Algebraic entropy and amenability of groups.

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Abstract

We prove that any finitely generated elementary amenable group of zero (algebraic) entropy contains a nilpotent subgroup of finite index or, equivalently, any finitely generated elementary amenable group of exponential growth is of uniform exponential growth.

1 Introduction.

Let G be a group generated by a finite set X. As usual, we denote by $||g||_X$ the *word length* of an element $g \in G$ with respect to X, i.e., the length of a shortest word over the alphabet $X \cup X^{-1}$ which represents g.

Recall that the *growth function* $\gamma_G^X : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by

$$\gamma_G^X(n) = card\,\{g \in G \ : \ ||g||_X \leq n\}.$$

Growth considerations in group theory have been introduced in 50-th by Efremovic [6], Švarc [27], and Følner [7], and (independently) in 60-th by Milnor [20] with motivations from differential geometry and theory of invariant means.

The *exponential growth rate* of G with respect to X is the number

$$\omega(G, X) = \lim_{n\to\infty} \sqrt[n]{\gamma_G^X(n)}.$$

The above limit exists by submultiplicativity of γ_G^X [30, Theorem 4.9]. The quantity

$$\omega(G) = \inf_X \omega(G, X)$$

is called a *minimal exponential growth rate* of G (the infimum is taken over all finite generating sets of G). Finally, the *(algebraic) entropy* of the group G is defined by the formula

$$h(G) = \log \omega(G).$$

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This notion of entropy comes from geometry and should not be confused with the notion of entropy for a pair (G, μ), where μ is a symmetric probability measure on a group G, as defined in [1]. In particular, if G is a fundamental group of a compact Riemannian manifold of unit diameter, then $h(G)$ is a lower bound for the topological entropy of the geodesic flow of the manifold [19]. The exponential growth rates appear also in the study of random walks on the Cayley graphs of finitely generated groups. We refer to [10], [16] and [14], for more details and backgrounds.

The group G is said to be of *exponential growth* if $\omega(G, X) > 1$, of *uniformly exponential growth* if $\omega(G) > 1$, and of *subexponential growth* if $\omega(G, X) = 1$. If there exist constants $C, d > 0$ such that $\gamma_G^X(n) \leq Cn^d$ for all $n \in \mathbb{N}$, then G is said to be *of polynomial growth*. These depend on G only, not on the finite generating set X. The famous Milnor problem [21] asks whether there exists a finitely generated group of *intermediate growth*, i.e., of subexponential growth but not of polynomial growth? The negative answers has been obtained by Wolf [31], Milnor [22], Tits [29], and Chou [4] for some particular classes of groups, but the discovery of finitely generated groups of intermediate growth is more recent and due to Grigorchuk [8].

Now it is an important open problem to know whether there exists a finitely generated group of exponential growth but not of uniformly exponential growth. This question goes back to the book [14] and can be found in [9] as well as in [10] and [16]. Let us mention some known results in this direction. There are many examples of classes of groups which are known to have uniformly exponential growth, for example, non–elementary hyperbolic groups [18], one–relator groups of exponential growth [12], and solvable groups which are not virtually nilpotent [24]. Amalgamated products and HNN–extensions were investigated in [3].

In order to explain the Hausdorff–Banach–Tarski paradox, von Neumann [23] introduced the class of amenable groups in 1929. He showed that all finite and abelian groups are amenable and the class of amenable groups, AG, is closed under four standard operations of constructing new groups from given ones:

(S) Taking of subgroups.

(Q) Taking of quotient groups.

(E) Group extensions.

(U) Direct limits (i.e., given a set of groups $\{G_\lambda\}_{\lambda \in \Lambda}$ such that, for any $\lambda, \mu \in \Lambda$, there is $\nu \in \Lambda$ satisfying $G_\lambda \cup G_\mu \subseteq G_\nu$, take $\bigcup\limits_{\lambda \in \Lambda} G_\lambda$).

As in [5], let EG be the class of *elementary amenable groups* that is the smallest class which contains all abelian and finite groups, and closed under (S)–(U). In particular, EG contains all solvable groups. However it is easy to construct a finitely generated group $G \in EG$ that is not solvable–by–finite.

The main result of this paper is the following.

Theorem 1.1. *Let G be a finitely generated elementary amenable group of zero entropy. Then G contains a nilpotent subgroup of finite index. In particular, any elementary amenable group of exponential growth is of uniformly exponential growth.*

This extends the result of Chou, saying that any elementary amenable group of subexponential growth contains a nilpotent subgroup of finite index, as well as the result of the author from [24], where the analog of Theorem 1.1 was proved in case of solvable groups.

In [26], Rosset proved that if G is a group of subexponential growth, H is a normal subgroup of G, and G/H is solvable, then H is finitely generated. The techniques developed in the present paper allows to obtain the following more general result on the structure of normal subgroups of groups with zero entropy.

Theorem 1.2. *Let G be a finitely generated group of zero entropy, H be a subgroup of G such that the quotient group G/H is elementary amenable. Then H is finitely generated.*

In particular, Theorem 1.2 provides a natural approach to prove that a group has uniform exponential growth. As an immediate consequence, we have

Corollary 1.3. *Suppose G is a finitely generated group of zero entropy. Then any term of the derived series of G is finitely generated.*

The paper is organized as follows. The outline of the proof of Theorem 1.1 will be given in the next section. In Section 3, we prove Theorem 1.1 in the particular case of so called AF–groups introduced in Section 2. The description of elementary amenable groups and some of their properties are considered in Section 4. The proof of Theorem 1.1 and Theorem 1.2 in general case is given in Section 5. Finally, some needed technical lemmas involving the commutator calculus will be discussed in Appendix.

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2 Outline of the proof.

Here we describe shortly the main idea of the proof of Theorem 1.1.

Denote by AF the class of all groups such that, for any $S \in AF$, there exists a finite subnormal series of type

$$\{1\} = S_0 \triangleleft S_1 \triangleleft \ldots \triangleleft S_k = S, \qquad (1)$$

where each factor S_{i+1}/S_i is either abelian or finite. It is not difficult to see that any group from AF is finitely generated and AF is closed under the operations (S), (Q), and (E). The class AF plays the same role with respect to elementary amenable groups as polycyclic groups with respect to solvable. More precisely, this relation can be expressed as follows.

Proposition 2.1. *Let $G \in EG$ be a finitely generated group. Suppose, in addition, that G is not virtually nilpotent. Then there exists a normal subgroup H of G such that $G/H \in AF$ and at least one of the following conditions holds.*

1) G/H has exponential growth.

2) H is not finitely generated.

It is easy to see that if a group G has a quotient group of uniform exponential growth, then it is of uniform exponential growth itself (Lemma 3.1, *1)* below). Thus the proof of Theorem 1.1 is divided into two parts depending on the condition 1) or 2) is true. The first case is relatively simple and will be considered in Section 3. Namely, we will prove

Proposition 2.2. *Any group $S \in AF$ is virtually solvable.*

Corollary 2.3. *A group $S \in AF$ has zero entropy if and only if it contains a nilpotent subgroup of finite index. In particular, any AF–group of exponential growth is of uniform exponential growth.*

This shows that G has uniform exponential growth in first case. The second case is more complicated. By Corollary 2.3, we can assume that G/H is virtually nilpotent. Moreover, since the property to be of uniform exponential growth is preserved under the taking of subgroups of finite index (Lemma 3.1, *2)*), we can assume the nilpotency of G/H. In this settings the following proposition plays the crucial role in our proof.

Proposition 2.4. *Let G be a finitely generated group such that there exists an exact sequence*

$$1 \longrightarrow K \longrightarrow G \longrightarrow N \longrightarrow 1,$$

where N is nilpotent of degree d and K is not finitely generated. Then we have

$$\omega(G) \geq \sqrt[\alpha]{2}, \tag{2}$$

where $\alpha = 12 \cdot 4^{d+2}$.

The proof of Proposition 2.4 involves some techniques based on commutator calculus. We give this proof in Section 5 modulo some auxiliary results which will be obtained in Appendix.

3 The entropy of AF–groups.

Let us recall some elementary properties of exponential growth rates needed for the sequel.

Lemma 3.1. *Let G be a finitely generated group. Then the following assertions are true.*

1) Suppose R is a normal subgroup of G; then $\omega(G/R) \leq \omega(G)$.

2) Suppose R is a subgroup of finite index in G; then $\omega(R) \leq \omega(G)^{(2[G:R]-1)}$.

Proof. The proof of claim *1)* is straightforward and left as an exercise to the reader. The claim *2)* is quite trivial also and follows from Proposition 3.3 of [28]. $\square$

The following observation is also quite trivial. The proof is left as an exercise to the reader.

Lemma 3.2. *Any periodic AF–group is finite.*

Recall that a subgroup H of a group G is called characteristic if for any automorphism ϕ of G one has $\phi(H) \leq H$. Evidently if G is a normal subgroup of a group F and H is a characteristic subgroup of G, then H is normal in F.

Recall also that a group P is called *polycyclic* if there is a subnormal series

$$1 = P_k \lhd P_{k-1} \lhd \ldots \lhd P_0 = P, \tag{3}$$

where P_{i-1}/P_i is cyclic for all $i = 1, \ldots, k$.

Lemma 3.3. *Let G be an extension of a finite group F by a polycyclic group P. Then G contains a characteristic polycyclic subgroup of finite index. In particular, G is virtually polycyclic.*

Proof. We proceed by induction on k, the length of the series of type (3) for the group P. The case $k = 0$ is trivial. Suppose now that $k > 1$. Denote by ϕ the natural homomorphism from G to P and consider R, the full preimage of P_1 under ϕ. By inductive hypothesis, R contains a characteristic polycyclic subgroup R_0 of finite index. Since R_0 is characteristic and R is normal in G, R_0 is normal in G. Clearly, G/R_0 is finite–by–cyclic. It is easy to check using standard arguments that there exists a cyclic subgroup T of finite index in G/R_0. Finally, we consider the full preimage U of T under the natural homomorphism $G \to G/R_0$. Obviously U is polycyclic, as it is an extension of a polycyclic group R_0 by a cyclic group T. Moreover, U has a finite index in G. Denote this index by m. Then the subgroup $G^m = \langle g^m \,:\, g \in G\rangle$ is polycyclic, as it is contained in U. Evidently G^m is characteristic and has a finite index in G by Lemma 3.2. $\square$

Proof of Proposition 2.2. Let S be an AF–group. We will prove the proposition by induction on the length of the subnormal series

$$\{1\} = S_0 \lhd S_1 \lhd \ldots \lhd S_k = S, \tag{4}$$

where each factor S_{i+1}/S_i is either abelian or finite. If $k = 0$, there is nothing to prove. Now suppose that $k > 0$ and that we have proved the proposition for any AF–group admitting a series of type (4) with at most $(k-1)$ terms. Consider S_{k-1}. By inductive assumption, S_{k-1} is virtually polycyclic. If S/S_{k-1} is finite, then S is virtually polycyclic in the obvious way. Now consider the case of abelian S/S_{k-1}. Denote by V a polycyclic subgroup of finite index in S_{k-1}

and by j the index $|S_{k-1} : V|$. Clearly, $S_{k-1}^j = \langle s^j \ : \ s \in S_{k-1} \rangle$ is contained in V and, therefore, is polycyclic. Moreover, S_{k-1}^j is a characteristic subgroup of S_{k-1}. Hence S_{k-1}^j is normal in S. Note that S/S_{k-1}^j is an extension of a finite group S_{k-1}/S_{k-1}^j by polycyclic (moreover, by finitely generated abelian). Therefore, S/S_{k-1}^j contains a polycyclic subgroup W of finite index by Lemma 3.3. Consider the full preimage W_0 of the group W under the natural homomorphism $S \to S/S_{k-1}^j$. Obviously W_0 is polycyclic and has finite index in S. $\square$

Proof of Corollary 2.3. The corollary immediately follows from previous lemma, assertion (2) of Lemma 3.1, and the following theorem, which is the main result of [24]. $\square$

Theorem 3.3. [24, Theorem 1.1.] *Let G be a finitely generated solvable group of zero entropy. Then G contains a nilpotent subgroup of finite index.*

4 Description and some properties of elementary amenable groups.

First we recall the description of elementary classes of groups given in [25]. In case of elementary amenable groups this description is slightly stronger than Chou one [4]. Lemma 4.1, Lemma 4.2, and Theorem 4.3 presented in this section can be found in [25]. Here we equip them with proofs for convenience of the reader.

Definition 4.1. Let B be a class of groups. The *elementary class of groups with the base* B is the smallest set of groups which contains B and is closed under the operations (S)–(U).

Now we fix B. Let $\mathcal{E}_0(B)$ consist of the trivial group only. Assume that $\alpha > 0$ is an ordinal and that we have defined $\mathcal{E}_\beta(B)$ for each ordinal $\beta < \alpha$. If α is a limit ordinal, set $\mathcal{E}_\alpha(B) = \bigcup\limits_{\beta<\alpha} \mathcal{E}_\beta(B)$, and if α is successor, let $\mathcal{E}_\alpha(B)$ be the class of groups which can be obtained from groups in $\mathcal{E}_{\alpha-1}(B)$ by applying operation (U) or the following operation once.

(E_0) Given a group, take its extension by a group from B.

Lemma 4.1. *The class $\mathcal{E}_\alpha$ is closed under operation (S) and (Q) (SQ–closed for brevity) for each ordinal α.*

Proof. We proceed by transfinite induction on α. The lemma is clear for $\mathcal{E}_0$. Assume that $\alpha > 0$ and that $\mathcal{E}_\beta$ is SQ–closed if $\beta < \alpha$. Let $G \in \mathcal{E}_\alpha$, C be a subgroup of G, and D be an image of G under some homomorphism ϕ. We have to show that $C, D \in \mathcal{E}_\alpha$. If α is a limit ordinal, then $G \in \mathcal{E}_\beta$ for some $\beta < \alpha$. Hence $C, D \in \mathcal{E}_\beta \subseteq \mathcal{E}_\alpha$ by inductive hypothesis.

If α is a successor ordinal, then either G is an extension of type

$$1 \longrightarrow E \longrightarrow G \xrightarrow{\theta} F \longrightarrow 1$$

for some $E \in \mathcal{E}_{\alpha-1}$ and $F \in B$, or any finitely generated subgroup of G belongs to $\mathcal{E}_{\alpha-1}$. In first case, C is isomorphic to the extension of $C_1 = C \cap E$ by $C_2 = \theta(C)$ and D is the extension of $D_1 = \phi(E)$ by $D_2 = \phi(F)$. By assumption, $C_1, D_1 \in \mathcal{E}_{\alpha-1}$. Since B is SQ–closed, C_2 and D_2 belong to B. Therefore, $C, D \in \mathcal{E}_\alpha$. In second case note that any finitely generated subgroup of C and D belongs to $\mathcal{E}_{\alpha-1}$ being a subgroup or a quotient of a finitely generated subgroup of G. Since any group is a direct limit of its finitely generated subgroups, we have $C, D \in \mathcal{E}_\alpha$. $\square$.

Lemma 4.2. *Suppose that a group G is an extension of a group $F \in \mathcal{E}_\alpha$ by a group $H \in \mathcal{E}_\beta$ for some ordinals α, β. Then there exists an ordinal γ such that $G \in \mathcal{E}_\gamma$.*

Proof. We proceed by transfinite induction on β. The case $\beta = 0$ is trivial. Suppose that $\beta > 0$ and we have proved the lemma for all ordinals $\delta < \beta$. If β is a limit ordinal, then $H \in \mathcal{E}_\epsilon$ for some $\epsilon < \beta$ and the required γ exists by inductive hypothesis. Now let β be a successor ordinal. There are two possibilities to obtain H from $\mathcal{E}_{\beta-1}$.

First assume that H is an extension of D by E, where $D \in \mathcal{E}_{\beta-1}$ and $E \in B$. Consider the full preimage of D under the natural homomorphism $G \longrightarrow H$ and denote it by D_0. Then D_0 is an extension of F by D and thus $D_0 \in \mathcal{E}_\xi$ for some ξ by the inductive hypothesis. Therefore, $G \in \mathcal{E}_{\xi+1}$ since $G/D_0 \cong H/D$ belongs to B.

Next, suppose that H is a direct limit of groups $\{H_\lambda\}_{\lambda\in\Lambda}$ and embeddings $H_\lambda \cup H_\mu \le H_\nu$. Consider the full preimages H_λ^0 of H_λ in G. By our assumption, $H_\lambda \in \mathcal{E}_{\beta-1}$ for all $\lambda \in \Lambda$. Hence $H_\lambda^0 \in \mathcal{E}_{\xi_\lambda}$ for some ξ_λ by inductive hypothesis. Let us take the sum $\xi = \sum\limits_{\lambda\in\Lambda} \xi_\lambda$. As is known, ξ is an ordinal, which is not smaller than any ξ_λ. This implies $\mathcal{E}_{\xi_\lambda} \subseteq \mathcal{E}_\xi$ for all $\lambda \in \mathbb{N}$ and thus $H_\lambda^0 \in \mathcal{E}_\xi$ for all λ. Note that the embeddings $H_\lambda \to H_\mu$ can be extended to the embeddings $H_\lambda^0 \to H_\mu^0$ in the obvious way and then the corresponding direct limit $G_0 = \bigcup\limits_{i=1}^{\infty} H_\lambda$ will be isomorphic to G. This implies $G \in \mathcal{E}_{\xi+1}$. $\square$

The following is an immediate consequence of Lemmas 4.1 and 4.2.

Theorem 4.3. *Let B be a class of groups. Assume that B is closed under the operations (S) and (Q). Then we have*

$$\mathcal{E}(B) = \bigcup_\alpha \mathcal{E}_\alpha(B),$$

where the union is taken over all ordinal numbers.

Example 4.1. Let us take $B = A \cup F$, where A and F are the classes of all abelian and finite groups respectively. Then the corresponding elementary class is precisely EG as follows from Theorem 4.3.

Instead of Proposition 2.1, we will prove more stronger result by transfinite induction on α.

Lemma 4.3. *Let $G \in EG_\alpha$ be a finitely generated group. Suppose, in addition, that G is not virtually nilpotent. Then there exists a characteristic subgroup H of G such that $G/H \in AF$ and at least one of the following conditions holds.*

1) G/H has exponential growth.

2) H is not finitely generated.

Proof. The case $\alpha = 0$ is trivial. Suppose that $\alpha > 0$. First assume that α is a limit ordinal. Then $G \in EG_\beta$ for some $\beta < \alpha$ and thus the assertion of the proposition holds by the inductive assumption.

Now let α be a non limit ordinal. Assume that $G = \bigcup\limits_{\lambda \in \Lambda} G_\lambda$, where $G_\lambda \in EG_{\alpha-1}$. Since G is finitely generated, we have $G = G_{\lambda_0}$ for some λ_0. Hence, the assertion of the proposition is true by inductive hypothesis again. Further, suppose that G is an extension of the form

$$1 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 1,$$

where $M \in EG_{\alpha-1}$ and L is abelian or finite. First we consider the case of abelian L. Take $G' = [G, G]$ and observe that $G' \le M$ and thus $G' \in EG_{\alpha-1}$ by Lemma 4.1. If G' is not finitely generated, we can take it as H. Otherwise, there are two possibilities. The first one is the case of virtually nilpotent G'. Clearly, then $G \in AF$. Taking into account that G is not virtually nilpotent, we conclude that G is of exponential growth by Corollary 2.3.

Now let G' is not virtually nilpotent. Then there exists a characteristic subgroup $H \le G'$ satisfying the requirements of the proposition. Clearly, H is a characteristic subgroup of G. It remains to notice that since $G'/H \in AF$, then $G/H \in AF$ and if G'/H is of exponential growth, then so is G/H.

Similarly, if L is finite, say $|L| = m$, then we take the subgroup $G^m = \langle g^m : g \in G\rangle$. Evidently $G^m \le M$ and the further proof is essentially the same as in the previous case. Additionally, we only need the fact that any finitely generated periodic group from EG is finite (see [4]). $\square$

5 The proof of the main theorem.

Consider a group G and a subset of elements $Q \subseteq G$. We denote by $\mathcal{L}(Q)$ the set of all words in the alphabet Q. For two words $u, v \in \mathcal{L}(Q)$, we write $u \equiv v$ to express the letter–for–letter equality, and $u = v$ if u and v represent the same element of G. Also we put $u^v \equiv v^{-1}uv$ and $[u, v] \equiv u^{-1}v^{-1}uv$. For any group

H, denote by $\gamma_i H$ the i-th term of the lower central series

$$H = \gamma_1 H \rhd \gamma_2 H \rhd \dots ,$$

where $\gamma_{i+1} H = [\gamma_i H, H]$. Recall that a group N is called *nilpotent of degree t* if $\gamma_{t+1} N = 1$. Finally, given a subsets $Y, Z \subseteq G$, let $\langle Y \rangle$ denote the subgroup generated by Y, and $\langle Y \rangle^Z$ the subgroup generated by all elements of type $z^{-1} y z$, where $y \in Y, z \in Z$. Thus $\langle Y \rangle^G$ is the normal closure of Y in G.

Definition 5.1. Let G be a group with a given finite generating set X. For any finite subset $Y = \{y_1, \dots, y_m\} \subseteq G$, we define its *depth* with respect to X as follows

$$depth_X(Y) = \max_{i=1,\dots,m} ||y_i||_X .$$

If H is a finitely generated subgroup of G, then we define its depth with respect to X by putting

$$depth_X(H) = \min_{H = gp(Y)} depth_X(Y),$$

where the minimum is taken over all finite generating sets of H.

Lemma 5.1. *Suppose that G is a group with a given finite generating set X and R is a finitely generated subgroup of G; then we have*

$$\omega(G, X) \geq (\omega(R))^{\frac{1}{depth_X(R)}} .$$

The proof is straightforward and left as an exercise to the reader. $\square$

Let us introduce some notation. As above suppose G is a group generated by a finite set X. Then we set $W_1(X) = X \cup X^{-1}$ and

$$W_i(X) = \{[u^{\pm 1}, v^{\pm 1}] \ : \ u \in W(i_1), v \in W(i_2), i_1, i_2 \in \mathbb{N}, i_1 + i_2 = i\}$$

for any $i > 1$. As usual, we write *weight* $(v) = i$ if $v \in W_i$. Also, consider the function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(1) = 1 \ \text{ and } \ f(n+1) = 2f(n) + 2 \tag{5}$$

for any $n \in \mathbb{N}$. It can easily be checked that $f(n) = 3 \cdot 2^{n-1} - 2$. The proof of the following lemma is quite trivial and left to the reader.

Lemma 5.2. *Let f be the function given by (5). Then for any $i, j \in \mathbb{N}$, one has* $2(f(i) + f(j)) \leq f(i+j)$.

Lemma 5.3. *For any group G with a given finite generating set X, one has*

$$depth_X(W_n(X)) \leq f(n). \tag{6}$$

Proof. We proceed by induction on n. The case $n = 1$ is trivial. Next, for $n > 1$, we observe that if $u \in W_{i_1}(X), v \in W_{i_2}(X)$ and $i_1 + i_2 = n$, then

$$\begin{aligned} ||[u^{\pm 1}, v^{\pm 1}]||_X \leq & \; 2(||u||_X + ||v||_X) \leq 2(depth_X(W_{i_1}(X)) + depth_X(W_{i_2}(X))) \\ \leq & \; 2(f(i_1) + f(i_2)) \leq f(n) \end{aligned}$$

by the inductive hypothesis and Lemma 5.2. □

As an exercise, one can show that if G is a non abelian free group and X is a basis in G, then $depth_X(W_n(X)) = f(n)$.

The following lemma will be proved in Appendix.

Lemma 5.4. *Suppose that G is a finitely generated group and, for some $s \in \mathbb{N}$, $s \geq 2$, all subgroups of type*

$$H_{v,w} = \langle v^{-l} w v^l \; : \; l \in \mathbb{Z} \rangle \tag{7}$$

are finitely generated for any $v \in \bigcup_{j=1}^{s-1} W_j$, $w \in \bigcup_{j=s}^{2s} W_j$. Then $\gamma_s(G)$ is finitely generated .

Proof of Proposition 2.4. Let X be some finite generating set of G. Let us put $s = d+1$. We would like to show that there is a subgroup $H_{v,w} \leq G$ of type (7) having no finite set of generators. Indeed, suppose that all $H_{v,w}$ are finitely generated. Then $\gamma_s G$ is finitely generated by Lemma 5.4. Clearly, $\gamma_s G \triangleleft K$. Therefore, $K/\gamma_s G$ is a subgroup of a finitely generated nilpotent group $G/\gamma_s G$ and thus is finitely generated. It follows that K is finitely generated and we get a contradiction.

Thus there exists $H_{v,w}$ which is infinitely generated. Consider the subgroup $H = \langle v, w \rangle$. For any sequence $\alpha = (\alpha_1, \dots, \alpha_p)$, where $\alpha_i \in \{0, 1\}$ for each $i = 1, \dots p$, $p \in \mathbb{N}$, we define an element $t(\alpha)$ by the formula

$$t(\alpha) = w^{\alpha_1} v w^{\alpha_2} v \dots w^{\alpha_p} v.$$

Suppose that $t(\alpha) = t(\beta)$ for some $\alpha = (\alpha_1, \dots, \alpha_p) \neq (\beta_1, \dots, \beta_q) = \beta$. Note that $H/H_{v,w}$ is cyclic and infinite (otherwise $H_{v,w}$ is finitely generated). Hence $vH_{v,w}$ has infinite order when regarded as an element of $H/H_{v,w}$. This implies $p = q$ and we have

$$w^{\alpha_1} v w^{\alpha_2} v \dots w^{\alpha_p} v =_H w^{\beta_1} v w^{\beta_2} v \dots w^{\beta_p} v. \tag{8}$$

Without loss of generality, we can assume $\alpha_1 \neq \beta_1$ and $\alpha_p \neq \beta_p$. Denote by w_l the element w^{v^l}. Then (8) can be rewritten as

$$(w_p)^{\alpha_1} (w_{p-1})^{\alpha_2} \dots (w_1)^{\alpha_p} =_H (w_p)^{\beta_1} (w_{p-1})^{\beta_2} \dots (w_1)^{\beta_p},$$

or, equivalently,

$$(w_p)^{\alpha_1 - \beta_1} = (w_{p-1})^{\beta_2} \dots (w_1)^{\beta_p} \left((w_{p-1})^{\alpha_2} \dots (w_1)^{\alpha_p} \right)^{-1}.$$

Note that $\alpha_1 - \beta_1 = \pm 1$. Therefore,

$$w_p \in \langle w_1, \dots, w_{p-1} \rangle. \tag{9}$$

Conjugating by v and using (9), we obtain

$$w_{p+1} = w_p^v \in \langle w_2, \dots, w_p \rangle \leq \langle w_1, \dots, w_{p-1} \rangle$$

and so on. By induction, $w_n \in \langle w_1, \dots, w_{p-1} \rangle$ for any $n \geq p$. Similarly, we can obtain $w_n \in \langle w_2, \dots, w_p \rangle$ for any $n \leq 1$. Hence $w_n \in \langle w_1, \dots, w_p \rangle$ for any $n \in \mathbb{Z}$ that contradicts to the assumption that $H_{v,w}$ is infinitely generated.

This shows that $t(\alpha) \neq t(\beta)$ whenever $\alpha \neq \beta$. Recall that $||v||_X \leq f(s-1)$ and $||w||_X \leq f(2s)$ by Lemma 5.3. Hence we have

$$||w^{\alpha_1} v w^{\alpha_2} v \dots w^{\alpha_p} v||_X \leq p(||w||_X + ||v||_X) \leq p(f(s-1) + f(2s)) \leq 2pf(2s).$$

Thus,

$$\begin{aligned} \gamma_H^X(n) \geq & \quad card\,\{t(\alpha) \ : \ ||t(\alpha)||_X \leq n\} \\ \geq & \quad card\left\{(\alpha_1, \dots, \alpha_p) \ : \ \alpha_1, \dots, \alpha_p \in \{0,1\},\ p \leq \left[\tfrac{n}{2f(2s)}\right]\right\} = 2^{\left[\frac{n}{2f(2s)}\right]}. \end{aligned}$$

Here $[x]$ means the integral part of x. This implies

$$\omega(H, X) \geq \sqrt[2f(2s)]{2}. \tag{10}$$

Note that $2f(2s) = 2(3 \cdot 2^{2s-1} - 2) \leq 6 \cdot 2^{2s-1} = 6 \cdot 2^{2d+1} = 12 \cdot 4^d$. Since (10) is true for arbitrary X, we obtain (2). $\square$

Proof of Theorem 1.2. The proof easily follows prom Proposition 2.4 and Theorem 1.1. We live details for the reader. $\square$

6 Appendix.

During this section we fix a group G generated by a finite set X and fix arbitrary finite subsets $V, W \in \mathcal{L}(X \cup X^{-1})$ (where $\mathcal{L}(X \cup X^{-1})$ denote the set of all words over $X \cup X^{-1}$). We assume in addition that V and W satisfy the following conditions.

(I) $X^{\pm 1} \in V$.

(II) The set V is ordered, i.e., $V = \{v_1, v_2, \dots, v_p\}$. Set $V_i = \{v_1, v_2, \dots, v_i\}$ for each $i = \overline{1,p}$; then either $[V_i, V_j] \subseteq V_{\min\{i,j\}-1}$ or $[V_i, V_j] \subseteq W$ for any $i, j = 1, \dots, p$.

(III) For any $i = 1, \dots, p$, $w \in W$, the normal closure $\langle w \rangle^{\langle v_i \rangle} = \langle v_i^{-l} w v_i^l \ : \ l \in \mathbb{Z} \rangle$ is finitely generated, i.e., there exists $L_i \in \mathbb{N}$ such that $\langle w \rangle^{\langle v_i \rangle} = \langle v_i^{-l} w v_i^l \ : \ |l| \leq L_i \rangle$.

In this settings we would like to show that $\langle W\rangle^G$ is finitely generated as a subgroup. To do this we need some auxiliary notion. Let $v \in \mathcal{L}(V^{\pm 1} \cup W^{\pm 1})$. Denote by $\lambda_i(v)$ the number of appearances of the letters $v_i^{\pm 1}$ in v. For instance, if $v \equiv v_1 v_2 v_1^{-1}$, then $\lambda_1(v) = 2$, $\lambda_2(v) = 1$. We note also that λ_i is defined just for words over $V^{\pm 1}$, not for elements of G. This remark becomes clear, if we consider, say, $v \equiv [v_1, v_2]$ such that $v = v_i$ for some $i > 2$. Evidently $\lambda_i(v_i) = 1$, but $\lambda_i(v) = \lambda_i([v_1, v_2]) = 0$.

Given X, G, V, and W as described above, we set

$$L = \max_{i=1,\dots,p} L_i$$

and

$$Z = \{v^{-1} w v \ : \ w \in W,\ v \in \mathcal{L}(V^{\pm 1}),\ \lambda_i(v) \le L\ \forall i = 1, \dots, p\}. \tag{11}$$

We will say that the above decomposition $z \equiv v^{-1} w v$, where $w \in W$, $v \in \mathcal{L}(V^{\pm 1})$, is a *canonical form* of the element $z \in Z$; clearly $\lambda_i(z) = 2\lambda_i(v)$.

The main goal of this section is to prove the following.

Proposition A.1. *In the above notation, we have $\langle W\rangle^G = \langle Z\rangle$.*

The proof will consist of four lemmas. First of all we introduce some auxiliary notation. Denote by $(x, y^{\pm 1})_1$ the commutators $[x, y^{\pm 1}]$, i.e., the set $\{[x, y], [x, y^{-1}]\}$, and put

$$(x, y^{\pm 1})_{i+1} = \{[c, y], [c, y^{-1}] \ : \ c \in (x, y^{\pm 1})_i\}.$$

Lemma A.2. *Let H be a group, $a, b \in H$. Then*

$$(a, b^{\pm 1})_n \subseteq \langle a^{b^l} \ : \ l = -n, \dots, n\}$$

for any $n \in \mathbb{N}$.

Proof. For $n = 1$, we have

$$(a, b^{\pm 1})_1 \equiv a^{-1} a^{b^{\pm 1}} \in \langle a, a^{b^{\pm 1}}\rangle.$$

Now suppose $n > 1$. By induction, we can assume that the assertion of the lemma is true for $(n-1)$, i.e.,

$$(a, b^{\pm 1})_{n-1} \in \langle a^{b^l} \ : \ l = -n+1, \dots, n-1\rangle. \tag{12}$$

Denote $a_l = a^{b^l}$ for brevity and consider an element $c = a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m} \in (a, b^{\pm 1})_{n-1}$. By (12), we can assume that $|l_j| \le n-1$ for each j. We obtain

$$[c, b] \equiv c^{-1} c^b = (a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m})^{-1} (a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m})^b = (a_{l_1}^{\alpha_1} \dots a_{l_m}^{\alpha_m})^{-1} (a_{l_1+1}^{\alpha_1} \dots a_{l_m+1}^{\alpha_m}).$$

Therefore, $[c, b] \in \langle a_{-n+1}, \dots, a_n\rangle$. Similarly we obtain $[c, b^{-1}] \in \langle a_{-n}, \dots, a_{n-1}\rangle$. The lemma is proved. $\square$

The following three lemmas will be proved by common induction on r.

Lemma A.3. *Let $0 \leq n_1 < n_2 < \ldots < n_m$ be a sequence of integers. Consider a word*

$$\bar{v} \equiv a_1 \ldots a_{n_1} v_r^{\epsilon_1} a_{n_1+1} \ldots a_{n_2} v_r^{\epsilon_2} \ldots a_{n_m-1} \ldots a_{n_m} v_r^{\epsilon_m}, \tag{13}$$

where $a_i \in (V \setminus \{v_r\})^{\pm 1}$, $\epsilon_i \in \mathbb{Z}$ for each $i = 1, \ldots, m$, and

$$\sum_{i=1}^{m} |\epsilon_i| \leq L + 1. \tag{14}$$

Then we have

$$\bar{v} = v_r^{\sigma} \cdot a_1 b_1 \cdot a_2 b_2 \cdot \ldots \cdot a_{n_m} b_{n_m}, \tag{15}$$

where $\sigma = \sum_{i=1}^{m} \epsilon_i$ and

$$b_i \in \left\langle V_{r-1} \cup \left(\bigcup_{j=-L}^{L} W^{v_r^j} \right) \right\rangle \tag{16}$$

for all i. In particular, $b_i \in \langle V_{r-1} \cup Z \rangle$.

Lemma A.4. *Suppose that $v^{-1}wv$ is a canonical form of an element $z \in Z$ and a is a word over $(V \setminus \{v_r\})^{\pm 1} \cup \left(\bigcup_{j=-L}^{L} W^{v_r^j} \right)^{\pm 1}$. Then we have*

$$z^a \in \left\langle \; y_0^{-1} w_0 y_0 \; : \; \begin{array}{l} y_0 \in \mathcal{L}(V^{\pm 1}), \; w_0 \in W, \\ \lambda_i(y_0) \leq L \; \forall \; i = 1, \ldots, r, \\ \lambda_i(y_0) \leq \lambda_i(v) + \lambda_i(a) \; \forall \; i = r+1, \ldots, p \end{array} \right\rangle. \tag{17}$$

In particular, if

$$\lambda_i(v) + \lambda_i(a) \leq L \tag{18}$$

for each $i = r+1, \ldots, p$, then $z^a \in \langle Z \rangle$.

Lemma A.5. *For any $z = v^{-1}wv \in Z$ and any $t \in (V_r)^{\pm 1}$, we have*

$$z^t \in \left\langle \; y_0^{-1} w_0 y_0 \; : \; \begin{array}{l} y_0 \in \mathcal{L}(V^{\pm 1}), \; w_0 \in W, \\ \lambda_i(y_0) \leq L \; \forall \; i = 1, \ldots, r, \\ \lambda_i(y_0) \leq \lambda_i(v) + \lambda_i(t) \; \forall \; i = r+1, \ldots, p \end{array} \right\rangle. \tag{19}$$

In particular, $z^t \in \langle Z \rangle$.

Proof. For all lemmas the case $r = 1$ is essentially the same as the inductive step. So we assume Lemmas A.3 – A.5 to be true for all positive integers $s < r$ whenever $r > 1$ and are going to do the inductive step.

Proof of Lemma A.3.

Using the formula $xy = yx[x,y]$, we can collect all appearances of the letter $v_r^{\pm 1}$ in the word $\bar{v}$ from right to left to obtain a word of the form (15). It is easy to check that each b_i will be a product of elements of the sets $(a_i, v_r^{\pm 1})_n$, where $n \leq \sum_{i=1}^{m} |\epsilon_i|$. For an element $u \in (a_i, v_r^{\pm 1})_n$, there are two possibilities.

(a) First assume that $u \in V$. Then $u \in V_{r-1}$ by condition (II) (see the beginning of the section).

(b) Suppose that $u \notin V$. Consider the minimal n_0 such that $(a_i, v_r^{\pm 1})_{n_0} \not\subseteq V$. Clearly, $a_i \in V^{\pm 1}$ implies that $n_0 \geq 1$. By Lemma A.2,

$$u \in \left((a_i, v_r^{\pm 1})_{n_0}, v_r^{\pm 1}\right)_{n-n_0} \subseteq \left\langle \left((a_i, v_r^{\pm 1})_{n_0}\right)^{v_r^l} \ : \ |l| \leq n - n_0 \right\rangle.$$

Using (14), we note that $n - n_0 \leq \sum_{i=1}^{m} |\epsilon_i| - n_0 \leq L$ and hence

$$u \in \left\langle \left((a_i, v_r^{\pm 1})_{n_0}\right)^{v_r^l} \ : \ |l| \leq L \right\rangle \leq \left\langle \bigcup_{j=-L}^{L} W^{v_r^j} \right\rangle.$$

Indeed, by minimality of n_0, we have $(a_i, v_r^{\pm 1})_{n_0 - 1} \in V$. Now using condition (II), we obtain $(a_i, v_r^{\pm 1})_{n_0} = \left((a_i, v_r^{\pm 1})_{n_0-1}, v_r^{\pm 1}\right) \subseteq W$

Thus in both cases

$$u \in \left\langle \bigcup_{j=-L}^{L} W^{v_r^j} \cup V_{r-1} \right\rangle,$$

and, therefore, the same is true for each b_i. The lemma is proved.

Proof of Lemma A.4.

Denote by Y the group situating at the right–hand side of (17). The proof will be by induction on the length of a. The case $|a| = 0$ is trivial. Now suppose $|a| = n + 1 \geq 1$. Then $a = a_0 a_1$, where $a_1 \in (V \setminus \{v_r\})^{\pm 1} \cup \left(\bigcup_{j=-L}^{L} W^{v_r^j} \right)^{\pm 1}$ and a_0 has length n. By inductive assumption, we have

$$z^a = z^{a_0 a_1} = (z_1 \ldots z_q)^{a_1} = z_1^{a_1} \ldots z_q^{a_1}, \tag{20}$$

where $z_j = y_j^{-1} w_j y_j$ are some elements such that $w_j \in W^{\pm 1}$, $y_j \in \mathcal{L}(V^{\pm 1})$, $\lambda_i(y_j) \leq \lambda_i(a_0) + \lambda_i(v)$ for all $i = r+1, \ldots, p$, and $\lambda_i(y_j) \leq L$ for all $i = 1, \ldots, r$.

Now let us consider $z_j^{a_1}$ for some j and prove that $z_j^{a_1} \in Y$. There are three possibilities.

(a) $a_1 \in \left(\bigcup\limits_{j=-L}^{L} W^{v_r^j} \right)^{\pm 1}$. Evidently, $a_1 \in Z$ in this case. Moreover, $\lambda_i(a_1) = 0$ for all $i \neq r$ and $\lambda_r(a_1) \leq 2L$. Therefore, $a_1 \in Y$ and hence $z_j^{a_1} \in Y$.

(b) $a_1 \in (V_{r-1})^{\pm 1}$. We note that this case is impossible if $r = 1$. If $r > 1$, we assume that Lemma A.5 has already been proved for all smaller volumes of the parameter. Thus we obtain $z_j^{a_1} \in Y$ applying Lemma A.5 for $t \equiv a_1$, $z \equiv z_j$.

(c) $a_1 \in (V \setminus V_r)^{\pm 1}$. Suppose $a_1 \equiv v_k$ for some $k \in \{r+1, \dots, p\}$. For the element $z_j^{a_1}$ consider its canonical form, the word $v_k^{-1} y_j^{-1} w_j y_j v_k$, obtained from the canonical form of the element z_j. We have

$$\lambda_k(v_k y_j) = \lambda_k(y_j) + 1 \leq \lambda_k(a_0) + \lambda_k(v) + 1 = \lambda_k(a) + \lambda_k(v).$$

Clearly, if $i \neq k$, then $\lambda_i(v_k y_j) = \lambda_i(y_j)$. This shows that $z_j^{a_1}$ lies in Y again.

Since $z_j^{a_1} \in Y$ is true for each factor of type $z_j^{a_1}$ in (20), we obtain $z^a \in Y$ and the proof of the lemma is completed.

Proof of Lemma A.5.

Denote by F the group situated at the right side of (19). In view of inductive arguments, it is sufficient to consider the case $t \equiv v_r^{\pm 1}$. Assume that $t \equiv v_r$ for convenience (the case $t \equiv v_r^{-1}$ is analogous). First suppose that $\lambda_r(v) < L$, i.e., $\lambda_r(z) \leq 2L - 2$. Note that

$$\lambda_i(z^{v_r}) = \begin{cases} \lambda_i(z), \text{ if } i \neq r, \\ \lambda_i(z) + 2, \text{ if } i = r. \end{cases}$$

Thus $\lambda_i(z^{v_r}) \leq L$ for $i = 1, \dots, r$, and $\lambda_i(z^{v_r}) = \lambda_i(z)$ for $i = r+1, \dots, p$. This means that $z^{v_r} \in F$.

Now let $\lambda_r(v) = L$. Then $\lambda_r(vv_r) = L+1$ and the word $\bar{v} \equiv vv_r$ has the form (13). Applying Lemma A.3, we obtain $\bar{v} = v_r^{\sigma} a$, where $|\sigma| = \left| \sum\limits_{i=1}^{m} \epsilon_i \right| \leq \lambda_r(\bar{v}) = L + 1$ and a satisfies the condition $\lambda_i(a) = \lambda_i(v) \leq L$ for all $i = r+1, \dots, p$ (obviously this condition follows from (16)). In case $|\sigma| \leq L$ we do nothing. If $|\sigma| = L + 1$, we apply condition (III) and obtain

$$z^{v_r} = w^{v_r^{\sigma} a} = \left(\prod_{i=-L}^{L} \left(w^{v_r^i} \right)^{\xi_i} \right)^a = \prod_{i=-L}^{L} \left(\left(w^{v_r^i} \right)^a \right)^{\xi_i}. \tag{21}$$

Finally, we consider the elements $(w^{v_r^j})^a$, where $|j| \leq L$. We would like to show that these elements lie in F. The element a satisfies the conditions of Lemma A.4, as it contains no appearances of the letters $v_r^{\pm 1}$. Thus $\left(w^{v_r^i} \right)^a \in F$ by Lemma A.4. It follows that $z^{v_r} \in F$. The same arguments show that $z^{v_r^{-1}} \in F$. The lemma is proved and the inductive step is completed. □

Proof of Proposition A.1. Lemma A.5 implies that $z^t \in \langle Z \rangle$ for any $z \in Z$, $t \in V$. Since $X^{\pm 1} \subseteq V$, we have $z^g \in \langle Z \rangle$ for any $g \in G$. This means that $\langle Z \rangle^G = \langle Z \rangle$ and we get what we want. $\square$

Proof of Lemma 5.4. The reader can easily check that sets $W = \bigcup_{j=s}^{2s} W_j$ and $V = \bigcup_{j=1}^{s-1} W_j$ satisfy hypothesis (I) – (III) listed at the beginning of this section. Indeed, (I) is obvious. To satisfy (II), we just need to order commutators in V in such a way that $weight(v_i) \leq weight(v_j)$ whenever $i \geq j$. Finally, (III) follows from the conditions of Lemma 5.4. It remains to note that $\gamma_s(G) = \langle W_s \rangle^G = \langle W \rangle^G$. $\square$

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