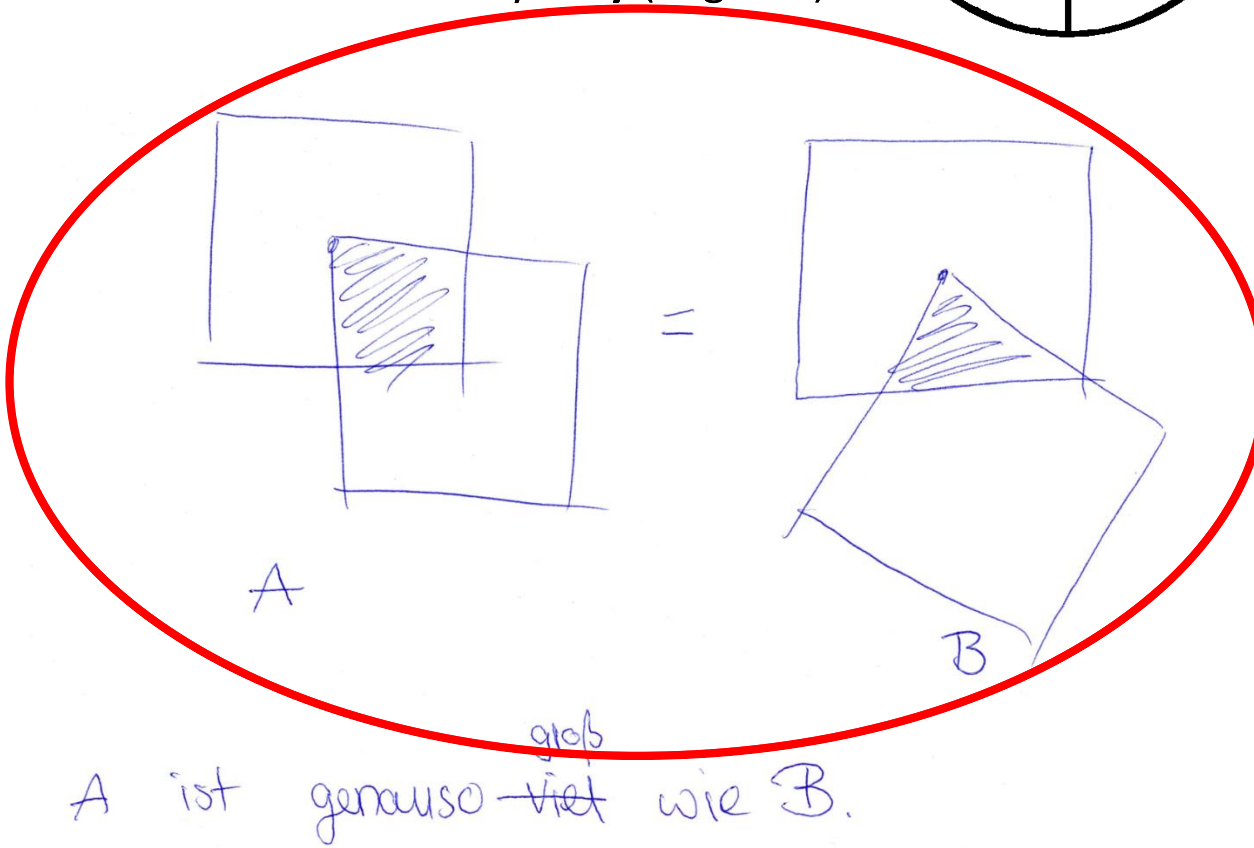
A ist genauso ~~viel~~ groß wie B.

Examples of Heuristics

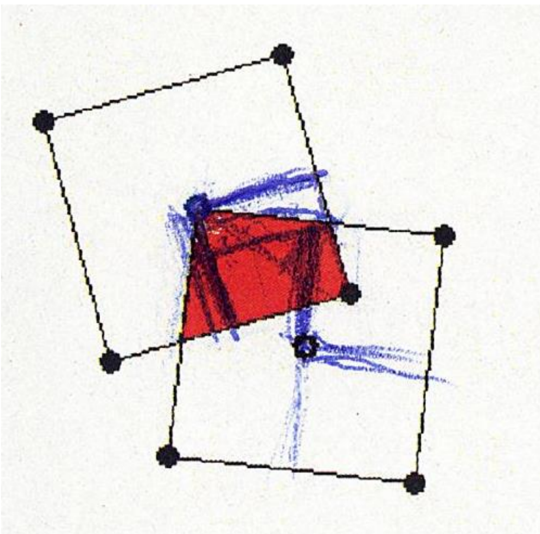


- Drawing figures

Student‘s solution by **Lucy** (5. grade)



Examples of Heuristics



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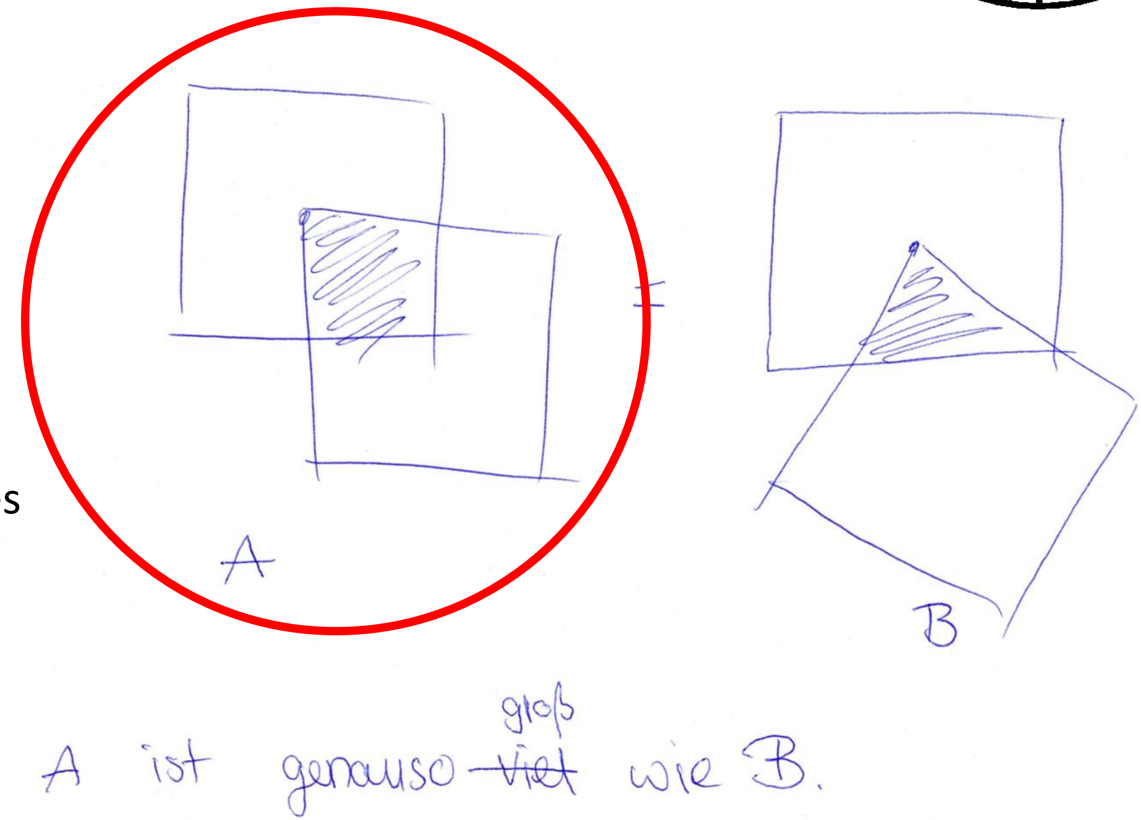
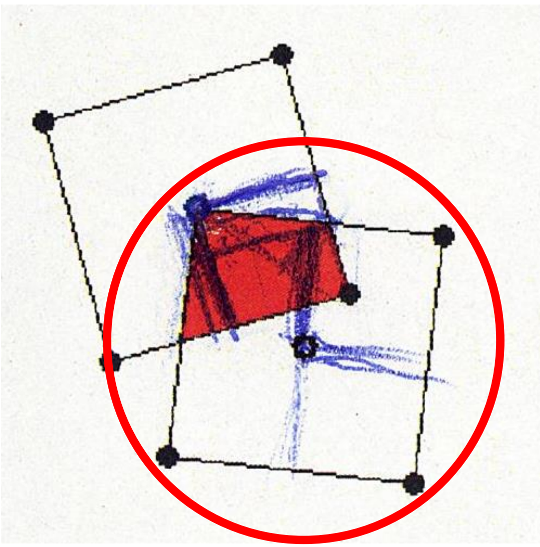


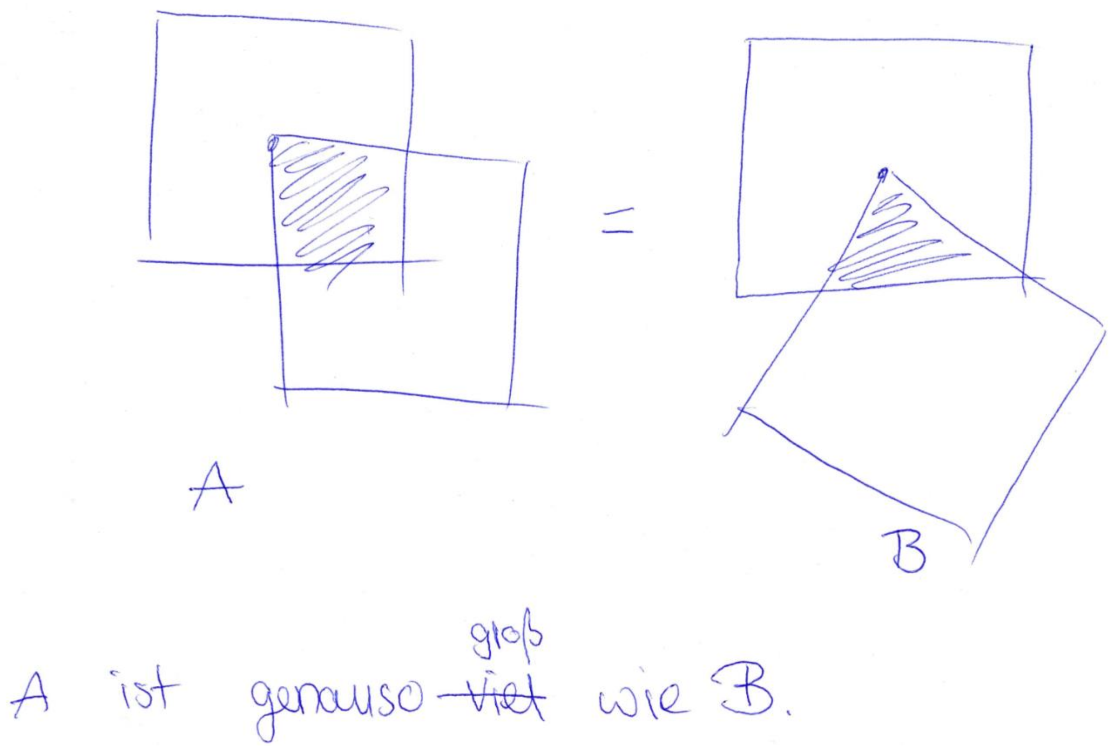
- Drawing figures
- Working with special cases

Examples of Heuristics

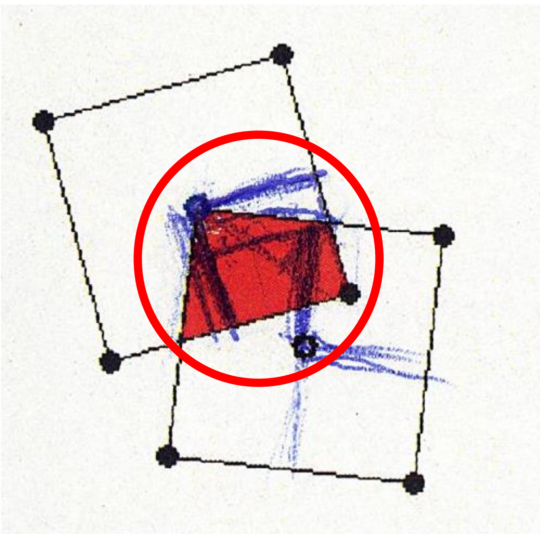


- Drawing figures
- Working with special cases
- Using auxiliary lines

Student‘s solution by **Lucy** (5. grade)

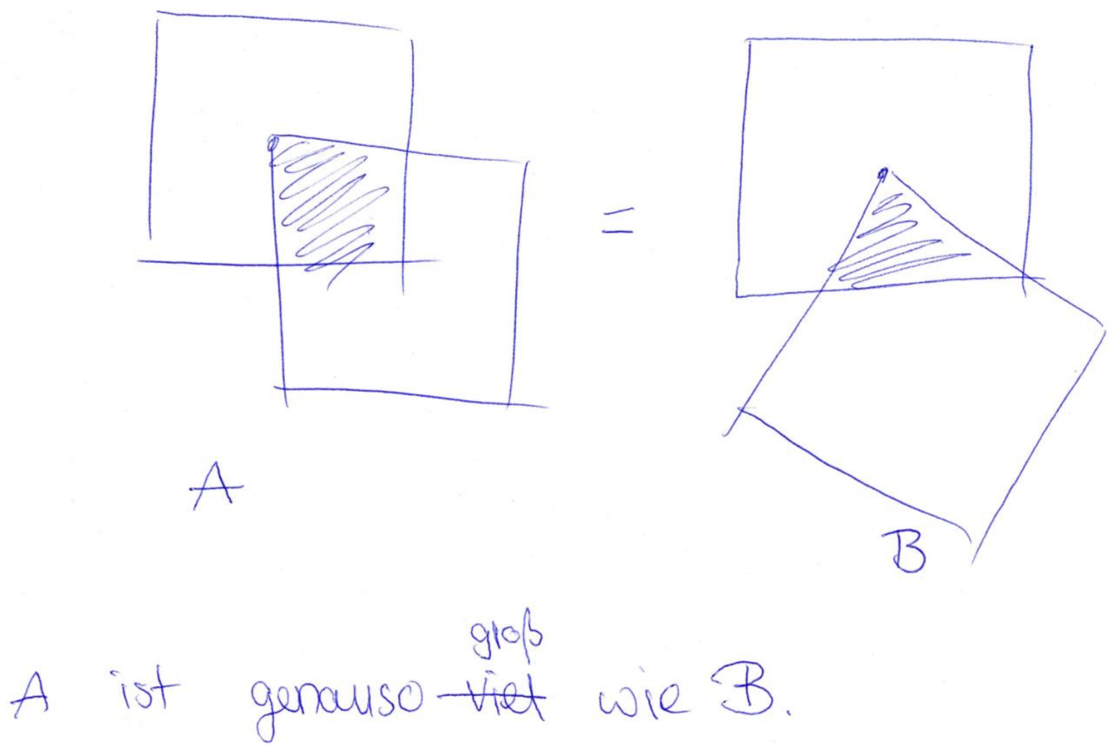


Examples of Heuristics



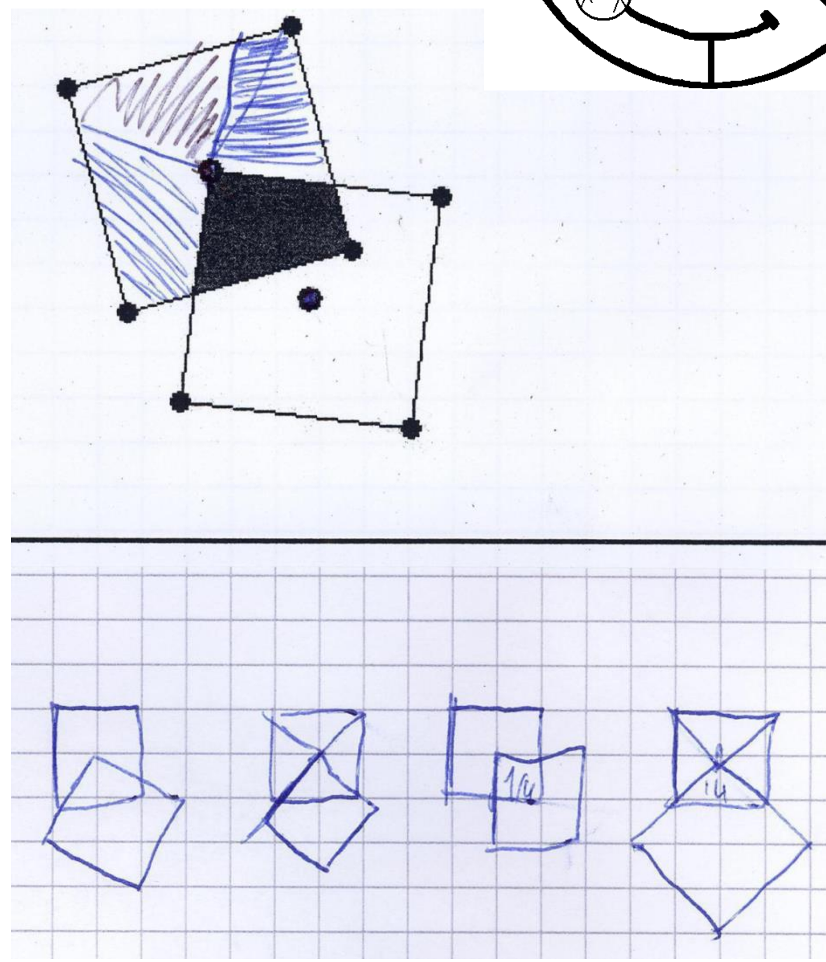
- Drawing figures
- Working with special cases
- Using auxiliary lines
- Disassemble

Student‘s solution by **Lucy** (5. grade)



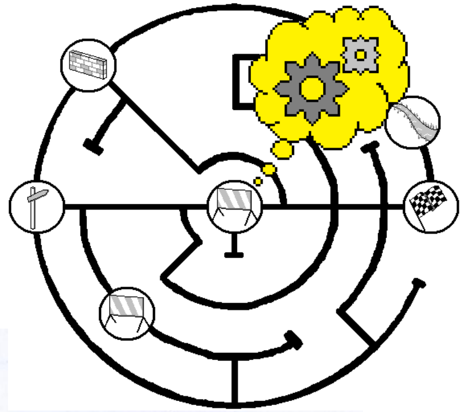
Examples of Heuristics

Student‘s solution by **Vincent** (5. grade)



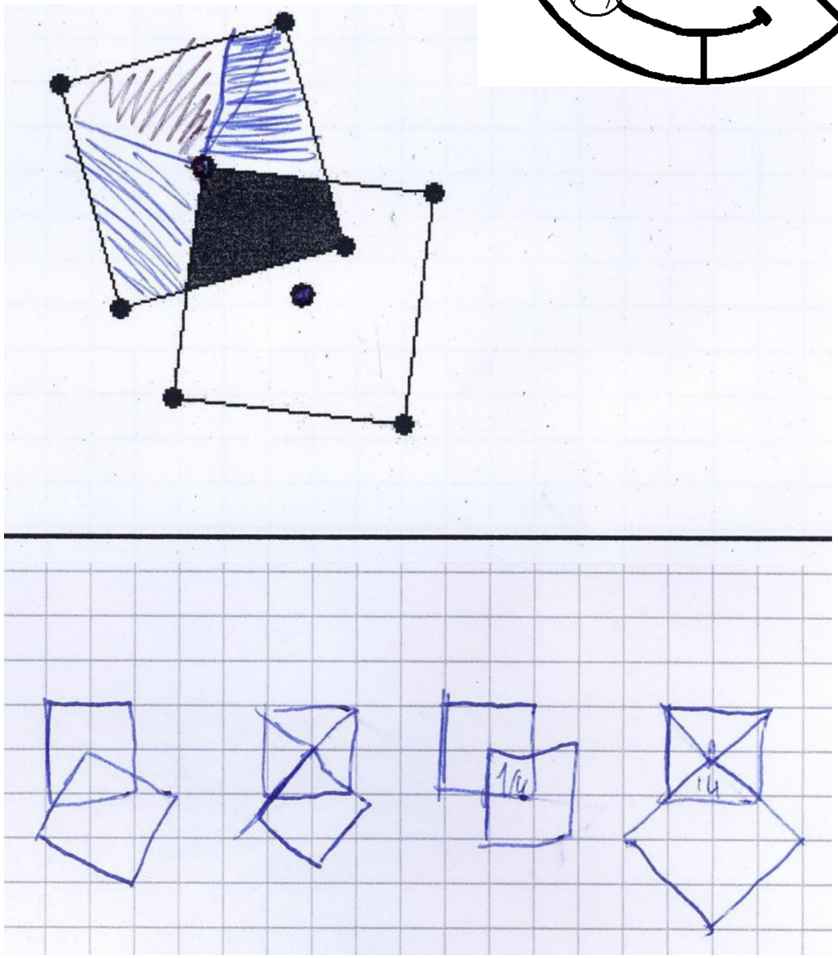
- Drawing figures
- Working with special cases
- Using auxiliary lines
- Disassemble

Heuristics: How they work



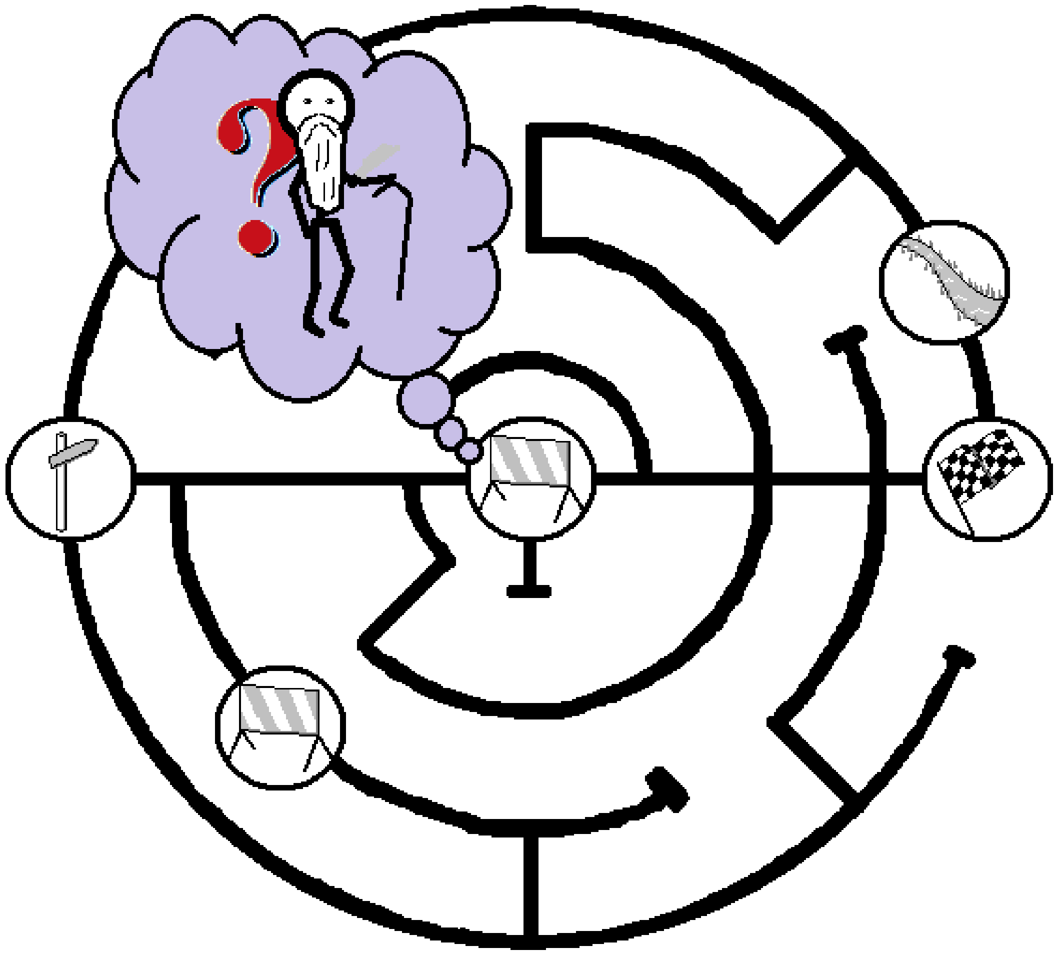
Heuristics can help to

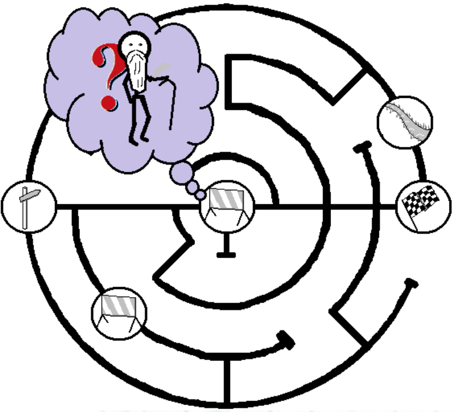
- better understand a task
- generate ideas for further actions
- find arguments for a solution.



Rott (2018)

Metacognition, Self-Regulation





Metacognition

Aufgabe 10 ★ ★ ★ ★ ★

What remainder does the following division provide

$$5^{15} : 6 = \blacksquare ?$$

a) 1 **b)** 2 **c)** 3 **d)** 4 **e)** 5

Metacognition

★ ★ ★ ★ ★

division provide

$= ■?$

3

e) 5

$$5^2 = 5 \cdot 5 = 25$$

$$5^3 = 25 \cdot 5 = 125$$

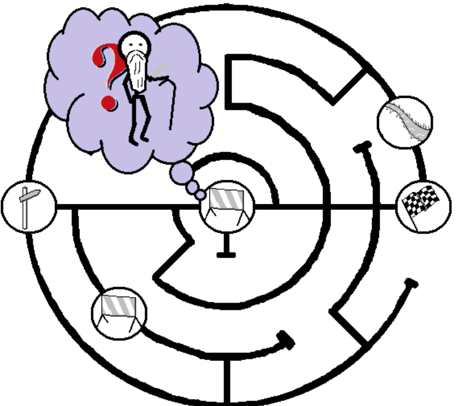
$$5^4 = 125 \cdot 5 = 625$$

$$5^5 = \underbrace{625 \cdot 5}_{3125} = 3125$$

$$5^6 = \underbrace{3125 \cdot 5}_{15625} = 15625$$

$$\cdots$$

$$5^{15} = ?$$



Metacognition

$5^2 = 5 \cdot 5 = 25$

$5^3 = 25 \cdot$

$5^4 = 125$

$5^5 = 625$

31

$5^6 = 3125$

15625

$5^2 : 6 = 25 : 6$ R1

$5^3 : 6 = 125 : 6$ R5

$5^4 : 6 = 625 : 6 = (600 + 25) : 6$ R1

$5^5 : 6 = 3125 : 6 = (3000 + 125) : 6$ R5

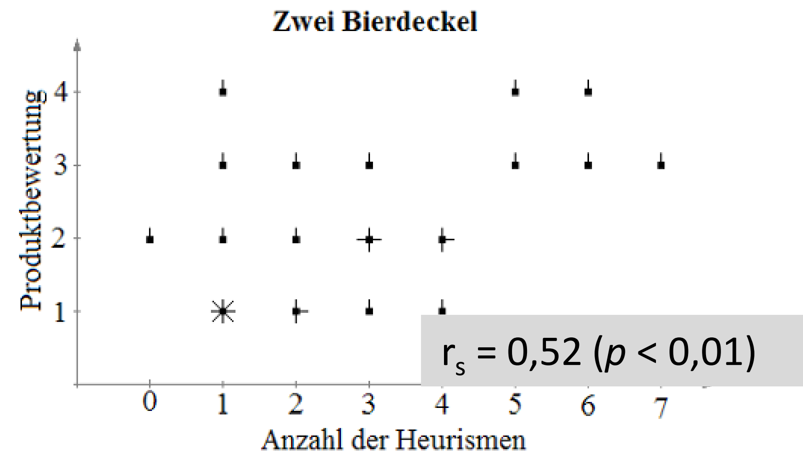
$\cdots$

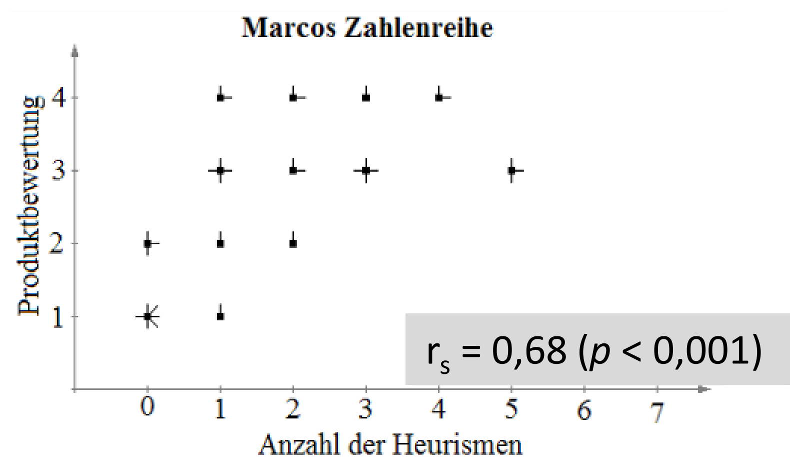
INSIGHTS INTO RESEARCH HEURISTICS

Heuristics: How they work

Many studies report the benefits of heuristics:

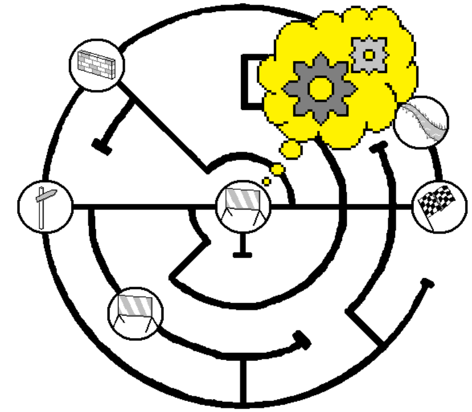
- Successful participants use more heuristics than less successful participants.
- Pupils and students who have undergone heuristic training perform better in problem-solving tests than participants without training.





Kilpatrick (1967); Lucas (1974); Kantowski (1974); Koichu, Berman & Moore (2007); Collet (2009); Rott (2013)

Heuristics

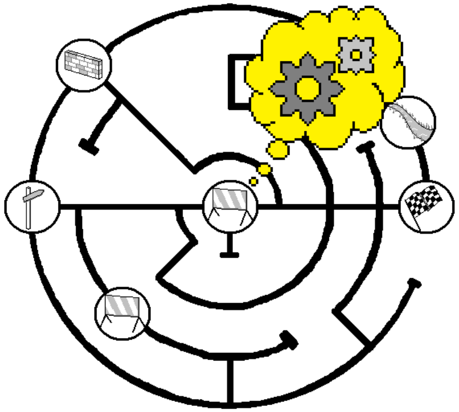


- There is a positive correlation between problem solving competence and the use of heurisms *("more is better")*.

- "students' use of heuristic strategies was positively correlated with performance on ability tests, and on specially constructed problem solving tests; however, the effects were relatively small." (Schoenfeld, 1992)

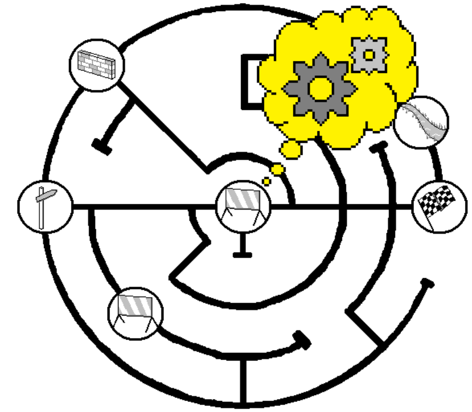
- We know from expertise research that experts can use heuristics more purposefully and thus use fewer heuristics than novices (Chi, 2006).

Heuristics



- Why are the training results so unsatisfactory and the learning and transfer effects so weak?
- One reason for this: Heuristics are *descriptive*, not *prescriptive* (vgl. Schoenfeld, 1992).

Heuristics
descriptive, not prescriptive

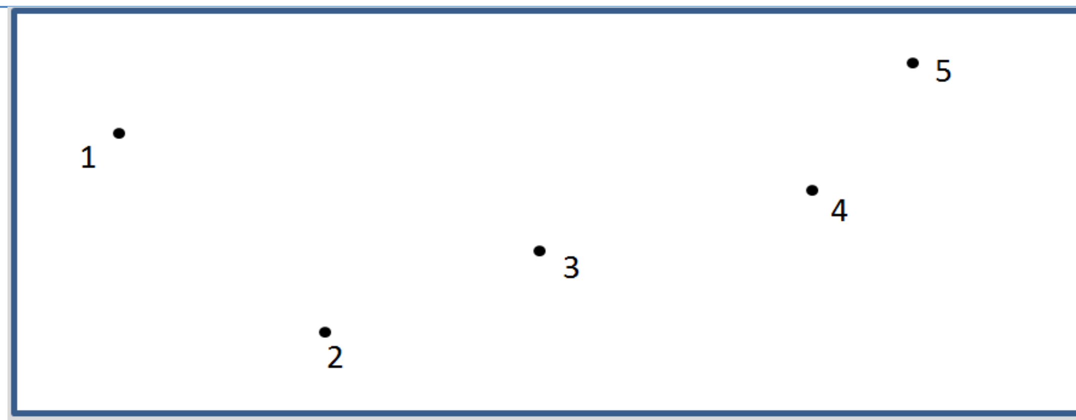


How many divisors does 10.000.000 have?

Proof that the following statement is true for $0 < a, b, c, d < 1$:

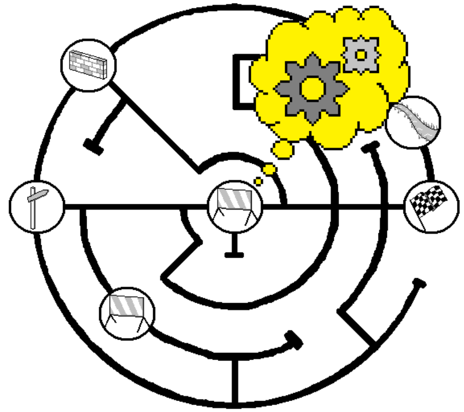
$$(1-a)(1-b)(1-c)(1-d) > 1-a-b-c-d$$

The map shows a piece of land.
There are five wells in this area.



Develop a division of the land into five areas, so that to each place in an area the well in that area is nearest.

Heuristics
descriptive, not prescriptive

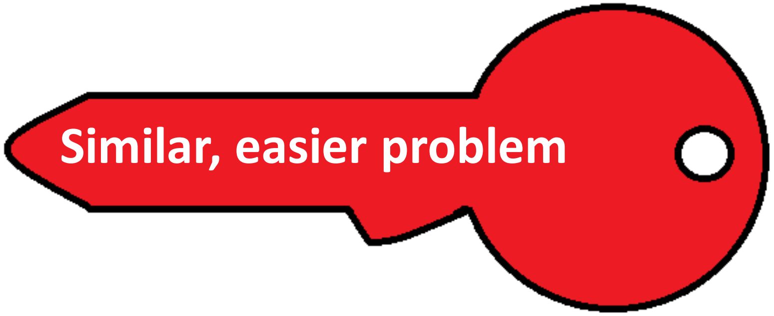


How many divisors does 10.000.000 have?

How many divisors does 100 have?

It is $100 = 10 \cdot 10 = 2^2 \cdot 5^2$

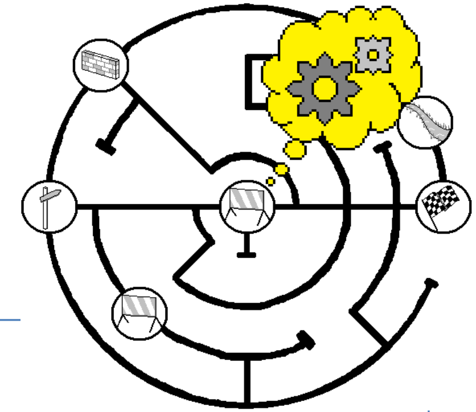
	2^0	2^1	2^2
5^0	1	2	4
5^1	5	10	20
5^2	25	50	100



There are $3 \cdot 3 = 9$ divisors.

For $10.000.000 = 2^7 \cdot 5^7$ there are $8 \cdot 8 = 64$, resp.

Heuristics
descriptive, not prescriptive

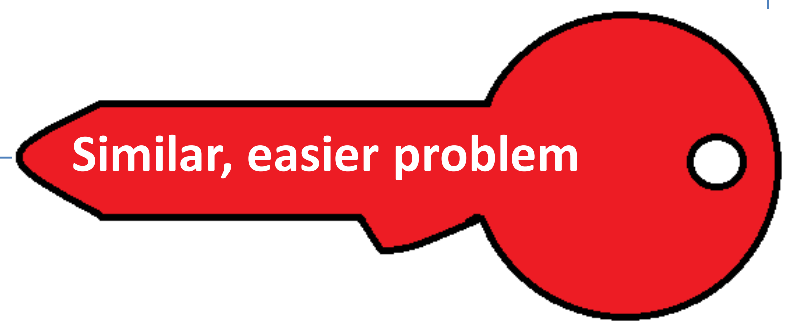


Proof that the following statement is true for
$0 < a, b, c, d < 1$:

$$(1-a)(1-b)(1-c)(1-d) > 1-a-b-c-d$$

Proof that the following statement is true for $0 < a, b < 1$:

$$(1-a)(1-b) > 1-a-b$$

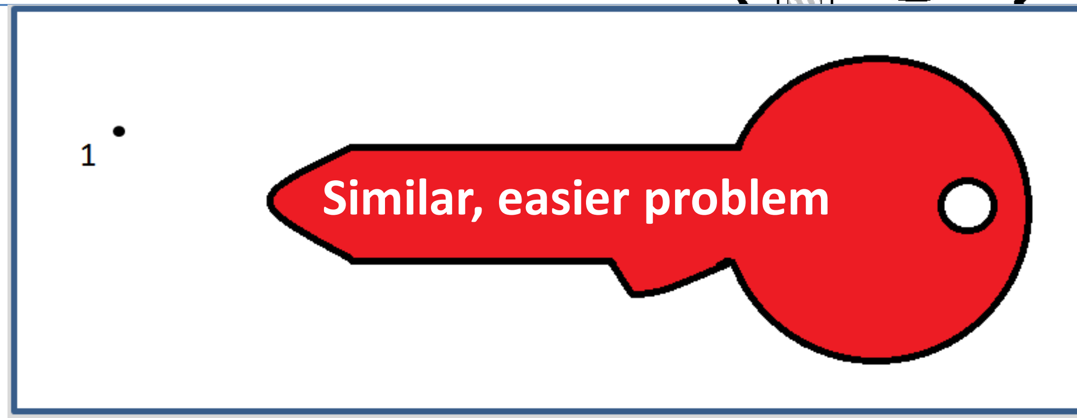


It is $(1-a)(1-b) = 1-a-b+ab > 1-a-b$

It is $(1-a-b)(1-c) = 1-a-b-c+ac+bc$

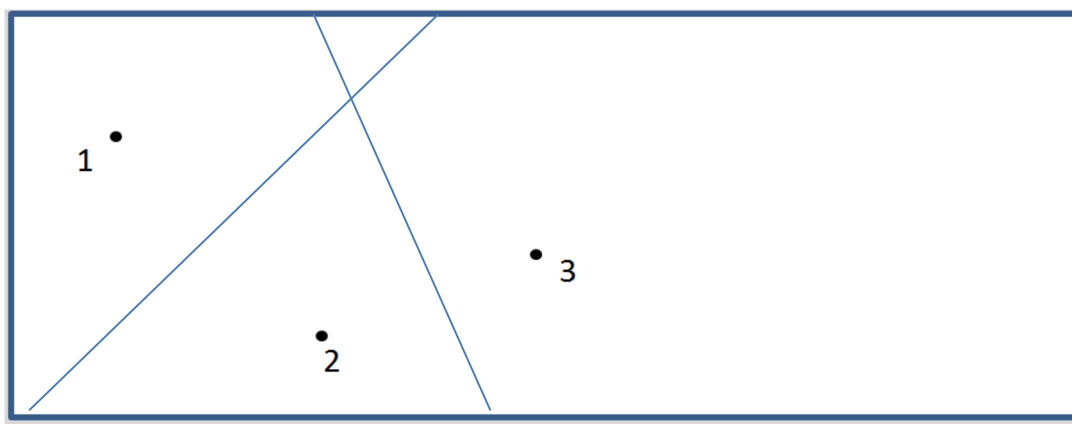
Heuristics
descriptive, not prescriptive

The map shows a piece of land.
There are five wells in this area.

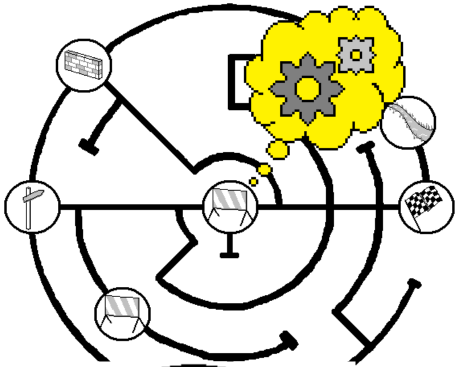


Develop a division of the land into five areas, so that to each place in an area the well in that area is nearest.

Consider only two or three wells
=> perpendicular bisector



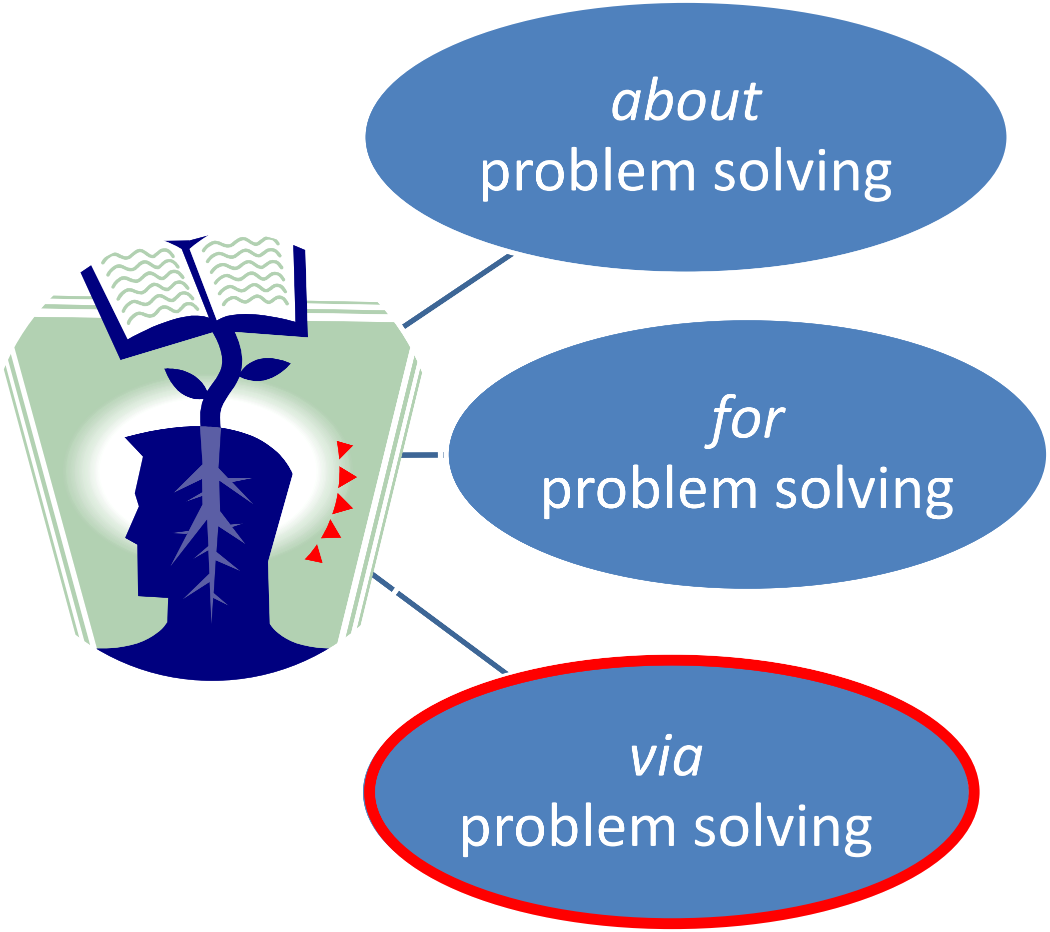
Heuristics
descriptive, not prescriptive



- A posteriori one can give the same name to these solution ideas for the three problems.
- A priori one can probably do little with hints like the following:

„If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? […] Could you solve a part of the problem? Keep only a part of the condition, drop the other part; […]“ (Pólya, 1945, xvii)

INSIGHTS INTO RESEARCH LEARNING VIA PROBLEM SOLVING



Schroeder & Lester (1989)

A typical teaching unit

Exploring	Elaborating	Deepening
• Discovering a topic • Motivating for the topic	• Systematizing • Formulating rules and algorithms	• Exercising • Applying knowledge to more complex tasks

Holzäpfel, L. & Leuders, T. (2015)

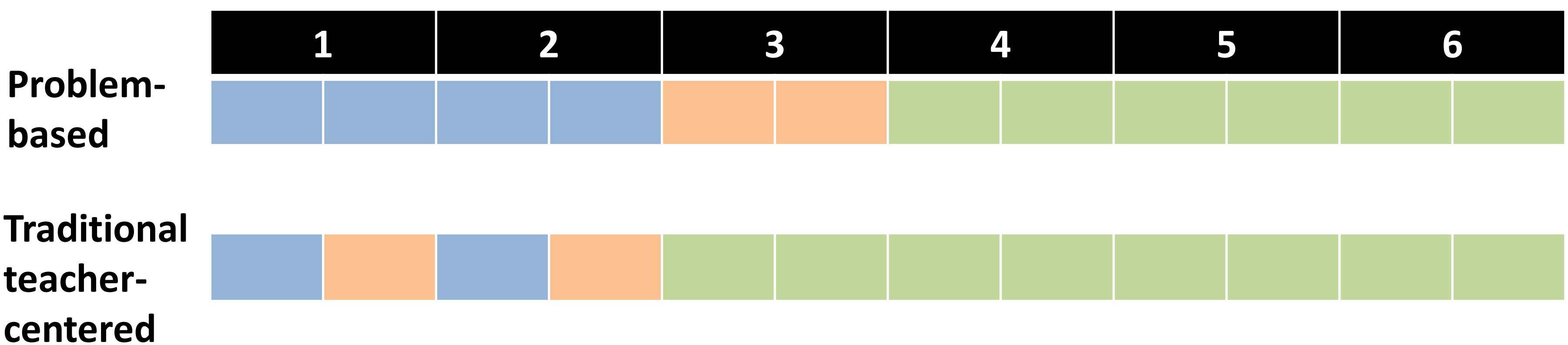
Learning via problem solving

Exploring	Elaborating	Deepening
• Discovering a topic • Motivating for the topic	• Systematizing • Formulating rules and algorithms	• Exercising • Applying knowledge to more complex tasks

"What are good problems for mathematics teaching? Especially, we need problems that fit the curricula and help teachers to teach via problem solving instead of 'matchstick problems' (a term coined by Thomas Jahnke)"
(Rott & Papadopoulos, 2019, p. 217)

A teaching experiment

Exploring	Elaborating	Deepening
• Discovering a topic • Motivating for the topic	• Systematizing • Formulating rules and algorithms	• Exercising • Applying knowledge to more complex tasks



Möller & Rott (2017)

A teaching experiment

Problem-based

Student-centered, discovery learning (cf. Bruner, 1974; Winter, 2016)

Traditional, teacher-centered

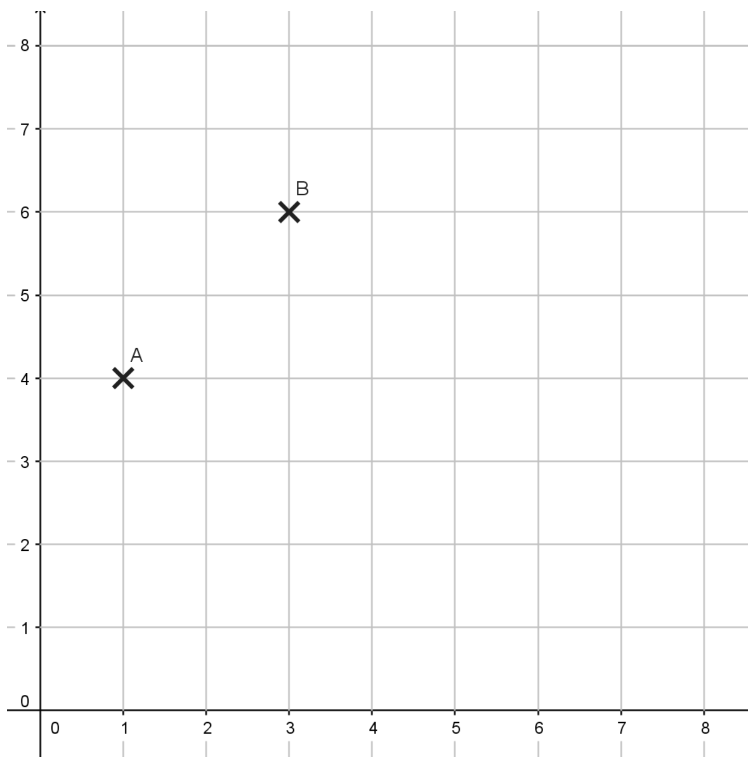
Effective acquisition of knowledge through lectures and explanations (Ausubel, 1968)

Möller & Rott (2017)

A teaching experiment

Problem-based

Student-centered, discovery learning (cf. Bruner, 1974; Winter, 2016)



Traditional, teacher-centered

Effective acquisition of knowledge through lectures and explanations (Ausubel, 1968)

Initial task
Two points A(1|4) and B(3|6) are given. Draw them in a coordinate system with the unit 1 cm.

Draw a point, which has the same distance to A and to B.

Find other points that have the same distance to A and to B and draw them. What do you notice? Note.

Möller & Rott (2017)

A teaching experiment

Problem-based

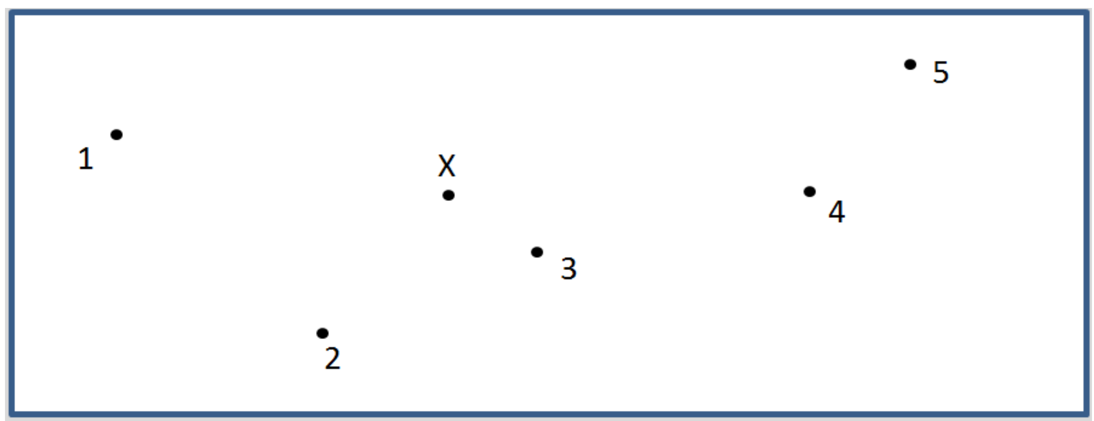
Student-centered, discovery learning (cf. Bruner, 1974; Winter, 2016)

Traditional, teacher-centered

Effective acquisition of knowledge through lectures and explanations (Ausubel, 1968)

Initial problem

The map shows a piece of land.
There are five wells in this area.
Imagine standing at X with a herd of thirsty sheep.
Which well are you going to?



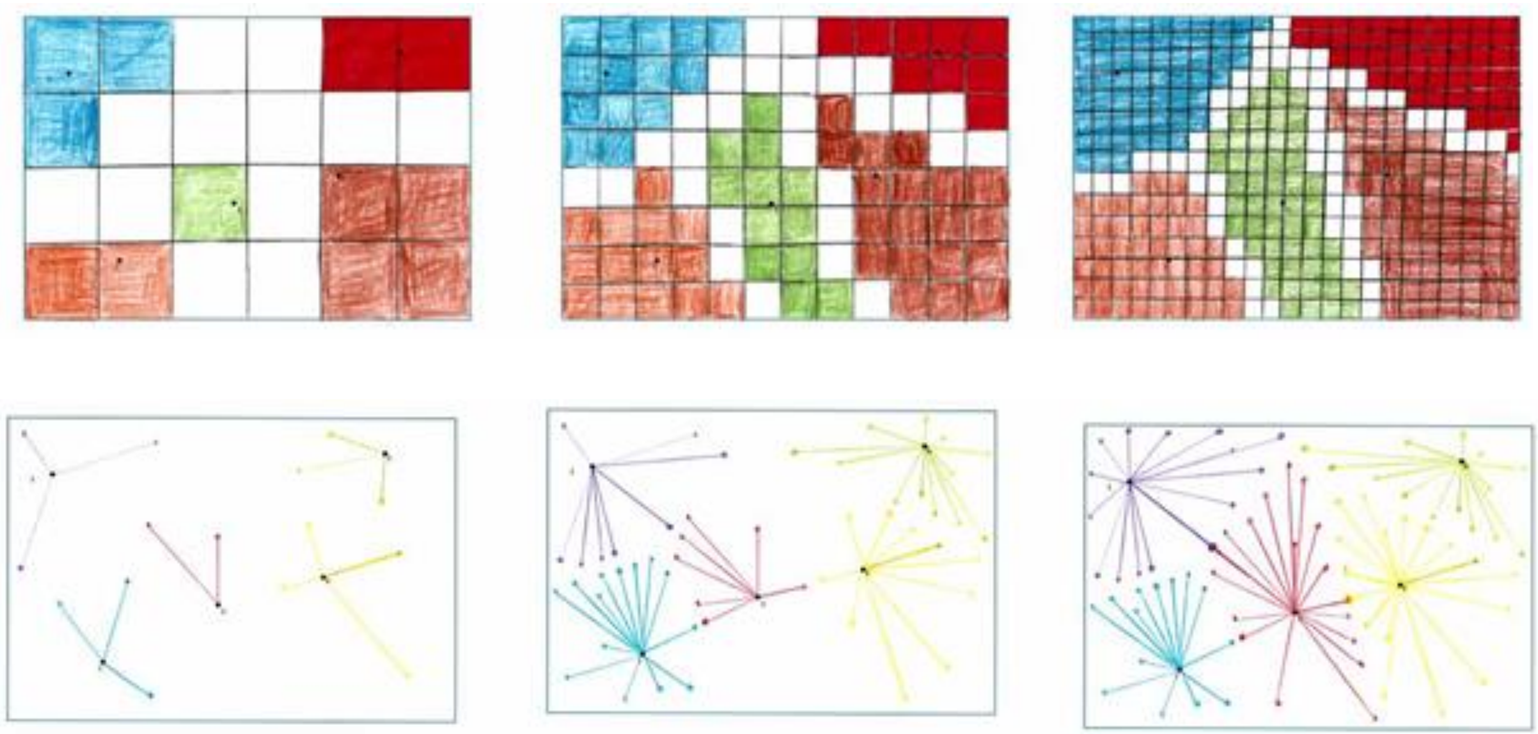
The choice was not difficult, you go to the nearest well. Now develop a division of the land into five areas, so that to each place in an area the well in that area is the nearest.

Möller & Rott (2017); Büchter & Leuders (2005); Holzäpfel, Rott & Dreher (2016)

Learning via problem solving

Problem-based

Looking at special cases

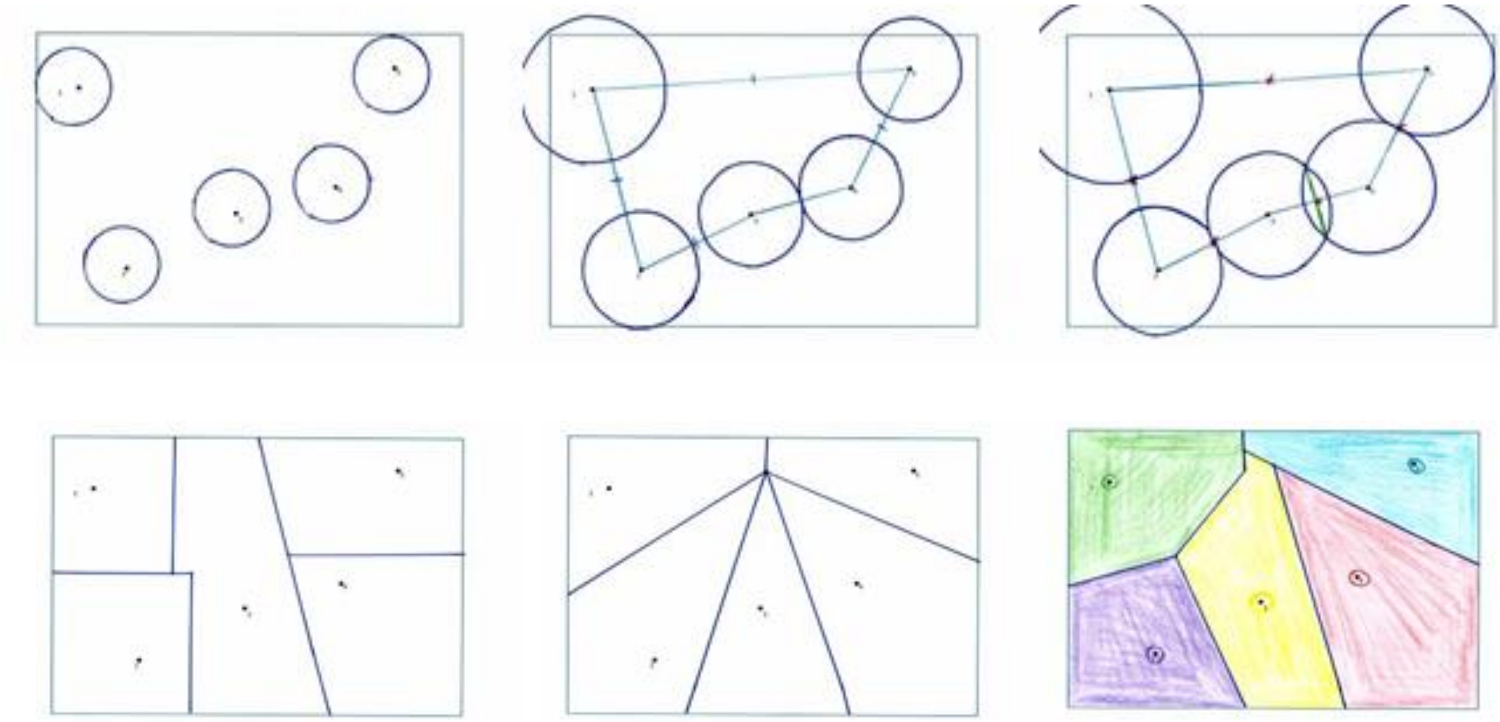


Möller & Rott (2017); Büchter & Leuders (2005); Holzäpfel, Rott & Dreher (2016)

Learning via problem solving

Problem-based

Going back to sth. known



Möller & Rott (2017); Büchter & Leuders (2005); Holzäpfel, Rott & Dreher (2016)

Learning via problem solving

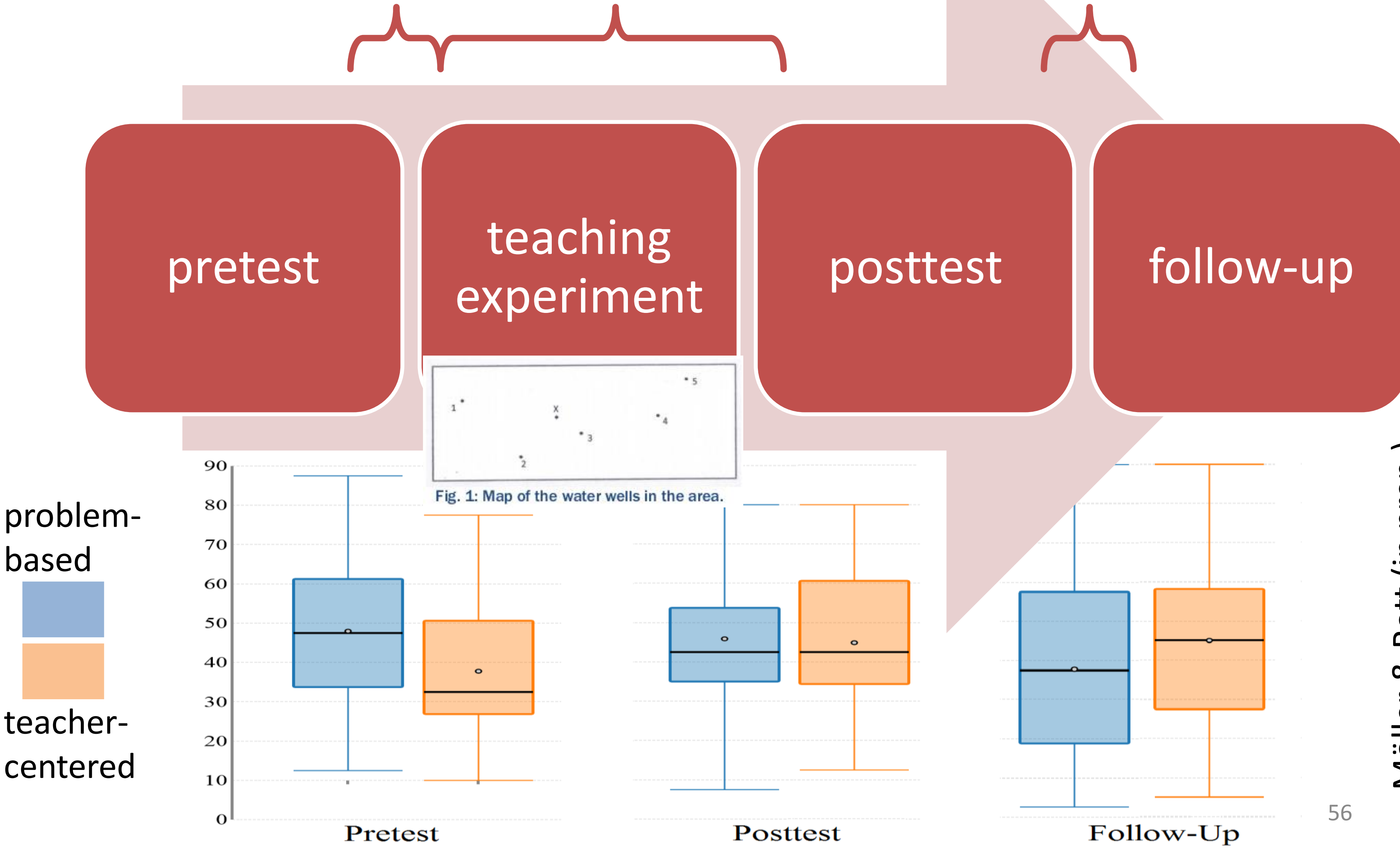


Fig. 1: Map of the water wells in the area.

Möller & Rott (in prep.)

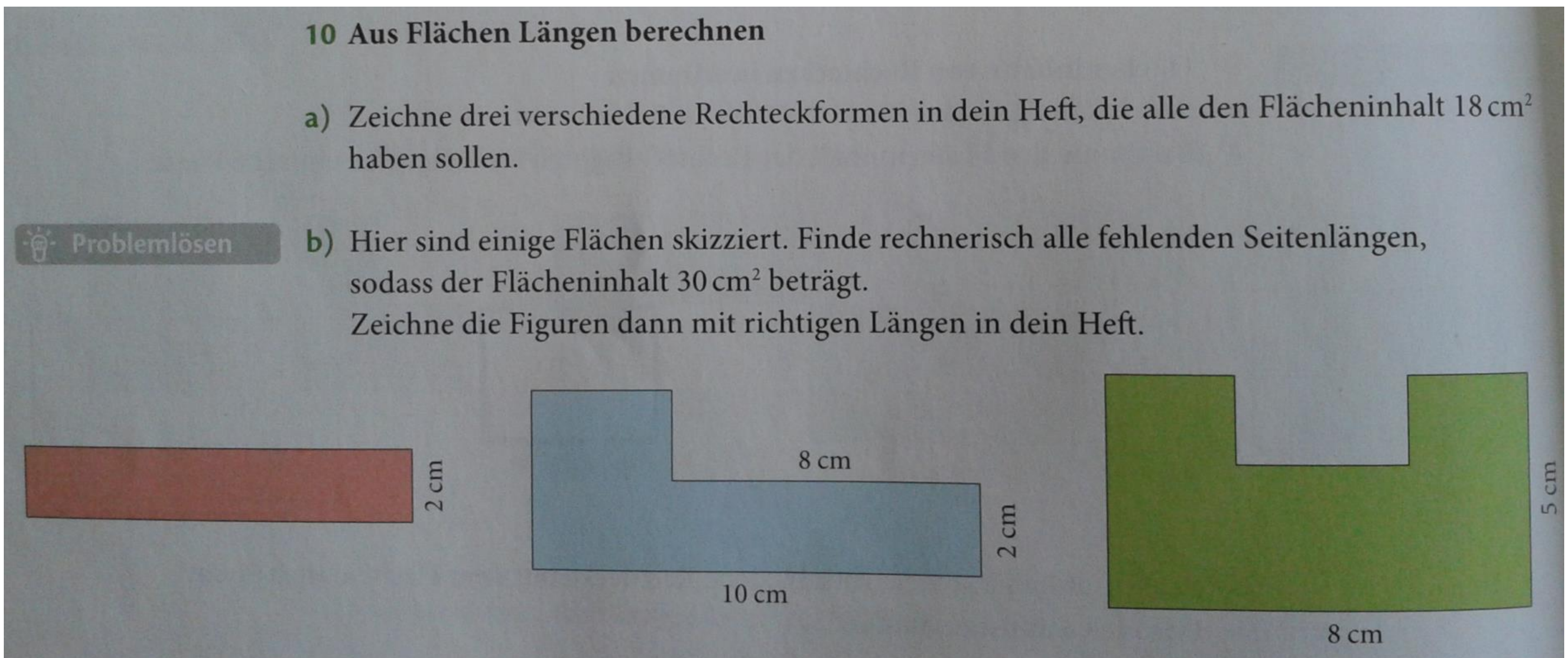
Learning via problem solving

Exploring	Elaborating	Deepening
• Discovering a topic • Motivating for the topic	• Systematizing • Formulating rules and algorithms	• Exercising • Applying knowledge to more complex tasks

“productive exercises”

a) Draw three different rectangular shapes in your booklet, each with a surface area of 18 cm².

b) Some areas are sketched here. Calculate all missing side lengths so that the area is 30 cm².



10 Aus Flächen Längen berechnen

a) Zeichne drei verschiedene Rechteckformen in dein Heft, die alle den Flächeninhalt $18\,cm^2$ haben sollen.

Problemlösen

b) Hier sind einige Flächen skizziert. Finde rechnerisch alle fehlenden Seitenlängen, sodass der Flächeninhalt $30\,cm^2$ beträgt.
Zeichne die Figuren dann mit richtigen Längen in dein Heft.

“productive exercises”

Find several solutions for the following equation:

- $\frac{\square}{\square} + \frac{\square}{\square} = \frac{1}{2}$

What numbers can you represent as the sum of unit fractions (numerator is 1)?

- $\frac{1}{\square} + \frac{1}{\square} = \square$

INSIGHTS INTO RESEARCH TESTING PROBLEM SOLVING

WYTIWYG

What you test is what you get

- If problem solving (or other process-related competencies like reasoning or modeling) is not part of written tests (by the teachers or centralized), students will not take it seriously.
- If problem solving is not part of centralized tests (and instead rote skills are tested), teachers will not pay much attention to problem solving but to routine procedures.

(cf. Maaß, 2007)

Control mechanisms for teaching contents

External influences and control mechanisms for teaching contents:

- (standardized) curricula
- school books
- school inspections
- a system of reporting
- standardized tests (for comparisons or for degrees/diplomas)

(Altrichter & Maag Merki, 2016)

Operationalizing PS competence

“Problem solving competence cannot be described as a unidimensional construct. At least analytical and dynamic aspects of problem solving competence have to be distinguished if all aspects of problem solving are to be covered. Analytical problem solving abilities are needed to structure, represent and integrate information. Dynamic problem solving includes aspects of self-regulated learning as well as the ability to adapt the problem solving process to a changing environment by continuously processing feedback information. The assessment of dynamic aspects of problem solving competence requires dynamic test environments. […]”

(Wirth & Klieme, 2003)

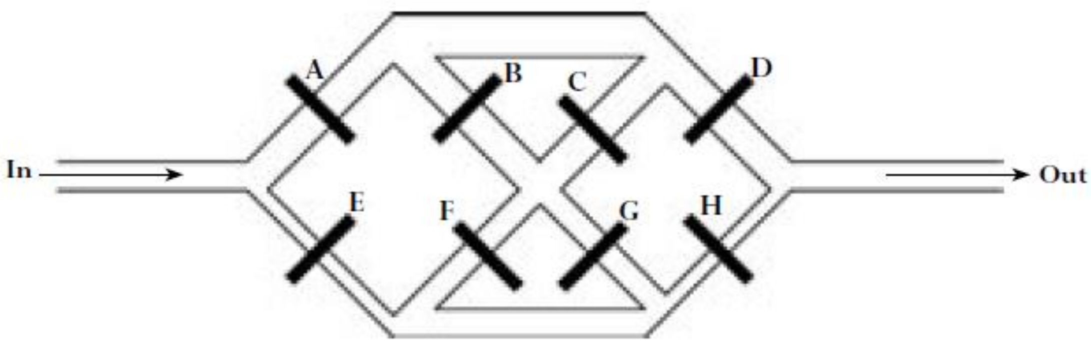
Trouble shooting – the **IRRIGATION** problem

vel 2 (*Irrigation*, Question 2 and

ı 1), 544 (*Irrigation*, Question 2)

Below is a diagram of a system of irrigation channels for watering sections of crops. The gates A to H can be opened and closed to let the water go where it is needed. When a gate is closed no water can pass through it.

This is a problem about finding a gate which is stuck closed, preventing water from flowing through the system of channels.



Michael notices that the water is not always going where it is supposed to.

He thinks that one of the gates is stuck closed, so that when it is switched to open, it does not open.

Trouble shooting – the **IRRI**

Below is a diagram of a system of irrigation channels f sections of crops. The gates A to H can be opened and c the water go where it is needed. When a gate is closed n pass through it.

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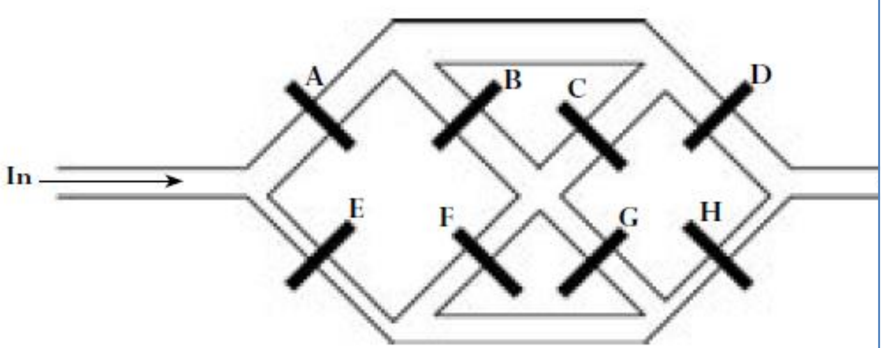

Michael notices that the water is not always going supposed to.

He thinks that one of the gates is stuck closed, so that switched to open, it does not open.

IRRIGATION – Question 1

Michael uses the settings given in Table 1 to test the gates.

Table 1. **Gate Settings**

A	B	C	D	E	F	G	H
Open	Closed	Open	Open	Closed	Open	Closed	Open

With the gate settings as given in Table 1, **on the diagram below** draw all the possible paths for the flow of water. Assume that all gates are working according to the settings.

IRRIGATION – Question 2

Michael finds that, when the gates have the Table 1 settings, no water flows through, indicating that at least one of the gates set to open is stuck closed.

Decide for each problem case below whether the water will flow through all the way. Circle "Yes" or "No" in each case.

Problem Case	Will water flow through all the way?
Gate A is stuck closed. All other gates are working properly as set in Table 1.	Yes / No
Gate D is stuck closed. All other gates are working properly as set in Table 1.	Yes / No
Gate F is stuck closed. All other gates are working properly as set in Table 1.	Yes / No

IRRIGATION – Question 3

Michael wants to be able to test whether **gate D** is stuck closed.

In the following table, show settings for the gates to test whether **gate D** is stuck closed when it is set to open.

Settings for gates (each one open or closed)

A	B	C	D	E	F	G	H

Operationalizability

- It is a research gap how problem solving competence, heuristic literacy etc. can best be measured.

- PISA offers a possible approach (measurement of partial competences; dynamic approaches) - but alternatives have to be discussed.

CONCLUSIONS

Conclusions

- Problem solving is a field of research in which there is a lot to do – even though there has been research for decades.
- In my eyes, we especially need good, content-relevant problems for teaching *via* problem solving (no more „matchstick problems“).
- Also, we need to closely look at how and when we test for problem-solving competency.

Conclusions

"Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice. […] if you wish to learn swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems." (Pólya 1962, p. v)

"Wanted a swimming-teacher who can swim himself." (Advertisement in a French provincial newspaper; Freudenthal 1973, p. 162)

"Isn't problem-solving practice all you really need?

This is the issue of learning to solve problems by solving problems. One does, of course. The acquisition of heuristic strategies from one's own problem-solving experience is a remarkably inefficient, hit-or-miss process. [...] In a word, the answer to [the question] is 'no'." (Schoenfeld 1985, p. 211 f.)

Thank You!

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