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Spherical designs and finite group representations (some results of E. Bannai)

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Abstract

We reprove several results of Bannai concerning spherical t-designs and finite subgroups of orthogonal groups. These include criteria in terms of harmonic representations of subgroups of $O(n)$ for the corresponding orbits to be t-designs ($t = 0, 1, 2, 3, \ldots$) in $\mathbb{S}^{n-1}$. We also discuss a conjecture of Bannai, dating from 1984, according to which t is bounded independently of the dimension n (for $n \geq 3$) for such designs. © 2003 Published by Elsevier Ltd.

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1. Introduction

Given a dimension $n \geq 2$ and an integer $t \geq 0$, a *spherical t-design* in dimension n is a nonempty finite subset X of the unit sphere $\mathbb{S}^{n-1}$ of the Euclidean space $\mathbb{R}^n$ such that

$$\frac{1}{|X|}\sum_{x\in X}\phi(x) = \int_{\mathbb{S}^{n-1}}\phi(y)\,\mathrm{d}\mu(y)$$

for all $\phi \in \mathcal{F}^{(t)}(\mathbb{S}^{n-1})$, the space of those real-valued continuous functions on the sphere which are restrictions of polynomial functions of degree at most t on $\mathbb{R}^n$. Here $|X|$ denotes the cardinality of X and μ the $O(n)$-invariant probability measure on $\mathbb{S}^{n-1}$, where $O(n)$ is the group of orthogonal transformations on $\mathbb{R}^n$.

The term "spherical design" goes back to [9] (see also [11] and [29]). There is an existence result for all values of n and t [21] (see also [1–3], and [31]), but explicit examples are in general not straightforward to construct when $n \geq 3$ and $t \geq 2$. However,

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for low values of t, results of Bannai provide spherical t-designs as orbits in $\mathbb{S}^{n-1}$ of finite subgroups of $O(n)$. The purpose of the present exposition is to prove some of these results of Bannai in a way we find simpler than in the original articles; in particular, we avoid the use of bases in spaces of harmonic polynomials. (As we were finishing this work, we found out an exposition overlapping substantially with ours in Section 2 of [18].)

More precisely, let G be a subgroup of $O(n)$. For an integer $k \geq 0$, let $\pi_G^{(k)}$ denote the natural linear representation of G in the space $\mathcal{H}^{(k)}(\mathbb{R}^n)$ of real-valued polynomials on $\mathbb{R}^n$ which are homogeneous of degree k and harmonic (see Section 2 below). We denote by 1_G the unit representation of G, and $\rho \not< \sigma$ means that the representation ρ of G is *not* a subrepresentation of the representation σ of G. A finite subgroup G of $O(n)$ is said to be *t-homogeneous* if the orbit Gx_0 of any point $x_0 \in \mathbb{S}^{n-1}$ is a spherical t-design.

Theorem 1 (Bannai). *Let G be a finite subgroup of $O(n)$ and let s, t be positive integers.*

(i) *If $1_G \not< \pi_G^{(k)}$ for $1 \leq k \leq t$, then G is t-homogeneous, and conversely.*
(ii) *If $n \geq 3$ and if $\pi_G^{(k)}$ is irreducible for $1 \leq k \leq s$, then G is $(2s)$-homogeneous.*
(iii) *If $n \geq 3$, if $\pi_G^{(k)}$ is irreducible for $1 \leq k \leq s$, and if $\pi_G^{(s)} \not< \pi_G^{(s+1)}$, then G is $(2s+1)$-homogeneous.*
(iv) *If there exists one orbit of G on $\mathbb{S}^{n-1}$ which is a spherical $(2t)$-design, then G is t-homogeneous.*

Moreover, for each integer $n \geq 3$, there exists an integer $t_{\max}(n)$ such that, whenever some finite subgroup of $O(n)$ is t-homogeneous, then $t \leq t_{\max}(n)$.

Claim (i) is essentially a reformulation of the definitions (it appears as Theorem 6.1 in [11]). Claims (ii) and (iii) appear in [4] and [5], with a slightly more restrictive hypothesis. (In particular, it was observed in [11] that the absolute irreducibility of $\pi_G^{(k)}$, assumed by Bannai, can be replaced by irreducibility; also, in (iii), the hypothesis $\pi_G^{(s)} \not< \pi_G^{s+1}$ is a weakening of the corresponding hypothesis by Bannai.) Claim (iv) and the bound $t \leq t_{\max}(n)$ appear in [6] and [7]. With appropriate definitions, claims (i)–(iv) carry over to *compact* subgroups of $O(n)$.

The converses of claims (ii) and (iii) do *not* hold, and the claims themselves do *not* hold for $n = 2$ (see below, the end of Section 2). As the group of $(t+1)$-roots of unity is a spherical t-design in $\mathbb{S}^1$ for each $t \geq 0$, the last claim in the theorem does *not* hold for $n = 2$.

(After submission of this paper, C. Pache has found that, in claims (ii) and (iii) of Theorem 1, it is enough to assume that $\pi_G^{(s)}$ is irreducible, instead of assuming that $\pi_G^{(k)}$ is irreducible for $1 \leq k \leq s$. See the Appendix below.)

Proposition 2 (Bannai). *Let H be a finite subgroup of $O(n)$, let X be a spherical $(2s)$-design in $\mathbb{S}^{n-1}$ which is H-invariant, and let λ denote the permutation representation of H defined by its action on X. Then λ contains $\bigoplus_{k=0}^{s} \pi_H^{(k)}$ as a subrepresentation.*

This is Theorem 2 in [6]. Claims (i)–(iii) of Theorem 1 are proved in Section 2. The other claims of Theorem 1 and Proposition 2 are proved in Section 3.

Theorem 1 can be illustrated by numerous examples. In particular, consider in $O(3)$ the subgroup $G(T)$ (resp. $G(C)$, $G(D)$) of orthogonal symmetries of a regular tetrahedron (resp. cube, dodecahedron). Then $G(T)$ is 2-homogeneous and has orbits which are spherical 3-designs and 5-designs; $G(C)$ is 3-homogeneous; $G(D)$ is 5-homogeneous and has orbits which are spherical 9-designs; see [12]. The group of automorphisms of a Leech lattice, which is a finite subgroup of $O(24)$, is 11-homogeneous and has orbits which are spherical 15-designs (see Example 8.5 in [9], as well as Section 7 in [12]). There are other examples in [22] and [23]. Constructions involving finite subgroups of $O(n)$ in the closely related subject of "spherical designs with weights" (better known as "cubature formulas on spheres") go back at least to [25].

We mention a few more examples in Section 4, which among other things makes precise the following statement (which rules out subgroups of $O(2)$).

Proposition 3. *There exists an infinite family of* 7*-homogeneous groups, and there exist* 11*-homogeneous groups.*

There is a *conjecture of Bannai*, according to which the last statement of Theorem 1 holds with a bound $t_{\max}$ independent of the dimension n, but we have not been able to make progress on this. At the end of the paper, we formulate questions related to Bannai's conjecture.

2. Sufficent conditions for t-homogeneity

Consider $n \geq 2$ and $t \geq 0$ as in the Introduction, a finite subgroup G of $O(n)$, a point x_0 in the unit sphere $\mathbb{S}^{n-1}$, and the orbit $X = Gx_0$. We have

$$\frac{1}{|X|}\sum_{x\in X}\phi(x) = \frac{1}{|G|}\sum_{g\in G}\phi(gx_0)$$

for all functions ϕ on $\mathbb{S}^{n-1}$. If G is more generally a compact subgroup of $O(n)$, we define an orbit Gx_0 to be a *spherical t-design* if

$$\int_G \phi(gx_0)\,\mathrm{d}g = \int_{\mathbb{S}^{n-1}} \phi(y)\,\mathrm{d}\mu(y) \tag{1}$$

for all $\phi \in \mathcal{F}^{(t)}(\mathbb{S}^{n-1})$, where $\mathrm{d}g$ denotes the normalized Haar measure on G.

Let p be the linear operator on $\mathcal{F}^{(t)}(\mathbb{S}^{n-1})$ defined by $(p(\phi))(y) = \int_G \phi(gy)\,\mathrm{d}g$. Then p is a projection onto the subspace of $\mathcal{F}^{(t)}(\mathbb{S}^{n-1})$ of G-invariant functions, here written $\mathcal{F}^{(t)}(\mathbb{S}^{n-1})^G$; observe that this space contains the space $\mathcal{F}^{(0)}(\mathbb{S}^{n-1})$ of constant functions, which we identify with $\mathbb{R}$. Thus, condition (1) reads

$$(p(\phi))(x_0) = \int_{\mathbb{S}^{n-1}} \phi(y)\,\mathrm{d}\mu(y) \tag{2}$$

for all $\phi \in \mathcal{F}^{(t)}(\mathbb{S}^{n-1})$.

Extending a previous definition, we say that a compact subgroup G of $O(n)$ is *t-homogeneous* if, for all $x_0 \in \mathbb{S}^{n-1}$, the orbit Gx_0 is a spherical t-design.

Given two vector spaces U and V, here over $\mathbb{R}$, the space of linear mappings from U to V is denoted below by $\mathcal{L}(U, V)$.

Whenever V is a space of functions on which a group G acts, we denote by V^G the space of G-invariant functions.

Proposition 4. *Let G be a compact subgroup of $O(n)$.*

(i) *For $t \geq 0$, the group G is t-homogeneous if and only if $\mathcal{F}^{(t)}(\mathbb{S}^{n-1})^G = \mathbb{R}$.*

In particular:

(ii) *the group G is 1-homogeneous if and only if G does not fix any vector $x \neq 0$ in $\mathbb{R}^n$;*
(iii) *the group G is 2-homogeneous if and only if the linear action of G on $\mathbb{R}^n$ is irreducible.*

Proof. Eq. (2) shows that G is t-homogeneous if and only if $p(\phi)$ is a constant function for all $\phi \in \mathcal{F}^{(t)}(\mathbb{S}^{n-1})$, and this establishes (i).

For claim (ii), observe that $\mathcal{F}^{(1)}(\mathbb{S}^{n-1}) = \mathbb{R} \oplus \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, so that $\mathcal{F}^{(1)}(\mathbb{S}^{n-1})^G = \mathbb{R}$ if and only if $\mathcal{L}(\mathbb{R}^n, \mathbb{R})^G = \{0\}$, if and only if $(\mathbb{R}^n)^G = \{0\}$.

For claim (iii), assume first that G is reducible; let V be a nontrivial G-invariant subspace of $\mathbb{R}^n$. If p denotes the orthogonal projection of $\mathbb{R}^n$ onto V and $\langle \cdot \mid \cdot \rangle$ the Euclidean scalar product, the function $\mathbb{R}^n \ni x \longmapsto \langle p(x) \mid p(x) \rangle \in \mathbb{R}$ is in $\mathcal{F}^{(2)}(\mathbb{S}^{n-1})^G$ and is not constant, so that $\mathcal{F}^{(2)}(\mathbb{S}^{n-1})^G \neq \mathbb{R}$.

Assume next that $\mathcal{F}^{(2)}(\mathbb{S}^{n-1})^G$ contains a nonconstant function ϕ. If the odd part $x \longmapsto \frac{1}{2}(\phi(x) - \phi(-x))$ is not zero, G is reducible by (ii). We may therefore assume without loss of generality that $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a homogeneous polynomial of degree 2. Let β be the symmetric bilinear form defined on $\mathbb{R}^n$ by $\beta(x, y) = \phi(x + y) - \phi(x) - \phi(y)$ and let $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be the operator defined by $\beta(x, y) = \langle x \mid B(y) \rangle$ for all $x, y \in \mathbb{R}^n$. Then B is self-adjoint, commutes with all elements of G, and any eigenspace of B in $\mathbb{R}^n$ is G-invariant. Moreover, since ϕ is not constant, B is not a scalar multiple of the identity, thus B has a nontrivial eigenspace, and the action of G on $\mathbb{R}^n$ is reducible. (This implication is a particular case of one in Theorem 7, proven below.) □

Let us now review some classical facts on spherical representations of $O(n)$. For $k \geq 0$, let $\mathcal{P}^{(k)}(\mathbb{R}^n)$ denote the space of real-valued polynomial functions on $\mathbb{R}^n$ which are homogeneous of degree k, and let

$$\mathcal{H}^{(k)}(\mathbb{R}^n) = \left\{ \phi \in \mathcal{P}^{(k)}(\mathbb{R}^n) \;\middle|\; \frac{\partial^2 \phi}{\partial x_1^2} + \cdots + \frac{\partial^2 \phi}{\partial x_n^2} = 0 \right\}$$

denote the space of *harmonic polynomials* of degree k. We will identify these spaces with spaces of continuous functions on $\mathbb{S}^{n-1}$.

Each of these spaces is also $O(n)$-invariant for the natural action. More precisely, for $k \geq 0$, the linear representation $\pi^{(k)}$ of $O(n)$ in $\mathcal{H}^{(k)}(\mathbb{R}^n)$ is defined by $(\pi^{(k)}(g)\phi)(x) = \phi(g^{-1}x)$; we denote by $\pi^{(k)}_{\mathbb{C}}$ the complexified representation, in the space $\mathcal{H}^{(k)}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$. The two following results are classical (see e.g. [26] or [30]).

(i) We have $O(n)$-invariant direct sums

$$\mathcal{F}^{(t)}(\mathbb{S}^{n-1}) = \mathcal{P}^{(t)}(\mathbb{R}^n) \oplus \mathcal{P}^{(t-1)}(\mathbb{R}^n) = \bigoplus_{k=0}^{t} \mathcal{H}^{(k)}(\mathbb{R}^n)$$

for all $t \geq 0$.

(ii) The representations $\pi_{\mathbb{C}}^{(k)}$, $k \geq 0$, are pairwise inequivalent irreducible complex representations of $O(n)$. A fortiori, the $\pi^{(k)}$ are pairwise inequivalent irreducible real representations of $O(n)$.

For any subgroup G of $O(n)$, we denote by $\pi_G^{(k)}$ the restriction of $\pi^{(k)}$ to G.

Proof of Theorem 1(i). This follows from Proposition 4(i) and from the direct sum decomposition in (i) above. □

For $\psi_1, \psi_2 \in \mathcal{P}^{(k)}(\mathbb{R}^n)$, we have a differential operator with constant coefficients $\psi_2(\partial/\partial x)$, and it happens that $\psi_2(\partial/\partial x)\psi_1$, a priori a function on $\mathbb{R}^n$, is in fact a constant function; moreover, if $[\psi_1 \mid \psi_2]$ denotes the value of this constant function, the assignment $(\psi_1, \psi_2) \longmapsto [\psi_1 \mid \psi_2]$ defines a scalar product on $\mathcal{P}^{(k)}(\mathbb{R}^n)$, and therefore also on $\mathcal{H}^{(k)}(\mathbb{R}^n)$ by restriction. The representation $\pi^{(k)}$ of $O(n)$ is orthogonal for this scalar product: $[\pi^{(k)}(g)\psi_1 \mid \pi^{(k)}(g)\psi_2] = [\psi_1 \mid \psi_2]$ for all $g \in O(n)$ and $\psi_1, \psi_2 \in \mathcal{H}^{(k)}(\mathbb{R}^n)$.

For $l, m \geq 0$, we have a linear mapping

$$\mu_{l,m} : \begin{cases} \mathcal{H}^{(l+m)}(\mathbb{R}^n) & \longrightarrow \mathcal{L}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(m)}(\mathbb{R}^n)) \\ \phi & \longmapsto \left(\psi \longmapsto \psi\left(\frac{\partial}{\partial x}\right)\phi\right). \end{cases} \tag{3}$$

Lemma 5. *For all $l, m \geq 0$, the mapping $\mu_{l,m}$ defined by (3) is injective.*

In the case $l = m$, the image of $\mu_{l,l}$ is inside the space $\mathcal{L}^{sa}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n))$ of operators on $\mathcal{H}^{(l)}(\mathbb{R}^n)$ which are self-adjoint with respect to the scalar product $[\cdot \mid \cdot]$.

Proof. Denote by $\omega \in \mathcal{P}^{(2)}(\mathbb{R}^n)$ the function defined by $\omega(x) = x_1^2 + \cdots + x_n^2$. It is a classical fact that any $\alpha \in \mathcal{P}^{(k)}(\mathbb{R}^n)$ can be written in a unique way as $\alpha = \theta_\alpha + \rho_\alpha \omega$ with $\theta_\alpha \in \mathcal{H}^{(k)}(\mathbb{R}^n)$ and $\rho_\alpha \in \mathcal{P}^{(k-2)}(\mathbb{R}^n)$, so that $\alpha \longmapsto \theta_\alpha$ is the orthogonal projection from $\mathcal{P}^{(k)}(\mathbb{R}^n)$ onto $\mathcal{H}^{(k)}(\mathbb{R}^n)$ with respect to $[\cdot \mid \cdot]$. Since the pointwise multiplication $\mathcal{P}^{(l)}(\mathbb{R}^n) \otimes \mathcal{P}^{(m)}(\mathbb{R}^n) \ni \psi \otimes \chi \longmapsto \psi\chi \in \mathcal{P}^{(l+m)}(\mathbb{R}^n)$ is clearly onto, it follows that the linear mapping $\mathcal{H}^{(l)}(\mathbb{R}^n) \otimes \mathcal{H}^{(m)}(\mathbb{R}^n) \ni \psi \otimes \chi \longmapsto \theta_{\psi\chi} \in \mathcal{H}^{(l+m)}(\mathbb{R}^n)$ is also onto.

Choose now $\phi \in \mathcal{H}^{(l+m)}(\mathbb{R}^n)$. For all $\psi \in \mathcal{H}^{(l)}(\mathbb{R}^n)$ and $\chi \in \mathcal{H}^{(m)}(\mathbb{R}^n)$, we have

$$\begin{aligned}[\mu_{l,m}(\phi)\psi \mid \chi] &= \chi\left(\frac{\partial}{\partial x}\right)\psi\left(\frac{\partial}{\partial x}\right)\phi = \theta_{\chi\psi}\left(\frac{\partial}{\partial x}\right)\phi + \rho_{\chi\psi}\left(\frac{\partial}{\partial x}\right)\omega\left(\frac{\partial}{\partial x}\right)\phi \\ &= \theta_{\chi\psi}\left(\frac{\partial}{\partial x}\right)\phi = [\phi \mid \theta_{\chi\psi}]\end{aligned}$$

since $\omega(\partial/\partial x)\phi = 0$. In particular, if $\phi \in \mathrm{Ker}(\mu_{l,m})$, then $[\phi \mid \theta] = 0$ for all $\theta \in \mathcal{H}^{(l+m)}(\mathbb{R}^n)$, and therefore $\phi = 0$. Thus $\mu_{l,m}$ is injective.

Assume now that $l = m$. For $\phi \in \mathcal{H}^{(2l)}(\mathbb{R}^n)$ and $\psi_1, \psi_2 \in \mathcal{H}^{(l)}(\mathbb{R}^n)$, we have

$$[\mu_{l,l}(\phi)\psi_1 \mid \psi_2] = \psi_2\left(\frac{\partial}{\partial x}\right)\psi_1\left(\frac{\partial}{\partial x}\right)\phi = [\mu_{l,l}(\phi)\psi_2 \mid \psi_1]$$

and the operator $\mu_{l,l}(\phi)$ is self-adjoint. □

The natural representation $\pi^{(l,m)}$ of $O(n)$ on $\mathcal{L}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(m)}(\mathbb{R}^n))$ is given by

$$\pi^{(l,m)}(g)\lambda = \pi^{(m)}(g) \circ \lambda \circ \pi^{(l)}(g^{-1})$$

for $\lambda \in \mathcal{L}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(m)}(\mathbb{R}^n))$, and $g \in O(n)$. Observe that the application $\mu_{l,m}$ of (3) is $O(n)$-equivariant for $\pi^{(l+m)}$ and $\pi^{(l,m)}$. Though it is not used below, it can also be observed that $\pi^{(l,m)}$ is equivalent to the tensor product of the representations $\pi^{(l)}$ and $\pi^{(m)}$ (note that $\pi^{(l)}$, being orthogonal, is equivalent to its own contragredient).

In the case $l = m$, the space $\mathcal{L}^{sa}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n))$ is $\pi^{(l,l)}(O(n))$-invariant. Let $\mathcal{L}_0(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n))$ denote the space of endomorphisms of trace zero. The space

$$\mathcal{L}_0^{sa}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n)) = \mathcal{L}^{sa}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n)) \cap \mathcal{L}_0(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n))$$

is also $\pi^{(l,l)}(O(n))$-invariant.

Lemma 6. *For $l \geq 1$, the image of $\mu_{l,l}$ is inside the space of self-adjoint operators of trace zero on $\mathcal{H}^{(l)}(\mathbb{R}^n)$, so that we have a mapping*

$$\mu_{l,l} : \mathcal{H}^{(2l)}(\mathbb{R}^n) \longrightarrow \mathcal{L}_0^{sa}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n))$$

which is $O(n)$-equivariant for the natural representations.

Proof. Consider the sequence

$$\mathcal{H}^{(2l)}(\mathbb{R}^n) \longrightarrow \mathcal{L}^{sa}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n)) \longrightarrow \mathbb{R},$$

where the first mapping is $\mu_{l,l}$ and the second is the trace. The two mappings are $O(n)$-equivariant, $\dim_{\mathbb{R}}(\mathcal{H}^{(2l)}(\mathbb{R}^n)) > 1$, and $\mathcal{H}^{(2l)}(\mathbb{R}^n)$ is $O(n)$-irreducible; it follows that the composition of these two mappings is zero. □

Theorem 7. *Let G be a compact subgroup of $O(n)$ and let $t \geq 0$ be an integer. Assume that each integer k such that $1 \leq k \leq t$ is given as a sum $k = l+m$ of nonnegative integers, and that*

$$\mathcal{L}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(m)}(\mathbb{R}^n))^G = \{0\} \text{ in case } l \neq m,$$

$$\pi_G^{(l)} \text{ is irreducible in case } l = m.$$

Then G is t-homogeneous.

Proof. By Theorem 1(i), it is enough to show that $\mathcal{H}^{(k)}(\mathbb{R}^n)^G = \{0\}$ whenever $1 \leq k \leq t$.

In the case $k = l + m$ with $l \neq m$, the existence of the $O(n)$-equivariant mapping (3), which is injective by Lemma 5, implies that $\mathcal{H}^{(k)}(\mathbb{R}^n)^G$ embeds in $\mathcal{L}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(m)}(\mathbb{R}^n))^G$, and is therefore $\{0\}$.

Similarly, in the case $k = l + l$ with $l \geq 1$, the space $\mathcal{H}^{(k)}(\mathbb{R}^n)^G$ embeds by Lemma 6 in

$$\mathcal{L}_0^{sa}(\mathcal{H}^{(l)}(\mathbb{R}^n), \mathcal{H}^{(l)}(\mathbb{R}^n))^G \approx \{0\},$$

where the last isomorphism follows from Schur's lemma. □

Here is the form of Schur's lemma used in the previous argument. Let V be a finite-dimensional real vector space given together with a Euclidean scalar product and let π be an orthogonal representation of some group G in V. Let $\mathcal{L}_0^{sa}(V, V)^G$ denote the space of those operators $A : V \longrightarrow V$ which are self-adjoint, of trace 0, and such that $A\pi(g) = \pi(g)A$ for all $g \in G$. If π is irreducible, then $\mathcal{L}_0^{sa}(V, V)^G = \{0\}$.

Proof of claims (ii) and (iii) in Theorem 1. Observe that two representations $\pi_G^{(j)}$ and $\pi_G^{(j')}$ with $j \neq j'$ are never equivalent, since their dimensions are distinct. Apply now Theorem 7 with $k = l + l$ if $k = 2l$ is even and $k = l + (l + 1)$ if $k = 2l + 1$ is odd. □

Consider an integer $m \geq 2$ and a dihedral subgroup G of order $2m$ if $O(2)$. On the one hand, G is $(m - 1)$-homogenous and is not m-homogenous. On the other hand, the largest integer s such that $\pi_G^{(k)}$ is irreducible for all $k \in \{1, \ldots, s\}$ is $s = (m/2) - 1$ is m if even and $s = m - 1$ if m is odd. It follows that neither claim (ii) nor (iii) of Theorem 1 carry over to the case $n = 2$.

On the converses of claims (ii) and (iii) in Theorem 1

The subgroup $U(n)$ of $O(2n)$ is transitive on the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{C}^n = \mathbb{R}^{2n}$, and therefore is t-homogeneous for all $t \geq 0$. However the representation $\pi_{U(n)}^{(2)}$ is reducible.

Indeed, let $\langle \, | \, \rangle_{\mathbb{C}}$ denote the scalar product on $\mathbb{C}^n$, so that the Euclidean scalar product on $\mathbb{R}^{2n} = \mathbb{C}^n$ is given by $\langle x \mid y \rangle = \Re(\langle x \mid y \rangle_{\mathbb{C}})$. There is a $U(n)$-invariant decomposition

$$\mathcal{H}^{(2)}(\mathbb{R}^{2n}) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H}_{\mathbb{C}}^{(2,0)}(\mathbb{C}^n) \oplus \mathcal{H}_{\mathbb{C}}^{(1,1)}(\mathbb{C}^n) \oplus \mathcal{H}_{\mathbb{C}}^{(0,2)}(\mathbb{C}^n)$$

where $\mathcal{H}_{\mathbb{C}}^{(p,q)}(\mathbb{C}^n)$ is the space of harmonic polynomial functions $\mathbb{C}^n \longrightarrow \mathbb{C}$ which are[1] homogeneous of degree p in $z_1, \ldots, z_n$ and homogeneous of degree q in $\overline{z}_1, \ldots, \overline{z}_n$. We have a $U(n)$-invariant direct sum

$$\mathcal{H}^{(2)}(\mathbb{R}^{2n}) = \left(\mathcal{H}_{\mathbb{C}}^{(1,1)}(\mathbb{C}^n) \cap \mathcal{H}^{(2)}(\mathbb{R}^{2n})\right)$$
$$\bigoplus \left(\left(\mathcal{H}_{\mathbb{C}}^{(2,0)}(\mathbb{C}^n) \oplus \mathcal{H}_{\mathbb{C}}^{(0,2)}(\mathbb{C}^n)\right) \cap \mathcal{H}^{(2)}(\mathbb{R}^{2n})\right).$$

The first factor contains functions of the form

$$(x_1, \ldots, x_{2n}) \longmapsto \langle \alpha \mid z \rangle_{\mathbb{C}} \langle \alpha \mid \overline{z} \rangle_{\mathbb{C}}$$

and the second factor contains functions of the form

$$(x_1, \ldots, x_{2n}) \longmapsto \langle \alpha \mid z \rangle_{\mathbb{C}}^2 + \langle \alpha \mid \overline{z} \rangle_{\mathbb{C}}^2$$

with $\alpha \in \mathbb{R}^{2n}$.

There is in [6] an example of a *finite* group showing that the converses of claims (ii) and (iii) in Theorem 1 do not hold.

[1] It is convenient to use on $\mathbb{R}^{2n} = \mathbb{C}^n$ not only the canonical coordinates $(x_1, \ldots, x_{2n})$, but also coordinates $(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$, with $z_j = x_j + ix_{n+j}$ and $\overline{z}_j = x_j - ix_{n+j}$ for $1 \leq j \leq n$. A smooth function ϕ ($\mathbb{R}$-valued or $\mathbb{C}$-valued) is then harmonic if $\sum_{1 \leq j \leq n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} \phi(z, \overline{z}) = 0$. Let $\mathcal{P}_{\mathbb{C}}^{(p,q)}(\mathbb{C}^n)$ denote the space of polynomial functions $\mathbb{R}^{2n} \longrightarrow \mathbb{C}$ which are homogeneous of degree p in $z_1, \ldots, z_n$ and homogeneous of degree q in $\overline{z}_1, \ldots, \overline{z}_n$. Then $\mathcal{H}_{\mathbb{C}}^{(p,q)}(\mathbb{C}^n)$ is the kernel of the Laplacian viewed as a linear mapping $\mathcal{P}_{\mathbb{C}}^{(p,q)}(\mathbb{C}^n) \longrightarrow \mathcal{P}_{\mathbb{C}}^{(p-1,q-1)}(\mathbb{C}^n)$.

3. Upper bounds on homogeneity

If H is a group which acts (here on the left) on a set X, we denote by $\mathbb{R}^{H\backslash X}$ the space of real-valued functions on the orbit space $H\backslash X$. As before, $n \geq 2$ and $s \geq 0$ are given integers.

Lemma 8. *Let $X \subset \mathbb{S}^{n-1}$ be a spherical $(2s)$-design. Then the linear mapping*

$$Ev_X : \begin{cases} \mathcal{F}^{(s)}(\mathbb{S}^{n-1}) & \longrightarrow \ \mathbb{R}^X \\ \phi & \longmapsto \ (x \longmapsto \phi(x)) \end{cases}$$

is injective. If X is moreover invariant by some finite subgroup H of $O(n)$, then Ev_X is H-equivariant, so that in particular the mapping

$$\begin{cases} \mathcal{F}^{(s)}(\mathbb{S}^{n-1})^H & \longrightarrow \ \mathbb{R}^{H\backslash X} \\ \phi & \longmapsto \ (Hx \longmapsto \phi(Hx)) \end{cases}$$

is also injective.

Proof. For ϕ in the kernel of the mapping $\mathcal{F}^{(s)}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}^X$, we have

$$\int_{\mathbb{S}^{n-1}} \phi^2(y)\,\mathrm{d}\mu(y) = \frac{1}{|X|}\sum_{x\in X}\phi^2(x) = 0$$

and therefore $\phi = 0$, so that Ev_X is injective. The other claims are straightforward to check. □

Though we will not need it here, observe that an immediate consequence of Lemma 8 is the well-known inequality

$$|X| \geq \dim_{\mathbb{R}}(\mathcal{F}^{(s)}(\mathbb{S}^{n-1})) = \binom{n+s-1}{s} + \binom{n+s-2}{s-1}$$

for any spherical $(2s)$-design. A second observation is that, in the case $-X = X$ is an antipodal spherical $(2s+1)$-design, an analogous lemma shows that the restriction mapping $\mathcal{P}^{(s)}(\mathbb{S}^{n-1}) \longrightarrow \mathbb{R}^Y$ is injective, where $Y \subset X$ is any subset such that $X = Y \cup (-Y)$ and $Y \cap (-Y) = \emptyset$; this in turn implies that

$$|X| = 2|Y| \geq 2\dim_{\mathbb{R}}(\mathcal{P}^{(s)}(\mathbb{R}^n)) = 2\binom{n+s-1}{s}$$

for any antipodal spherical $(2s+1)$-design (see Theorem 5.11 of [9] for a proof *not* using the hypothesis $-X = X$).

Proof of Theorem 1(iv) and Proposition 2. If H is a finite group which is transitive on a spherical $(2s)$-design X, Lemma 8 implies that $\mathcal{F}^{(s)}(\mathbb{S}^{n-1})^H = \mathbb{R}$. Thus H is s-homogeneous by part (i) of Theorem 1.

Proposition 2 follows from the H-equivariance in Lemma 8, and from the fact that the direct sum $\mathcal{F}^{(s)}(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{s} \mathcal{H}^{(k)}(\mathbb{R}^n)$ is H-invariant (indeed $O(n)$-invariant). □

Proof of the last claim of Theorem 1. Let $c(n)$ be a bound for the theorem of Jordan on normal Abelian subgroups of finite linear groups. Let G be a finite subgroup of $O(n)$.

We can choose an Abelian subgroup A of G of index at most $c(n)$. Consider a point $x_0 \in \mathbb{S}^{n-1}$, its orbit $X = Gx_0$, and assume that X is a spherical t-design for some $t \geq 0$. Observe that X is invariant by A and that $|A\backslash X| \leq c(n)$. Thus

$$\dim_{\mathbb{R}}(\mathcal{F}^{([t/2])}(\mathbb{S}^{n-1})^A) \leq |A\backslash X| \leq c(n)$$

by Lemma 8 (here $[t/2]$ denotes the integer part of $t/2$).

Assume now that $n \geq 3$. As the representation of the Abelian group A on $\mathbb{R}^n$ is reducible, there exists a polynomial $f \in \mathcal{H}^{(2)}(\mathbb{R}^n)^A$ which is not zero. More precisely, in appropriate coordinates, we can set

$$f(x_1, \ldots, x_n) = (n-2)(x_1^2 + x_2^2) - 2(x_3^2 + \cdots + x_n^2).$$

As f defines a continuous function on $\mathbb{S}^{n-1}$ which is not constant (and therefore which takes infinitely many values), the only polynomial expression of the form $c_0 + c_1 f + c_2 f^2 + \cdots + c_k f^k$ which is zero on $\mathbb{S}^{n-1}$ is that with $c_0 = c_1 = \cdots = c_k = 0$. In other words, the functions $1, f, f^2, \ldots, f^k$ are linearly independent, and in $\mathcal{F}^{(2k)}(\mathbb{S}^{n-1})^A$. In particular, we have

$$\left[\frac{t}{4}\right] + 1 \leq \dim_{\mathbb{R}}(\mathcal{F}^{([t/2])}(\mathbb{S}^{n-1})^A) \leq c(n)$$

and $t \leq 4c(n) - 1$. □

Remark. In the proof above, we can set

$$c(n) = (\sqrt{8n} + 1)^{2n^2} - (\sqrt{8n} - 1)^{2n^2};$$

see, e.g. [10]. (There in [32] a discussion of the bound in Jordan's theorem using the classification of finite simple groups.) Observe however that this bound holds for the index of a *normal Abelian* subgroup of G, whereas we have only used that A is an *Abelian* subgroup of G.

4. Examples of t-homogeneous groups for $3 \leq t \leq 11$

Let G be a finite subgroup of $O(n)$. The dimensions $h_G^{(k)}$ of the spaces $\mathcal{H}^{(k)}(\mathbb{R}^n)^G$ of G-invariant harmonic polynomials can be computed from the adapted Molien–Poincaré series

$$\sum_{k=0}^{\infty} h_G^{(k)} T^k = \frac{1}{|G|} \sum_{g \in G} \frac{1 - T^2}{\det(1 - gT)}$$

which is an equality between formal power series; see No. V.5.3 in [8], or [24]. Thus, at least in principle, the maximal t for which $G < O(n)$ is t-homogeneous can be found out with computations involving the action of G on $\mathbb{R}^n$ only, not on $\mathcal{H}^{(k)}(\mathbb{R}^n)$ for $k \geq 2$. (Actual computations are however known to be "in general" as complicated as possible: see [15].)

In the case of an irreducible finite group W generated by $l \geq 2$ reflections, this series is of the form

$$\sum_{k=0}^{\infty} h_G^{(k)} T^k = \prod_{i=2}^{l} \frac{1}{1 - T^{m_i+1}}$$

where the increasing sequence $m_1 = 1 < m_2 \leq \cdots \leq m_l$ is that of the *Coxeter exponents* of the Coxeter group W; see No. V.6.2 in [8] or Chapter 3 in [16]. It follows that W is m_2-homogeneous, and that some orbits of W on $\mathbb{S}^{l-1}$ are spherical m_3-designs when $l \geq 3$. In particular, with standard notation for the types of finite Coxeter groups, we have the following list.

$W(A_l) \approx \mathrm{Sym}(l+1)$ is 2-homogeneous for $l \geq 2$, and $m_3 = 3$ for $l \geq 3$.
$W(B_l) \approx (\mathbb{Z}/2\mathbb{Z})^l \rtimes \mathrm{Sym}(l)$ is 3-homogeneous for $l \geq 2$, and $m_3 = 5$ for $l \geq 3$.
$W(D_l) \approx (\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes \mathrm{Sym}(l)$ is 3-homogeneous for $l \geq 4$, and $m_3 = \min\{5, l-1\}$.
$W(I_2^p) \approx (\mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ is $(p-1)$-homogeneous for $p = 5$ and[2] $p \geq 7$.
$W(E_6)$ is 4-homogeneous, and $m_3 = 5$.
$W(E_7)$, $W(F_4)$, $W(H_3)$ are 5-homogeneous, and $m_3 = 7, 7, 9$ respectively.
$W(E_8)$ is 7-homogeneous, and $m_3 = 11$.
$W(H_4)$ is 11-homogeneous, and $m_3 = 19$.

Some root systems are example of exceptional orbits: those of types A_2, D_4, and E_6 are spherical 5-designs which are orbits of t-homogeneous groups for $t = 2, 3, 4$ respectively. (Let us mention that *infinite* Coxeter groups are also relevant to spherical designs [14].)

There are a large number of pairs $G < O(n)$ where $n \geq 3$ and G is a finite subgroup of $O(n)$ with $\pi_G^{(1)}$ and $\pi_G^{(2)}$ irreducible, such that the group generated by G and $\{\pm 1\}$ is 5-homogeneous. Cases with G quasi-simple (i.e. simple modulo its centre) have been classified in [18] (see also [19]). In particular, there exist several infinite families of such examples. There exist also an infinite family and some isolated examples of pairs $G < O(n)$, with $n \geq 3$, such that the finite group G is 7-homogeneous (see [29], in particular Remark 18.10); the groups of the infinite family are the automorphism groups of the so-called *Barnes–Wall lattices*. We know two examples of 11-homogeneous groups, which are $W(H_4) < O(4)$, see above, and the group of automorphisms of the Leech lattice already mentioned in the introduction (known as the Conway group, sometimes denoted by $\cdot 0$, and of which the quotient $Co_1 = \cdot 0/\{\pm 1\}$ is the largest of the three simple Conway groups). In consequence, we will tentatively phrase as follows a quantitative version of the conjecture of Bannai stated in the introduction.

(Q1) *Do there exist infinitely many pairs $G < O(n)$, with $n \geq 3$, such that the finite group G is t-homogeneous for some $t > 7$?*

(Q2) *Do there exist examples $G < O(n)$, with $n \geq 3$, such that the finite group G is t-homogeneous for some $t > 11$?*

[2] Also for $p = 6$, except that I_2^6 is then conventionally written G_2.

5. Analogous questions for lattice designs

Let L be a lattice in $\mathbb{R}^n$ with minimal norm $N(L) = \min\{\langle x \mid x\rangle : x \in L, x \neq 0\}$. Set $\rho = \sqrt{N(L)}$ and let

$$X_L = \{x \in \mathbb{S}^{n-1} : \rho x \in L\}$$

be its set of normalized *short vectors*. A natural question to ask is: for which t is the set X_L a spherical t-design? We report below some information communicated to us by Venkov.

The theory of extreme lattices (namely of lattice sphere packing of highest density) motivates the study of *strongly eutactic* and *strongly perfect* lattices, defined as those for which X_L is respectively a spherical 3-design and a spherical 5-design. (A lattice is *extreme* if the density of the corresponding lattice sphere packing is a local maximum in the space of all lattices of the same rank. It is a theorem of Voronoï that a lattice is extreme if and only if it is "eutactic" and "perfect"; see for example Chapter 4 of [20].) The number of similarity classes of strongly perfect lattice is finite in any dimension, and it is conjectured that there exists at least one in any large enough dimension. There are exactly 10 similarity classes of strongly perfect lattices in dimensions ≤ 11, traditionally denoted by A_1, A_2, D_4, E_6, E_6^*, E_7, E_7^*, E_8, K'_{10}, and K'^*_{10}. For the 29 known similarity classes of strongly perfect lattices in dimensions $12 \leq n \leq 23$, see Tables 19.1 and 19.2 in [29].

Consider a unimodular even integral lattice $L < \mathbb{R}^n$ such that $\langle x \mid x\rangle \geq 4$ for all $x \in L, x \neq 0$. Then X_L is always a spherical 5-design. If $n = 24$, then L is a Leech lattice, and the spherical 11-design X_L appears already above as the orbit of the Conway group $\cdot 0$. Otherwise, $n \geq 32$. For $n = 32$, it is known that there are more than 10^7 nonsimilar lattices of this kind [17]; the group $\mathrm{Aut}(L)$ is far from being transitive on X_L except in a very small number of cases, indeed there are such L with $\mathrm{Aut}(L) = \{\pm 1\}$; in particular, there is a large number of spherical 5-designs in $\mathbb{S}^{31}$ which are *not* simply related to orbits of finite subgroups of $O(32)$.

We have already mentioned the infinite family of Barnes–Wall lattices, for which X_L is a spherical 7-design. There are known lattices for which X_L is a spherical 11-design, such as the Leech latttice in dimension $n = 24$, and three lattices in dimension $n = 48$; for some of this, see [28]. It is an open problem to know if there are such lattices in dimension $n = 72$, and it is conjectured that there are none unless $n \equiv 0 (\mathrm{mod}\ 24)$.

Conversely, it is also possible to define lattices in terms of appropriate spherical designs. More precisely, if $X \subset \mathbb{S}^{n-1}$ is a finite subset linearly generating $\mathbb{R}^n$ such that $\langle x \mid y\rangle \in \mathbb{Q}$ for all $x, y \in X$, and if $\langle X\rangle_{\mathbb{Z}}$ denotes the additive subgroup of $\mathbb{R}^n$ generated by X, then the appropriate homothetic image $L_X = \rho\langle X\rangle_{\mathbb{Z}}$ is an integral lattice in $\mathbb{R}^n$. In particular, let G be a finite subgroup of $O(n) \cap GL(n, \mathbb{Q})$, let $x_0 \in \mathbb{S}^{n-1}$, and let $X = Gx_0$. Then L_X is a G-invariant integral lattice. If G is t-homogeneous, then the homothetic images in $\mathbb{S}^{n-1}$ of all nonempty layers $\{x \in L \mid \langle x \mid x\rangle = r\}, r > 0$, are spherical t-designs.

Consider a lattice $L < \mathbb{R}^n$. Let $x_0 \in L$ be primitive (namely $x_0 \neq 0$, and $\mathbb{R}x_0 \cap L = \mathbb{Z}x_0$). Set $r_0^2 = \langle x_0 \mid x_0\rangle$ and $X = \{x \in \mathbb{S}^{n-1} \mid r_0 x \in L\}$. Then $r_0 L_X$ is obviously contained in L, and the inclusion is in general strict. There are a few exceptional cases such that $r_0 L_X = L$ for all choices of x_0. For example, this is so for the root lattice of type E_8, for the Leech lattice, and for the Thompson–Smith lattice $T < \mathbb{R}^{248}$, of which the automorphism group modulo its centre $\{\pm 1\}$ is the Thompson group (the sporadic simple finite group of order near 9×10^{16}). All layers of T are known to be spherical 7-designs. See [27] and Section 10 in [13].

It seems therefore natural to formulate the two following questions, analogous to those of Section 4.

(Q3) *Do there exist infinitely many similarity classes of lattices $L < \mathbb{R}^n$ such that X_L is a spherical t-design, for $t > 7$?*

(Q4) *Do there exist examples of lattices $L < \mathbb{R}^n$ such that X_L is a spherical t-design, for $t > 11$?*

There is a lattice $L = K'_{21}$ (notation of [29]) in dimension $n = 21$ such that $N(L) = 4$, for which the set of short vectors $\{x \in L \mid \langle x \mid x \rangle = 4\}$ provides a spherical 5-design, but such that some layers $\{x \in L \mid \langle x \mid x \rangle = r\}$ for $r > 4$ provide only spherical 3-designs. (L is the only known strongly perfect lattice of which the dual K'^{*}_{21} is not strongly perfect.) Thus, it also makes sense to ask questions similar to (Q3) and (Q4), involving *all* layers of the lattices, rather than just the layer of short vectors.

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Appendix

The purpose of this Appendix is to present another proof of claims (ii) and (iii) of Theorem 1, which shows the following sharper result.

Theorem A.1. *Let G be a finite subgroup of $O(n)$ and let s be a positive integer.*

(ii) *If $n \geq 3$ and if $\pi_G^{(s)}$ is irreducible, then G is $(2s)$-homogeneous.*

(iii) *If $n \geq 3$, if $\pi_G^{(s)}$ is irreducible, and if $\pi_G^{(s)} \not< \pi_G^{(s+1)}$, then G is $(2s+1)$-homogeneous.*

The main ingredient of the proof below is a family $(\nu_{l,m})_{l,m\geq 0}$ of mappings which can be seen as a variation on the family $(\mu_{l,m})_{l,m\geq 0}$ constructed just before Lemma 5. For a vector space V, we denote by $\mathrm{Sym}^2(V)$ the symmetric square of V; for $u, v \in V$, we denote by $u \otimes_s v$ the element $1/2(u \otimes v + v \otimes u)$ of $\mathrm{Sym}^2(V)$. We define

$$\nu_{l,m} : \begin{cases} \mathcal{H}^{(l)}(\mathbb{R}^n) \otimes \mathcal{H}^{(m)}(\mathbb{R}^n) & \longrightarrow \mathcal{P}^{(l+m)}(\mathbb{R}^n) \\ \phi \otimes \psi & \longmapsto \phi\psi \end{cases}$$

if $l \neq m$, and

$$\nu_{l,l} : \begin{cases} \mathrm{Sym}^2(\mathcal{H}^{(l)}(\mathbb{R}^n)) & \longrightarrow \mathcal{P}^{(2l)}(\mathbb{R}^n) \\ \phi \otimes_s \psi & \longmapsto \phi\psi \end{cases}$$

for $l \geq 0$.

Lemma A.2. *For $n \geq 3$ and for each $l, m \geq 0$, the image of $\nu_{l,m}$ contains $\mathcal{H}^{(i)}(\mathbb{R}^n)$ for every i such that $|l-m| \leq i \leq l+m$ and $i \equiv l+m \pmod 2$.*

Proof. It is known that, for each nonnegative integer k, there exists a unique polynomial $Q^{(k)}(T) = \sum_{i=0}^{[k/2]} q_i^{(k)} T^{k-2i} \in \mathbb{R}[T]$ of degree k, such that, for every $x \in \mathbb{R}^n$, the function $Q_x^{(k)}$ defined by $Q_x^{(k)}(y) := \sum_{i=0}^{[k/2]} q_i^{(k)} \langle x \mid x\rangle^i \langle x \mid y\rangle^{k-2i} \langle y \mid y\rangle^i$ lies in $\mathcal{H}^{(k)}(\mathbb{R}^n)$, and such that

$$\int_{\mathbb{S}} Q_x^{(k)}(y) f(y)\, \mathrm{d}\mu(y) = f(x), \qquad \forall x \in \mathbb{S}^{n-1};$$

see [9]. (For $x \in \mathbb{S}^{n-1}$, observe that $Q_x^{(k)}$ is the unique homogeneous function of degree k such that $Q_x^{(k)}(y) = Q^{(k)}(\langle x \mid y\rangle)$ for every $y \in \mathbb{S}^{n-1}$.) The polynomials $Q^{(k)}(T)$ are related to the usual Gegenbauer (or ultraspherical) polynomials C_k^{λ} by

$$Q^{(k)}(T) = \frac{n+2k-2}{n-2} C_k^{(n-2)/2}(T).$$

There exist constants $q_i^{(l,m)}$ such that

$$Q^{(l)}(T)Q^{(m)}(T) = \sum_{i \geq 0} q_i^{(l,m)} Q^{(i)}(T),$$

with $q_i^{(l,m)} \neq 0$ if and only if $|l-m| \leq i \leq l+m$ and $i \equiv l+m \pmod 2$, when $n > 2$; see for example [33, Formula (5.7)]. Let us choose a point $e \in \mathbb{S}^{n-1}$. We conclude the proof with the following lemma, applied to the group $G = O(n)$, the image W of $\nu_{l,m}$, the vector $v = Q_e^{(l)} Q_e^{(m)}$, and the components $v_i = q_i^{(l,m)} Q_e^{(i)}$. □

Lemma A.3. *Let $V = \bigoplus_{i \in I} V_i$ be a finite direct sum of pairwise inequivalent irreducible representations of a compact group G. Let W be a subrepresentation of V. Let $v = \sum_{i \in I} v_i \in W$ with $v_i \in V_i$. Then, for every $i \in I$ such that $v_i \neq 0$, we have $V_i \subset W$.*

Proof. Let $\pi : G \to GL(V)$ be a representation satisfying the hypotheses of the lemma, and, for every $i \in I$, let χ_i be the character of the subrepresentation V_i. Since the V_i's are pairwise inequivalent, the projection $p_i : V \to V_i$ of kernel $\bigoplus_{j \neq i} V_j$ can be written

$$p_i = \int_G \overline{\chi_i(g)} \pi(g)\, \mathrm{d}g.$$

It follows that $p_i W \subset W$. So we have $v_i = p_i v \in W \cap V_i$. Since V_i is irreducible, we have $V_i \subset W$ whenever $v_i \neq 0$. □

Lemma A.4. *Let V and W be two real finite-dimensional orthogonal representations of a group G. Then*

(i) *V and W are disjoint if and only if $(V \otimes W)^G = \{0\}$;*
(ii) *V is irreducible if and only if $\dim_{\mathbb{R}}(\mathrm{Sym}^2(V)^G) = 1$.*

Proof. We remark first that a real finite-dimensional orthogonal representation is equivalent to its contragredient, so $V \otimes W$ is equivalent to $\mathcal{L}(V, W) = V^* \otimes W$, and

$\mathrm{Sym}^2(V)$ is equivalent to $\mathcal{L}^{sa}(V)$, the space of self-adjoint operators on V. Clearly, V and W are disjoint if and only if $\mathcal{L}(V, W)^G = \{0\}$, if and only if $(V \otimes W)^G = \{0\}$. Also, V is irreducible if and only if $\mathcal{L}^{sa}(V)^G = \mathbb{R}$, if and only if $\dim_{\mathbb{R}}(\mathrm{Sym}^2(V)^G) = 1$. □

Proof of Theorem A.1(ii) and (iii). Let us suppose that $\pi_G^{(s)}$ is irreducible. We have to show that $\mathcal{H}^{(k)}(\mathbb{R}^n)^G = \{0\}$ for $1 \le k \le s$.

By Lemma A.4, $\mathrm{Sym}^2(\mathcal{H}^{(s)}(\mathbb{R}^n))$ is of dimension 1, and, by Lemma A.2, $\bigoplus_{j=0}^{s} \mathcal{H}^{(2j)}(\mathbb{R}^n)^G$ is a subrepresentation of $\mathrm{Sym}^2(\mathcal{H}^{(s)}(\mathbb{R}^n))^G$. Since $\mathcal{H}^{(0)}(\mathbb{R}^n)^G = \mathbb{R}$, we have $\mathcal{H}^{(2j)}(\mathbb{R}^n)^G = \{0\}$ for $1 \le j \le s$.

Since $n \ge 3$, if $k < l$, then $\dim_{\mathbb{R}}(\mathcal{H}^{(k)}(\mathbb{R}^n)) < \dim_{\mathbb{R}}(\mathcal{H}^{(l)}(\mathbb{R}^n))$. Therefore, if $\pi^{(s)}$ is irreducible, $\pi^{(s)}$ and $\pi^{(s-1)}$ are necessarily disjoint. Thus, by Lemma A.4, $(\mathcal{H}^{(s)}(\mathbb{R}^n) \otimes \mathcal{H}^{(s-1)}(\mathbb{R}^n))^G = \{0\}$. Now, by Lemma A.2, $\bigoplus_{j=1}^{s} \mathcal{H}^{(2j-1)}(\mathbb{R}^n)^G$ is a subrepresentation of $(\mathcal{H}^{(s)}(\mathbb{R}^n) \otimes \mathcal{H}^{(s-1)}(\mathbb{R}^n))^G$. Therefore, $\mathcal{H}^{(2j-1)}(\mathbb{R}^n)^G = \{0\}$ for $1 \le j \le s$. This terminates the proof of claim (ii).

Claim (iii) is proven by a similar argument. □

Remark. For $n = 3$ and for every $l \ge 0$, the mapping $\nu_{l,l}$ is a bijection, since the dimensions of its source and of its target are the same. It follows that

$$\mathrm{Sym}^2(\mathcal{H}^{(l)}(\mathbb{R}^n)) \qquad \text{and} \qquad \mathcal{P}^{(2l)}(\mathbb{R}^n) = \bigoplus_{i=0}^{l} \mathcal{H}^{(2i)}(\mathbb{R}^n)$$

are equivalent representations of $O(n)$. As a consequence, the converse of Theorem A.1(ii) does hold for $n = 3$. In particular, for $G < O(n)$, if $\pi_G^{(s)}$ is irreducible, then $\pi_G^{(k)}$ is irreducible for every $k \le s$.

The statement

> *for G a finite (or a compact) subgroup of $O(n)$, if $\pi_G^{(s)}$ is irreducible, then $\pi_G^{(k)}$ is irreducible for every $k \le s$*

is true for $n = 3$ (see above), but is false for $n = 2$. We do not know whether it holds or not for $n \ge 4$.

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