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Modules with isomorphic multiples and rings with isomorphic matrix rings. A survey. (English)

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This slim volume is packed with information about various kinds of cancellation problems. It is particularly strong on examples, but there are also some proofs, and many results with pointers to further reading.

In what follows, we shall use R and S to denote rings; P, Q, M will denote R-modules; n will be a positive integer; nP will denote the direct sum of n copies of P; and $M_n(R)$ will denote the ring of all $n \times n$ matrices over R. Although some information is given about the usual cancellation problem (when does $P \oplus M \cong Q \oplus M$ imply $P \cong Q$), this is not the main question studied. Most of the material concerns the two following cancellation problems.

Problem of n-cancellation: When does $nP \cong nQ$ as modules imply $P \cong Q$?

Problem of M_n-cancellation: When does $M_n(R) \cong M_n(S)$ as rings imply $R \cong S$?

These problems tend to be rather subtle. For instance if $nP \cong nQ$ with $n \geq 2$ and P and Q satisfy some reasonable conditions such as being finitely-generated and/or projective, then it can be difficult to distinguish P and Q if in fact they are not isomorphic.

Many examples are known of the failure of n-cancellation. Perhaps the most basic type of example, and the first of many given here, is to take a Dedekind domain R of class number 2, let P be a non-zero non-principal ideal of R, and then it is standard that $2P \cong 2R$ but P is not isomorphic to R as R-modules. Other examples are given over integral group rings, polynomials in two indeterminates over a non-commutative division ring, and so on, with related information concerning the failure of cancellation in the more standard sense. In addition to many very interesting but specific examples (including the well-known one involving the coordinate ring of the 2-sphere), the author gives some general methods for constructing examples of the failure of n-cancellation and positive results about when n-cancellation is valid. The rings considered are not always commutative or Noetherian, and for instance much information is given about positive results and counterexamples over von Neumann regular rings.

Concerning M_n-cancellation, the first question considered is that of how to recognize when a ring is a full $n \times n$ matrix ring for some $n \geq 2$. A good illustration of the subtlety of this question is the following statement which was proved by Levy, Robson, and Stafford. Let p be an odd prime number; let H be the ring of quaternions with integer coefficients; let R_p be the ring of all 2×2 matrices with diagonal entries in H and off-diagonal entries in pH; then R_p is a full 2×2 matrix ring over some "hidden" ring if and only if $p \equiv 1 \pmod 4$. The author considers various recognition theorems and examples of hidden matrices based either on tiled matrix rings (such as R_p) or skew polynomial rings. Then examples of the failure of M_n-cancellation are given, the first one here and the earliest one known to the author being due to P. M. Cohn. Many other examples are mentioned, some involving such natural rings as the first Weyl algebra

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and maximal $\mathbb{Z}$-orders. There are also positive results about when M_n-cancellation is valid, and a discussion of the connection between the problems of n-cancellation and M_n-cancellation.
This is an excellent survey which contains much that should interest anyone working in any branch of ring theory.

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