A brief introduction to dynamical zeta functions

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Abstract. In this short text we first introduce dynamical zeta functions and transfer operators, illustrating and motivating these notions with a simple one-dimensional dynamical system. Then we present a commented list of useful references.

1. Introduction and motivation

Transfer operators.

We shall take the point of view that dynamical zeta functions are useful objects to describe the spectrum of transfer operators. To define a *transfer operator,* we use two ingredients:

- a map $f : X \to X$ of a topological or metric space X to itself (the *dynamical system*), with the property that $f^{-1}(x)$ is an at most countable set for each $x \in X$;
- a *weight* $g : X \to \mathbb{C}$.

In order to get interesting results, one usually requires additional assumptions: e.g., the map f is supposed to be locally expanding or hyperbolic, and both the map f and the weight g should satisfy some smoothness condition (for example Hölder or Lipschitz continuity if X is a metric space, differentiability or analyticity if X has a manifold structure).

A *transfer operator* is a linear operator $\mathcal{L}_g$ which acts on a suitable vector space of functions $\varphi : X \to \mathbb{C}$ (e.g., the Banach space of bounded functions if $\sum_{y \in f^{-1}(x)} |g(y)| < \infty$ for all x) according to the formula:

$$\mathcal{L}_g \varphi(x) = \sum_{y \in f^{-1}(x)} \varphi(y) g(y) . \tag{1}$$

Before being used in the framework of dynamical systems, transfer operators appeared in statistical mechanics (see the classical treatise on thermodynamic formalism [31]).

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Example 1. Let us take $X = I = [0,1]$ the unit interval and $f(x) = 2x \pmod 1$. We consider first the unweighted case, i.e., we set $g \equiv 1$. Then it is easy to see that the two-dimensional vector space V_2 of functions $\varphi : I \to \mathbb{C}$ (viewing as equivalent two functions which differ on an at most countable set) such that $\varphi(x) = \varphi_L \in \mathbb{C}$ for $0 \le x < 1/2$, and $\varphi(x) = \varphi_R \in \mathbb{C}$ for $1/2 \le x \le 1$, is preserved by $\mathcal{L}_g$. In the basis just described for V_2, one checks that the matrix of $\mathcal{L}_g$ is simply $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, the eigenvalues of which are 0 and 2. Now we make the following trivial observation: since $\mathrm{tr}\, A^n = 2^n$ is exactly the cardinality of the set $\mathrm{Fix}\, f^n := \{x \in I \mid f^n(x) = x\}$ of fixed points of f^n, the unweighted zeta function defined formally by the power series

$$\zeta(z) = \exp \sum_{n \ge 1} \frac{z^n}{n} \# \mathrm{Fix}\, f^n \tag{2}$$

satisfies

$$\zeta(z) = \exp \sum_{n \ge 1} \frac{z^n}{n} \mathrm{tr}\, A^n = \frac{1}{\det(1 - zA)} \tag{3}$$

(where 1 is just the two by two identity matrix).

Invariant function spaces.

In our first, rather trivial, example the (unweighted) dynamical zeta function (2) was a rational function with poles the inverses of the eigenvalues of the transfer operator (1) acting on a finite dimensional vector space. For dynamical systems having a finite Markov grammar (e.g. Axiom A diffeomorphisms), this result (in particular the formula (3), with $\#\mathrm{Fix}\, f^n$ replaced by $\sum_{x \in \mathrm{Fix}\, f^n} \prod_{k=0}^{n-1} g(f^k(x))$) may be generalized *whenever the weight g is constant or locally constant* (see e.g. [20]). However, for non-constant weights there is in general no obvious finite dimensional space preserved by the transfer operator. One needs to find an adequate invariant Banach space (when possible a Hilbert space). It turns out, unfortunately, that the transfer operator acting on such a Banach space, although bounded, is usually not compact. We illustrate this remark with our second (and last) example, which shows also why it is natural to consider non-constant weights.

Example 2.a. Again we take $X = I$, which we assume to be partitioned into two sub-intervals $L = [0, c[$ and $R = [c, 1]$. We consider now a function $f : I \to I$ such that the restrictions of f to L and R are monotone and C^2, with $f(R) = I$, and such that f has a C^2 extension $\bar{f}$ to $\bar{L}$ with $\bar{f}(\bar{L}) = I$. Finally, we impose the following important *locally expanding* condition: there exists $\lambda > 1$ with $|\bar{f}'| \ge \lambda$ on $\bar{L}$ and $|f'| \ge \lambda$ on R. (From now on we neglect in this example all difficulties related to the boundary point c.) Now, if we take $g = 1/|f'|$ for our weight, we obtain the key property that Lebesgue measure dx on I is preserved by the dual of $\mathcal{L}_g$. By definition this means that for any $\varphi \in L^1(dx)$ we have

$$\int \varphi(x)\, dx = \int \mathcal{L}_g(\varphi)(x)\, dx \tag{4}$$

(just use the change of variable formula in an integral), which we abbreviate as $\mathcal{L}_g^*(dx) = dx$. It follows that if we find a positive fixed point $\varphi_0 \in L^1(dx)$ for

$\mathcal{L}_g$, then we may construct an absolutely continuous f-invariant measure for f by setting $d\mu_0 = \varphi_0 \, dx$. Indeed, for any bounded φ we get by using (4) that

$$\begin{aligned} \int \varphi(f(x))\, \varphi_0(x)\, dx &= \int \mathcal{L}_g((\varphi \circ f)\, \varphi_0)(x)\, dx \\ &= \int \varphi(x) \mathcal{L}_g(\varphi_0)(x)\, dx \\ &= \int \varphi(x)\varphi_0(x)\, dx \,. \end{aligned} \tag{5}$$

(In the second equality of (5) we just used formula (1) for $\mathcal{L}_g$.) It is therefore natural to try to understand the spectrum and eigenfunctions of $\mathcal{L}_g$ acting on a suitable space for $g = 1/|f'|$. Clearly, finite-dimensional spaces of locally constant functions, as the vector pace V_2 in Example 1, will not be preserved in general. Unfortunately, the Banach space $L^1(dx)$, although invariant under $\mathcal{L}_g$, is "too big:" the spectrum of $\mathcal{L}_g$ acting on $L^1(dx)$ consists in the entire closed unit disc (each point of the open disk is actually an eigenvalue of infinite multiplicity, see e.g. [44]). It turns out that the spectrum of $\mathcal{L}_g$ acting on the Banach space of continuous functions (with supremum norm) is also the entire unit disc. However, by combining the contraction property of the two inverse branches of f with the smoothness of f and g, we will see below that $\mathcal{L}_g$ preserves the "smaller" Banach space $C^1(I)$ of C^1 functions (endowed with the norm $\sup|\varphi| + \sup|\varphi'|$) and that its spectrum for this space has a *gap.* (We shall explain how this spectral property is useful to show exponential mixing properties for differentiable observables.)

Quasicompactness.

For L a bounded linear operator acting on a Banach space B, we define the *essential spectral radius* of L to be the smallest nonnegative number ρ such that the spectrum of L outside of the disc $\{z \in \mathbb{C} \mid |z| \le \rho\}$ consists of isolated eigenvalues of finite multiplicity. (This definition does not exclude the case where these eigenvalues accumulate on the circle $|z| = \rho$.)

In many interesting situations, including the framework of Example 2, one can find a Banach space $\mathcal{B}$ of functions $\varphi : X \to \mathbb{C}$ which is invariant under $\mathcal{L}_g$ and such that:

- there is an upper bound R for the spectral radius of $\mathcal{L}_g : \mathcal{B} \to \mathcal{B}$ and an upper bound $R_{\text{ess}} < R$ for the essential spectral radius of $\mathcal{L}_g : \mathcal{B} \to \mathcal{B}$;
- if one assumes additionally that the weight g is real-valued and strictly positive, then $R > 0$ is actually an eigenvalue of $\mathcal{L}_g$, with a positive eigenfunction φ_0, and a positive eigenfunctional $d\nu_0$ for $\mathcal{L}_g^*$ which is a Borel measure. (In particular, a trivial modification of the the algebra in (5) implies that $d\mu_0 = \varphi_0 d\nu_0$ is an f-invariant finite measure.)

For positive weights, it is often possible to prove under additional assumptions that the real positive eigenvalue R is simple and is the only eigenvalue of modulus R. In this case, one says that the transfer operator has a *spectral gap,* and one has

$$\tau := \sup\{|z| \mid z \in \text{spectrum}\, \mathcal{L}_g \,, z \neq R\} < R \,. \tag{6}$$

Example 2.b. In the setting of Example 2.a, one can prove that the spectral radius of $\mathcal{L}_{1/|f'|}$ acting on $C^1(I)$ is equal to 1 and that the essential spectral radius is not larger than $1/\lambda < 1$ (see [44] for a better bound). Also, one sees that there is a positive fixed function $\varphi_0 \in C^1(I)$. (The fixed eigenfunctional $d\nu_0$ is just Lebesgue measure dx as already mentioned.) Before discussing this example further, we note that a crucial bound in obtaining these results is the following inequality: for any $1 < \Lambda < \lambda$, there is a constant $C > 0$ so that for all $\varphi \in C^1(I)$ and all $n \geq 1$:

$$\sup\left|\frac{d}{dx}(\mathcal{L}_g^n\varphi)(x)\right| \leq \frac{C}{\Lambda^n}\sup\left|\frac{d}{dx}\varphi(x)\right| + C\sup|\varphi|\,. \tag{7}$$

(See e.g. [58] for various occurrences of this bound.) To prove (7), expand

$$\mathcal{L}_g^n\varphi(x) = \sum_{y:f^n(y)=x}\frac{\varphi(y)}{|(f^n)'(y)|}\,, \tag{8}$$

and differentiate the right-hand-side of (8) with respect to the variable x, applying the Leibniz formula to each summand. The contraction factor $1/\Lambda^n$ comes from the interior derivative which appears when differentiating $\varphi(y)$ with respect to x. The proof of (7) is a simple occurrence of a general phenomenon: the two building blocks of transfer operators are *composition* (here, with contracting local inverse branches) and *multiplication* by a weight, so that change of variables and the Leibniz formula (or integration by parts) are the key tools used to obtain bounds.

In fact, one may also show in the situation of Example 2.a that the eigenvalue 1 is simple and is the only eigenvalue of modulus 1. The simplicity of the eigenvalue is related to the fact that $d\mu_0 = \varphi_0\,dx$ is the unique absolutely continuous invariant measure of f and that $(f, d\mu_0)$ is ergodic. The existence of the gap (i.e., the fact that $\tau < 1$ in the notation of (6)) is linked to mixing properties of $(f, d\mu_0)$. Specifically, for any $1 > \tilde\tau > \tau$ there is a constant $C > 0$ so that if $\psi_1 \in L^1(dx)$ and $\psi_2 \in C^1(I)$, the *correlation function*

$$C_{\psi_1,\psi_2}(k) = \int \psi_1 \circ f^k\,\psi_2\,d\mu_0 - \int\psi_1\,d\mu_0\int\psi_2\,d\mu_0 \tag{9}$$

satisfies

$$\begin{aligned}
|C_{\psi_1,\psi_2}(k)| &= \left|\int\left[\mathcal{L}_g^k(\psi_1\circ f^k\,\psi_2\,\varphi_0) - (\int\psi_2\,d\mu_0)\,\psi_1\,\varphi_0\right]dx\right| \\
&= \left|\int\left[\mathcal{L}_g^k(\psi_2\,\varphi_0) - (\int\psi_2\,d\mu_0)\;\varphi_0\right]\psi_1\,dx\right| \\
&\leq \sup\left|\mathcal{L}_g^k(\psi_2\,\varphi_0) - \varphi_0\cdot\int(\psi_2\varphi_0)\,dx)\right|\int|\psi_1|\,dx \\
&\leq C\,\tilde\tau^n\,(\sup|\psi_2| + \sup|\psi_2'|)\int|\psi_1|\,dx\,.
\end{aligned} \tag{10}$$

Therefore the correlation functions associated to the unique absolutely continuous invariant measure $d\mu_0$ and C^1 observables decay exponentially fast (with uniform rate τ). In the last inequality of (10) we used the spectral decomposition of $\mathcal{L}_g$ given by $\{|z| \mid z \in \operatorname{spectrum} \mathcal{L}_g, |z| < 1\} \cup \{1\}$. (We refer e.g. to [58] for more details.)

As a last comment, we mention that whenever the map f is piecewise C^r (for some $r \geq 3$), then the weight $g = 1/|f'|$ is piecewise C^{r-1}, and the transfer operator acts on the Banach space $C^{r-1}(I)$. One can prove that $\varphi_0 \in C^{r-1}(I)$ and that the essential spectral radius of $\mathcal{L}_g$ is not larger than $1/\lambda^{r-1}$. When f is piecewise analytic, the operator $\mathcal{L}_g$ is a compact operator when acting on holomorphic functions on a complex neighbourhood of I (with a bounded extension to the boundary, and using the supremum norm). We refer to [43] for the analytic case, and to [47], [48] for the differentiable case where locally expanding maps on more general manifolds are considered (with $g = 1/|\det Df|$ or more general smooth weights).

Weighted dynamical zeta functions.

We return again to the very general framework of a transfer operator $\mathcal{L}_g$ constructed from a dynamical system f and a weight g. Having fixed a Banach space $\mathcal{B}$ for which both claims of the subsection *Quasicompactness* can be proved, our aim is to find, for various relevant classes of f, g, a suitable definition of a (generalized) trace "tr" $\mathcal{L}_g$ for $\mathcal{L}_g$ (and all its powers). Recall that $\mathcal{L}_g$ acting on $\mathcal{B}$ will *not* be a compact operator in general. In particular, we shall *not* require that this generalized trace be related to the sum of the eigenvalues of $\mathcal{L}_g$ (which might form an uncountable set). Instead, our wish is that the *generalized Fredholm determinant,* defined formally by

$$d_g(z) = \exp - \sum_{n \geq 1} \frac{z^n}{n} \text{"tr"} \mathcal{L}_g^n , \tag{11}$$

has the properties that:

- $d_g(z)$ defines an analytic function in a disc $|z| < R_{\text{ess}}^{-1}$, where R_{ess} is an upper bound for the essential spectral radius of $\mathcal{L}_g : \mathcal{B} \to \mathcal{B}$;
- the zeroes of $d_g(z)$ in this disc are exactly the inverses of the eigenvalues of $\mathcal{L}_g : \mathcal{B} \to \mathcal{B}$ in the corresponding annulus (the order of the zero coinciding with the algebraic multiplicity of the eigenvalue).

If the two above properties hold, the function $d_g(z)$ clearly deserves to be called a generalized Fredholm determinant. We shall see that sometimes the function $\zeta_g(z) = 1/d_g(z)$ has the form

$$\zeta_g(z) = \exp \sum_{n \geq 1} \frac{z^n}{n} \sum_{x \in \operatorname{Fix} f^n} \prod_{k=0}^{n-1} g(f^k(x)) \tag{12}$$

which is a general expression for the *dynamical zeta function* associated to the dynamics f and the weight g. Note that there are variants for the definition of $\zeta_g(z)$, in particular when the number of elements in $\operatorname{Fix} f^n$ can be infinite (even uncountable, see [71, 72]).

We now illustrate some possible definitions in the simple setting of Example 2.a.

Example 2.c. We consider again our interval map f and the weight $g = 1|f'|$, assuming that f is piecewise C^r for some $r \geq 2$. There are basically two ways to define the trace in this framework:

a) The *counting trace* is defined by taking

$$\mathrm{tr}^c \mathcal{L}_g = \sum_{y \in \mathrm{Fix}\, f} g(y) \tag{13}$$

(so that $\mathrm{tr}^c \mathcal{L}_g^n = \sum_{y \in \mathrm{Fix}\, f^n} \prod_{k=0}^{n-1} g(f^k y)$, note that the number of terms in the sum is finite but grows exponentially in n). With this definition, we just get $d_g^c(z) = 1/\zeta_g(z)$ with ζ_g defined in (12), and one may prove ([48]):

Theorem 1. *The power series for $\zeta_g(z)$ defines an analytic function in the disc of radius 1, with a meromorphic extension to the disc of radius λ where its poles are exactly the inverses of the eigenvalues of $\mathcal{L}_g : C^{r-1}(I) \to C^{r-1}(I)$ of modulus larger than $1/\lambda$ (with correct multiplicity).*

b) The *flat trace* is defined by setting

$$\mathrm{tr}^\flat \mathcal{L}_g = \sum_{y \in \mathrm{Fix}\, f} \frac{g(y)}{(1 - 1/|f'(y)|)} \tag{14}$$

(and thus $\mathrm{tr}^\flat \mathcal{L}_g^n = \sum_{y \in \mathrm{Fix}\, f^n} (\prod_{k=0}^{n-1} g(f^k y))/(1 - 1/|(f^n)'(y)|)$). Using this definition of the trace, we get a determinant $d_g^\flat(z)$ such that [48]:

Theorem 2. *The power series for $d_g^\flat(z)$ is analytic in the disc of radius λ^{r-1} where its zeroes are exactly the inverses of the eigenvalues of $\mathcal{L}_g : C^{r-1}(I) \to C^{r-1}(I)$ of modulus larger than $1/\lambda^{r-1}$ (with correct multiplicity).*

The flat determinant therefore “sees” more of the spectrum of $\mathcal{L}_g$ than the counting determinant (or zeta function) whenever $r - 1 > 1$. We refer e.g. to the survey [8] for a heuristic justification of the formula for the flat trace and references to papers of Atiyah and Bott which inspired the terminology.

Versions of Theorem 1 and Theorem 2 on the counting trace and the flat trace stated in Example 2.c apply to any locally λ-expanding C^r map $f : X \to X$ and C^{r-1} weight g (see [47], [48]). The spectral radius is then bounded by the exponential of the topological pressure of $\log |g|$.

2. Commented bibliography

The bibliography that we have assembled here is certainly not complete nor systematic. Although it clearly reflects the tastes (and sometimes the ignorance) of the author, we hope that it can serve as a useful introduction to the reader who wishes to enter the field. We have divided the list of references into several sublists, hoping to make it more accessible. Many topics have been completely omitted, for example random dynamical systems. Finally, we have for the sake of conciseness stated most results in a rather vague and intuitive way: we refer to the cited papers (or to the surveys [7-13]) for precise formulations.

0. Foundations.

The first sublist contains on the one hand two of the earliest references to dynamical zeta functions in the literature ([1], [6]) and on the other a few useful books in functional analysis ([2] and [5] contain general background, and [3], [4] specialized topics useful in particular to read the references in sublists 4.A, 5.A and 7). We have not attempted to list any elementary books on dynamical systems or ergodic theory.

1. Surveys.

This list should also include the book of Parry and Pollicott [28] and the first chapter of Ruelle's recent book [66] which contains a pleasant and broad-viewed introduction to dynamical zeta functions. Reference [11] is a beautiful and comprehensive account of the physicist's viewpoint, and contains many applications, in particular quantum chaos. Reference [7] contains an elementary exposition of Parry and Pollicott's "prime-orbit theorem" for Axiom A flows. Surveys [8] and [13] summarize many of the known results. We plan to include the more recent rigorous breakthroughs in [9].

2. Applications.

There should be many more references here! We have mainly selected a striking application to Feigenbaum period-doubling ([15], [18]) and a link with Riemann zeta functions [17].

3. Subshifts of finite type and Axiom A.

This subsection is the longest one and could actually be much longer: we have abstained from trying to give references to the original papers (in particular by Sinai, Ruelle, Bowen, Ratner) establishing the basic results in the ergodic theory of hyperbolic diffeomorphisms and flows, since there exist some very good books describing this material (notably [19], [28], and [31]). However, we have listed a few original papers specific to dynamical zeta functions, although the contents of some of them have been presented in [28].

References [19], [21] contain the basic theory of Axiom A diffeomorphisms and flows, showing how many of their ergodic properties can be studied (via Markov partitions and symbolic dynamics) by understanding subshifts of finite type (and their suspensions under Lipschitz or Hölder return times) with Lipschitz or Hölder weights. The observation of Bowen and Lanford that the unweighted zeta function of a subshift of finite type is rational is contained in [20], and the application by Manning to Axiom A diffeomorphisms (whose unweighted zeta functions (2) are also rational) appeared in [27].

A succession of results on transfer operators and zeta functions for subshifts of finite type f and Lipschitz (or Hölder) weights g is contained in [29], [30] (Pollicott), [33], [34] (Ruelle), and [25], [26] (Haydn). Many of them are collected in the book [28]. They give bounds on the spectral radius and essential spectral radius of the transfer operator (1) acting on Lipschitz functions, and say that the weighted zeta function (12) is meromorphic in a disc where its poles are the inverse eigenvalues of $\mathcal{L}_g$ in the corresponding annulus. The unweighted zeta function of a flow with countably closed orbits is defined formally by

$$\zeta^*(s) = \prod_{\tau} (1 - e^{-s\ell(\tau)})^{-1} \tag{15}$$

(where the product is over all closed orbits τ, with primitive length $\ell(\tau)$). The zeta functions $\zeta^*(s)$ of flows obtained by suspending subshifts of finite type under a Lipschitz return time r are shown to be analytic in a half-plane $\Re s > \eta$ (η being actually the topological entropy of the flow), and meromorphic in a larger half-plane $\Re s > \delta$ (the poles there corresponding to values of s such that the operator $\mathcal{L}_{e^{-sr}}$ weighted with $g = \exp(-sr)$ has the number 1 in its spectrum). Simplified sketches of some of the proofs may be found in the review [8].

Gallavotti [24] constructed examples of suspensions under Lipschitz return times where a non-polar singularity was present arbitrarily close to the vertical $\Re s = \delta$. Ruelle [32] and Pollicott constructed examples of mixing suspensions for which poles accumulated on the vertical $\Re s = \eta$, showing that topologically mixing Axiom A flows need not have exponential decay of correlations for the measure of maximal entropy (or other Gibbs measures).

Very recently, Dolgopyat [23] found sufficient conditions under which the poles of a mixing Axiom A flow cannot accumulate on the verical $\Re s = \eta$, and showed that generically the accumulation of these poles when it occurs cannot be "too fast," ensuring rapid decay (in the sense of Schwartz) of correlations for Gibbs measures associated to Lipschitz interactions (in particular the SRB measure). He also showed [22] that in the case of Anosov flows with C^1 stable and unstable foliations (in particular the geodesic flows on surfaces of negative curvature) the conditions forbidding the accumulation of poles were satisfied.

4. The smooth expanding case.

4.A. Analytic expanding case.

Over 20 years, ago Ruelle [43] observed that Grothendieck's theory of nuclear operators [3, 4] could be applied to transfer operators when both the dynamics and the weight were analytic (assuming that the dynamics is uniformly locally expanding). In this case, the operators (acting on a suitable space of bounded holomorphic functions) are compact, and the flat trace

$$\mathrm{tr}^\flat \mathcal{L}_g = \sum_{x \in \mathrm{Fix}\,(x)} \frac{g(x)}{\mathrm{Det}\,(1 - Df^{-1}(x))} \tag{16}$$

is actually the sum of the eigenvalues of $\mathcal{L}_g$. The corresponding determinant $d_g^\flat(z)$ is an entire functions with zeroes the inverses of the eigenvalues of $\mathcal{L}_g$. Ruelle applied these powerful results to Anosov diffeomorphisms or flows, under the strong assumption that their stable/unstable foliations are analytic. This assumption was required because the strategy then was to reduce to the expanding situation by projecting along stable manifolds. (It is unfortunately very rarely satisfied: the invariant foliations are usually only Hölder.) Extensions of these results (in particular a correction of the asymptotics of the eigenvalues) were obtained by Fried [36].

D. Mayer [40, 41] used this approach to study the Gauss map $x \mapsto \{1/x\}$ ($0 < x \leq 1$). One striking result in [40] is the proof of the reality of the spectrum of some weighted transfer operators associated to the Gauss map: this is obtained by considering the action of the operator on suitable Hilbert spaces of Hardy functions and proving self-adjointness. The survey [42] discusses some applications.

Papers [35] and [37-39] contain some rather strong results on the spectrum of transfer operators, mostly for rational or polynomial one-dimensional maps with rational weights.

4.B. Differentiable expanding case.

When the smoothness assumption on the locally expanding dynamics f and the weight g is weakened from analytic to differentiable (i.e., C^r for some $r \geq 1$) one must abandon the beautiful classical techniques of Grothendieck and work essentially as in the case of subshifts of finite type (i.e., prove bounds "by hand," using Taylor expansions to approach the transfer operators by finite rank operators). This was first done in the C^∞ case by Tangerman [49] who showed that the corresponding zeta function (12) was meromorphic in the whole complex plane. Later, Ruelle [47, 48] studied the case of finite differentiability, proving generalized versions of Theorems 1 and 2 above (also for "mixed" transfer operators, obtained by summing – or integrating – over a family of contractions not necessarily related to a single map). Fried [45] pushed the analysis a bit further, obtaining in particular results on the asymptotics of the distribution of eigenvalues.

We also mention in this subsection the articles of Collet-Isola [44] and Holschneider [46] who obtain exact formulas (as opposed to upper bounds) for transfer operators acting on classes of smooth functions. The introduction of wavelet techniques by Holschneider is novel, and allows to consider a large class of Banach spaces.

5. The smooth hyperbolic case.

Let us consider now a real analytic transformation $f : M \to M$ of a real analytic manifold and a real analytic weight $g : M \to \mathbb{C}$, assuming that f satisfies some uniform hyperbolicity condition (such as Anosov). As noted above, if the foliations of f are analytic too, one can reduce to the analytic expanding setting of 4.A, in particular the flat determinant $d_g^\flat(z)$ from the traces (14) is an entire function (and the zeta function (12), which can be written as a quotient of two finite products of flat determinants for modified weights, is meromorphic in the whole complex plane). Foliations being generically only Hölder, this reduction to the noninvertible expanding case only produces zeta functions meromorphic in a disc (and noncompact transfer operators), just like in the case of subshifts of finite type with Lipschitz weight. It is therefore desirable to introduce transfer operators associated to the "full" invertible hyperbolic system. The price to pay is that one needs to introduce Banach spaces of "functions" (actually distributions) that are smooth along unstable manifolds but rough (i.e., functionals over smooth functions) along stable manifolds (because the inverse map is an expansion along the stable directions). For Axiom A dynamical systems this was done first by Rugh [51, 52] (diffeomorphisms of surfaces or flows on 3-manifolds) who applied Grothendieck's theory to show that the associated transfer operators are nuclear, and that the flat determinants $d_g^\flat(z)$ are entire. This approach was subsequently extended by Fried

[50] to arbitrary dimensions. In particular, it follows from [50] that the geodesic flow corresponding to an analytic metric of negative curvature on a compact analytic Riemann manifold extends to a meromorphic function on the whole complex plane. The relation between the zeroes of $d^\flat_g(z)$ and the correlation spectrum has been partially explored [53].

More recently, Kitaev [54] has used a similar approach to study the flat determinants associated to C^r Anosov diffeomorphisms and C^r weights, without assuming additional smoothness of the foliations. (In fact, Kitaev studies mixed transfer operators obtained by summing over maps which preserve the same cone fields.) Here, everything must be done "by hand" and the Banach space of distributions on which the operator acts is again one of the key difficulties. We also mention Liverani's paper [55] on the decay of correlation of certain differentiable hyperbolic systems (using Hilbert-type metrics on Birkhoff cones of functions), although it is not connected to the zeta-function approach, because the philosophy involved in the definition of the Birkhoff cones is very similar to the ideas of Rugh and Kitaev.

6. The one-dimensional case.

The theory for piecewise monotone interval maps *without* assuming the existence of a finite Markov partition, was developed in parallel to that of Axiom A systems. The natural Banach space preserved by the transfer operator is the space of functions of bounded variation (as was already observed by Lasota and Yorke, see the very nice survey [58] for references). There is no flat trace in general, and one simply uses the counting trace and the "ordinary" zeta function (12). Hofbauer and Keller [59, 60] obtained crucial results, using the countable Markov towers of Hofbauer, under certain conditions. A rather general result which is the analogue of Theorem 1 in the Introduction above (assuming that the weight is of bounded variation and the partition of intervals of monotonicity is generating) was later obtained by Baladi and Keller [57], using again the Hofbauer tower technique. In [66] Ruelle subsequently used another approach (working with Markovian extensions with finitely many symbols, and setting the weight equal to zero to describe the infinite grammar of the original dynamics) and obtained further results (in particular relating the spectral radius to topological entropies). Reference [65] contains in particular the extension to the case where a finite or countable set of branches (not necessarily associated to a single interval map) is considered (see [75] and especially [76], [71] for variants and simplifications of the proof in [65]). The case of one complex dimension is considered in [67] (see also [71]).

The case of an interval map (e.g. with two branches) which is expanding except at a single neutral fixed point ($f(x) = x$ and $f'(x) = 1$) is interesting in view of applications to physics. Prellberg [61] and then Isola [64] show how to study the original map with the help of an auxiliary induced map which is piecewise (uniformly) expanding (with countably many intervals of monotonocity). Isola [61] proves in certain cases that correlation functions associated with the unique absolutely continuous invariant measure decay polynomially for suitable observables.

The case of Collet-Eckmann (or Benedicks Carleson)-type unimodal maps with a critical point (in particular from the logistic family) was studied independently by Keller-Nowicki [62] and Lai-Sang Young [69] who used tower extensions to remove the singularities in the weight $g = 1/|f'|$ of the transfer operator. Ruelle [68] introduced some methods from several complex variables to study directly the zeta function. Pollicott [63] pushed this approach further.

7. The one-dimensional case: kneading operator approach.

One of the main results in the classical paper [74] by Milnor and Thurston is a formula for the (unweighted) "negative" zeta function of a piecewise monotone interval map

$$\zeta^{-}(z) = \exp \sum_{n \geq 1} \frac{z^n}{n} 2\, \#\mathrm{Fix}^{-} f^n \,, \tag{17}$$

where $\mathrm{Fix}^{-} f^n$ is the number of fixed points of f^n which lie in an interval of monotonicity of f^n where f^n is decreasing (such intersections are transverse and therefore always form a finite set if f has finitely many intervals of monotonicity). This formula simply said that $1/\zeta^{-}(z)$ is the determinant of a finite matrix (the *kneading matrix*) whose coefficients are power series (the *kneading invariants*)in z associated to the orbits of the turning point.

In [72], [75], [70], and finally [73] this result of Milnor and Thurston was extended by Baladi and Ruelle to the weighted case, where the weight g is supposed to be of bounded variation, and where the branches considered do not necessarily come from a single map. (The transfer operators act on functions of bounded variation.) Since this situation includes the case where infinitely many (even uncountably many) fixed points exist, a new trace must be introduced, which is called the *sharp trace.* Ruelle [76] also considered similar cases where the weight is smoother. We refer to the reviews [9, 10] for more details.

Complex analogues of this approach are presented in [71].

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