Heterogeneous Optimized Schwarz Methods for Coupling Helmholtz and Laplace Equations

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1 Introduction

Optimized Schwarz methods have increasingly drawn attention over the last two decades because of their improvements in terms of robustness and computational cost compared to the classical Schwarz method. Their optimized transmission conditions have been obtained through analytical or numerical procedures in many different situations, involving mostly the same partial differential equation on each subdomain, see [6, 3, 7] and references therein. When dealing with heterogeneous problems, a domain decomposition approach which allows one to exploit different solvers adapted to the different physical problems is important. Due to their favorable convergence properties in the absence of overlap, and their capability to take physical properties at the interfaces into account, optimized Schwarz methods are a natural framework for such heterogeneous domain decomposition methods, where the spatial decomposition is simply provided by the multi-physics of the problem.

We introduce and analyze here heterogeneous optimized Schwarz methods with zeroth order optimized transmission conditions for the coupling between the hard to solve Helmholtz equation [5] and the Laplace equation. It is a simplified instance of the coupling of parabolic and hyperbolic operators, which might arise in Maxwell equations. The Helmholtz equation is used in the time harmonic regime of a wave equation and the Laplace operator represents the parabolic part. We consider a bounded domain $\Omega \subset \mathbb{R}^2$, with sufficiently regular boundary, divided into two subdomains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$, and $\Sigma_j = \partial\Omega_j \setminus \Gamma$. Our model problem is

$$\begin{aligned} (-\Delta - q\omega^2)u &= f \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + i\omega u &= 0 \qquad \text{on } \Sigma_1, \end{aligned} \tag{1}$$

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$$u = 0 \quad \text{on } \Sigma_2,$$

where $\omega > 0$ is the Helmholtz frequency, and $q \in L^\infty(\Omega)$ satisfies $q = 1$ in Ω_1 and $q = 0$ in Ω_2. Since the well-posedness of the problem is not straightforward due to the indefinite nature of the Helmholtz part, we first analyze it in more detail adapting arguments presented by Després in [2].

Lemma 1. *The norm $||u||^2 = \int_\Omega |\nabla u|^2 + \omega \int_{\Sigma_1} |u|^2$ is equivalent to the canonical norm on $H^1(\Omega)$ if $|\Sigma_1| > 0$.*

Proof. We first observe that $H^1(\Omega)$ is the direct sum of $\bar{V} = \left\{v \in H^1(\Omega) : \int_\Omega v = 0\right\}$ and $\tilde{V} = \left\{v \in H^1(\Omega) : v \text{ is constant in } \Omega\right\}$, $H^1(\Omega) = \tilde{V} \oplus \bar{V}$. Then, on the one hand, it easy to see that for all $v \in \tilde{V}$, there exist a constant $C = \sqrt{\frac{\omega|\Sigma_1|}{|\Omega|}}$ such that

$$C||v||_{H^1(\Omega)} \leq ||v|| \leq C||v||_{H^1(\Omega)}. \tag{2}$$

On the other hand, for every $v \in \bar{V}$, we first use the Poincaré inequality to get

$$||v||^2_{H^1(\Omega)} \leq (1+C)\int_\Omega |\nabla v|^2 \leq (1+C)\left(\int_\Omega |\nabla v|^2 + \omega \int_{\Sigma_1} |v|^2\right) = (1+C)||v||^2. \tag{3}$$

Exploiting the continuity of the trace operator, we obtain

$$||v||^2 = \int_\Omega |\nabla v|^2 + \omega \int_{\Sigma_1} |v|^2 \leq \int_\Omega |\nabla v|^2 + \omega \int_{\partial\Omega} |v|^2 \leq \max(1, C_{\partial\Omega}\omega)(\int_\Omega |\nabla v|^2 + \int_\Omega |v|^2). \tag{4}$$

Having proved that the two norms are equivalent on the subspaces $\bar{V}$ and $\tilde{V}$ with $\tilde{V} \oplus \bar{V} = H^1(\Omega)$, the two norms are also equivalent on $H^1(\Omega)$.

Let us define $V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Sigma_2\}$, with $||\cdot||_V = ||\cdot||_{H^1(\Omega)}$, and consider problem (1) in the variational form

$$\text{Find } u \in V : \ a(u,v) - b(u,v) =_{V^{-1}} \langle f, v\rangle_V \quad \forall v \in V, \tag{5}$$

where $a(u,v) = \int_\Omega \nabla u \nabla \bar{v} + i\omega \int_{\Sigma_1} u\bar{v}$, $b(u,v) = \omega^2 \int_{\Omega_2} u\bar{v}$ and $f \in V^{-1}$. To use Fredholm theory, we now show that the bilinear form b is a compact pertubation of a.

Lemma 2. *Let $\mathscr{B}$ be an operator from V to V such that*

$$a(\mathscr{B}u, v) = b(u,v) \quad \forall v \in V, \tag{6}$$

then $\mathscr{B}$ is a continuous compact operator.

Proof. We first prove continuity, i.e. $\exists C > 0 : \forall u \in V, ||\mathscr{B}u||_V \leq C||u||_V$. From the definition of $\mathscr{B}$, and applying Lax-Milgram to (6), we have $||\mathscr{B}u||_V \leq \frac{1}{\alpha}||b(u)||_{V^{-1}}$, where $b(u) : V \to \mathbb{R}$ is the functional defined by $_{V^{-1}} < b(u), v >_V := b(u,v)$. Then we have $\forall v \in V$

$$|_{V^{-1}} < b(u), v >_V | := |b(u,v)| = \omega^2 | \int_{\Omega_2} u\bar{v}| \leq \omega^2 ||u||_{L^2(\Omega_2)} ||v||_{L^2(\Omega_2)} \leq \omega^2 ||u||_{L^2(\Omega_2)} ||v||_V.$$

We thus conclude that $||b(u)||_{V^{-1}} \leq \omega^2 ||u||_{L^2(\Omega_2)}$, and hence have the bound

$$||\mathscr{B}u||_V \leq \frac{1}{\alpha}\omega^2 ||u||_V.$$

To prove compactness, let u_n be a bounded sequence in V, i.e $\exists C > 0 : \forall n, ||u_n||_V < C$. From weak compactness of V it follows that there exists a subsequence u_{n_j} such that $u_{n_j} \rightharpoonup u$ for some u. Hence u_{n_j} converge strongly to u in $L^2(\Omega)$. Considering $a(\mathscr{B}u_{n_j} - \mathscr{B}u, \mathscr{B}u_{n_j} - \mathscr{B}u) = b(u_{n_j} - u, \mathscr{B}u_{n_j} - \mathscr{B}u)$ we have letting $n \to \infty$ and using the Cauchy-Schwarz inequality

$$\left| \int_\Omega |\nabla(\mathscr{B}u_{n_j} - \mathscr{B}u)|^2 + i\omega \int_{\Sigma_1} |\mathscr{B}u_{n_j} - \mathscr{B}u|^2 \right| \leq \omega^2 ||u_{n_j} - u||_{L^2(\Omega_2)} ||\mathscr{B}u_{n_j} - \mathscr{B}u||_{L^2(\Omega_2)}. \tag{7}$$

We observe that $\mathscr{B}u_{n_j} \rightharpoonup \mathscr{B}u$ in V because $u_{n_j} \rightharpoonup u$ in V and $\mathscr{B}$ is a continuous operator [1]. Hence, both u_{n_j} and $\mathscr{B}u_{n_j}$ converge strongly in $L^2(\Omega)$. In particular we have that $a(\mathscr{B}u_{n_j} - \mathscr{B}u, \mathscr{B}u_{n_j} - \mathscr{B}u) \to 0$ which implies $||\mathscr{B}u_{n_j} - \mathscr{B}u|| \to 0$. With Lemma 1, we have that $\mathscr{B}u_{n_j} \to \mathscr{B}u$ in V and thus $\mathscr{B}$ is a compact operator.

Since $\mathscr{B}$ is a compact operator, thanks to Fredholm alternative, existence of the solution of problem (5) follows from uniqueness. We need two further Lemmas to prove uniqueness. We denote with $\gamma_j u$ and $S_j u$ the trace of u and the trace of the normal derivative on the j-th interface and we introduce the space $E(\Omega,\Delta) := \{u \in H^1(\Omega) : -\Delta u \in L^2(\Omega)\}$.

Lemma 3 (Grisvard, Theorem 1.5.3.11, page 61, [9]). *Let Ω be an open bounded subset of $\mathbb{R}^2$ whose boundary is a curvilinear polygon of class $C^{1,1}$ with interfaces $\Sigma_j, j = 1,..,N$. The mappings $u \to \gamma_j u$ and $u \to S_j u$ have a unique continuous extension from $E(\Omega,\Delta)$ to respectively $H^{\frac{1}{2}}(\Sigma_j)$ and $H^{-\frac{1}{2}}(\Sigma_j)$. Moreover for every $u \in E(\Omega,\Delta)$ and $v \in H^1(\Omega)$ with $\gamma_j v \in H^{\frac{1}{2}}(\Sigma_j)\ \forall j$, the Green's formula holds:*

$$(-\Delta u, v) = (\nabla u, \nabla v) - \sum_{j=1}^N \langle S_j u, \overline{\gamma_j v}\rangle. \tag{8}$$

Lemma 4 (Després, Corollary 2.1, page 22, [2]). *Let Ω be an open bounded arc-connected subset of $\mathbb{R}^2$ and assume that Γ is a nonempty open subset of $\partial\Omega$ of class C^2 and $q \in L^\infty(\Omega)$. If $u \in H^2(\Omega)$ satisfies*

$$(-\Delta - q\omega^2)u = 0 \text{ on } \Omega, \quad u|_\Gamma = \partial_n u|_\Gamma = 0, \tag{9}$$

then u=0 in Ω.

Theorem 1. *Under the hypotheses of Lemmas 3 and 4, $u \equiv 0$ is the only solution of the boundary value problem (1) with $f = 0$.*

Proof. Choosing $v \in D(\Omega)$, the space of $C^\infty(\Omega)$ functions with compact support, in the weak formulation of eq. (1) we obtain $-\Delta u - q\omega^2 u = 0$. Hence, since $u \in V$, $\Delta u \in L^2(\Omega)$ and $u \in E(\Omega, \Delta)$. Exploiting Green's formula and choosing $v = u$ we get

$$\int_\Omega |\nabla u|^2 - \omega^2 \int_{\Omega_1} |u|^2 + i\omega \int_{\Sigma_1} |u|^2 = 0. \tag{10}$$

Considering the imaginary part we have $\int_{\Sigma_1} |u|^2 = 0$, which implies $u = 0$ on Σ_1. We now have homogeneous Dirichlet data on the whole domain $\partial\Omega = \Sigma_1 \cup \Sigma_2$. Regularity results for Dirichlet problems in smooth domains state that $u \in H^2(\Omega)$. Exploiting again the Green's formula and $-\Delta u - q\omega^2 u = 0$ in Ω , we obtain

$$_{H^{-\frac{1}{2}}(\Sigma_1)}\langle \frac{\partial u}{\partial n}, v\rangle_{H^{\frac{1}{2}}(\Sigma_1)} + iw \int_{\Sigma_1} uv = 0. \tag{11}$$

Since $u = 0$ on Σ_1, we can conclude that $\partial_n u = 0$ on Σ_1 and by the unique continuation principle in Lemma 4, the result follows.

2 Heterogeneous Optimized Schwarz methods

In order to make analytical calculations, we simplify the analysis and set $\Omega = \mathbb{R}^2$, with Ω_1 being the left half plane and Ω_2 the right half plane. The heterogeneous optimized Schwarz method is given by

$$\begin{aligned} (-\omega^2 - \Delta)u_1 &= f \text{ in } \Omega_1, \quad (\partial_x + S_1)(u_1^n)(0,\cdot) = (\partial_x + S_1)(u_2^{n-1})(0,\cdot), \\ -\Delta u_2 &= f \text{ in } \Omega_2, \quad (\partial_x + S_2)(u_2^n)(0,\cdot) = (\partial_x + S_2)(u_1^{n-1})(0,\cdot), \end{aligned} \tag{12}$$

where the S_j, $j = 1,2$ are linear operators along the interface in the y direction. The system is closed by the Sommerfeld radiation condition $\lim_{x\to-\infty} \sqrt{|x|}\frac{x}{|x|}(\partial_x u_1^n - i\omega u_1^n) = 0$ and by the boundedness condition $\lim_{x\to+\infty} u_2^n = 0$. The goal is to find which operators lead to the fastest convergence. We define the errors $e^j := u - u^j$, and taking the Fourier transform of the error equations in the y direction, we obtain

$$\begin{aligned} (-\omega^2 - \partial_{xx} + k^2)(\hat{e}_1^n) &= 0 & k \in \mathbb{R},\ x < 0, \\ (\partial_x + \sigma_1(k))(\hat{e}_1^n)(0,k) &= (\partial_x + \sigma_1(k))(\hat{e}_2^{n-1})(0,k), & k \in \mathbb{R}, \\ (-\partial_{xx} + k^2)(\hat{e}_2^n) &= 0 & k \in \mathbb{R},\ x > 0, \\ (\partial_x + \sigma_2(k))(\hat{e}_2^n)(0,k) &= (\partial_x + \sigma_2(k))(\hat{e}_1^{n-1})(0,k), & k \in \mathbb{R}, \end{aligned} \tag{13}$$

where $\sigma_j(k)$ are the Fourier symbols of the operators S_j. Solving the equations in (13) and imposing the radiation/boundedness conditions, we get

$$\hat{e}_1^n = \hat{e}_1^n(0,k)e^{\lambda(k)x}, \quad \hat{e}_2^n = \hat{e}_2^n(0,k)e^{-|k|x},$$

where $\lambda(k) := i\sqrt{\omega^2 - k^2}$ if $k < \omega$ and $\lambda(k) := \sqrt{k^2 - \omega^2}$ if $k \geq \omega$. Applying the transmission conditions, it follows that

$$\hat{e}_1^n = \rho(k)\hat{e}_1^{n-2}, \qquad \hat{e}_2^n = \rho(k)\hat{e}_2^{n-2},$$

where

$$\rho(k) = \frac{-|k| + \sigma_1(k)}{\lambda(k) + \sigma_1(k)} \frac{\lambda(k) + \sigma_2(k)}{-|k| + \sigma_2(k)}.$$

Next, to approximate the optimal choice for $\sigma_1(k)$ and $\sigma_2(k)$ which would require non local operators, we set $\sigma_1 = -\sigma_2 = p(1+i)$. This choice is motivated by [4] where the single and double sided optimizations were studied and compared for the time harmonic Maxwell equations. Since both σ_j and $\lambda(k)$ contain complex numbers, we have to study the modulus of the convergence factor,

$$|\rho(k,p)|^2 = \begin{cases} \dfrac{((k-p)^2+p^2)}{((k+p)^2+p^2)} \dfrac{((\sqrt{k^2-\omega^2}-p)^2+p^2)}{((\sqrt{k^2-\omega^2}+p)^2+p^2)} & k \geq \omega, \\ \dfrac{((k-p)^2+p^2)}{((k+p)^2+p^2)} \dfrac{((\sqrt{\omega^2-k^2}-p)^2+p^2)}{((\sqrt{\omega^2-k^2}+p)^2+p^2)} & k < \omega. \end{cases} \tag{14}$$

Since we are interested in minimizing the convergence factor over all relevant numerically represented frequencies, we study now the minimax problem

$$\min_{p \geq 0} \max_{k \in [k_{\min}, k_{\max}]} |\rho(k,p)|^2, \tag{15}$$

where $k_{\min}$ is the minimum frequency and $k_{\max}$ is the maximum frequency supported by the numerical grid.

Theorem 2. *Assuming that $k_{\max} > 2\omega$, the solution of the minimax problem (15) is given by $p^* = \frac{\omega}{\sqrt{2}}$ if $|\rho(k_{\max}, p^* = \frac{\omega}{\sqrt{2}})|^2 \leq \frac{(\sqrt{2}-1)^2+1}{(\sqrt{2}+1)^2+1}$, and otherwise it is given by the unique p^* such that $|\rho(k=\omega, p^*)|^2 = |\rho(k_{\max}, p^*)|^2$.*

Proof. We consider $p > 0$, because for $p = 0$ the convergence factor is equal to 1, and for $p < 0$ it is greater than one, while for values of $p > 0$, the convergence factor is always less than 1. We introduce a change of variables which will be useful in the computations, namely $x = \sqrt{k^2-\omega^2}$ if $k \geq \omega$ and $x = \sqrt{\omega^2-k^2}$ for $k < \omega$. Problem (15) then becomes

$$\min_{p>0} \max \left(\max_{[0,\sqrt{\omega^2-k_{\min}^2}]} G(x,p), \max_{[0,\sqrt{k_{\max}^2-\omega^2}]} F(x,p) \right), \tag{16}$$

where

$$G(x,p) = \frac{((x-p)^2+p^2)}{((x+p)^2+p^2)} \frac{((\sqrt{\omega^2-x^2}-p)^2+p^2)}{((\sqrt{\omega^2-x^2}+p)^2+p^2)},$$

$$F(x,p) = \frac{((x-p)^2+p^2)}{((x+p)^2+p^2)} \frac{((\sqrt{x^2+\omega^2}-p)^2+p^2)}{((\sqrt{x^2+\omega^2}+p)^2+p^2)}.$$

First, we observe that $\frac{\partial G}{\partial x}_{|x=0} = \frac{\partial F}{\partial x}_{|x=0} = -\frac{(2((\omega-p)^2+p^2))}{(p((\omega+p)^2+p^2))} < 0$ for all $p > 0$ and $G(0,p) = F(0,p)$. Indeed, $x = 0$ ($k = \omega$) is a cusp for $\rho^2(k,p)$ and hence it is a local maximum which needs to be minimized. The minimum of $G(0,p)$ with respect to the variable p is given by $\bar{p} = \frac{\omega}{\sqrt{2}}$ and $G(x=0, p=\frac{\omega}{\sqrt{2}}) = \frac{(\sqrt{2}-1)^2+1}{(\sqrt{2}+1)^2+1} \approx 0.176$. We thus have found a lower bound for the value of the minimax problem. Next, we study how $G(x,p)$ behaves in the rest of the interval, and start by restricting our attention to the case $p \geq \bar{p}$. Computing the partial derivative with respect to x of $G(x,p)$, we find that it has a unique zero x_1 given by the root of the non linear equation

$$x(4p^4+x^4)(2p^2+x^2-\omega^2) = ((\omega^2-x^2)^2+4p^2)(2p^2-x^2)\sqrt{\omega^2-x^2}. \qquad (17)$$

To proof uniqueness, it is enough to notice that the LHS is zero for $x = 0$ and strictly increasing on x, if $p \geq \bar{p}$, while the RHS is greater than zero for $x = 0$ and strictly decreasing in x. Therefore $G(x,p)$ decreases until $x < x_1$ and then increases monotonically. If $x_1 > \sqrt{\omega^2-k_{\min}^2}$ then the $\max_{[0,\sqrt{\omega^2-k_{\min}^2}]} G(x,p) = G(0,p)$, otherwise if $x_1 \leq \sqrt{\omega^2-k_{\min}^2}$ it is sufficient to notice that $G(\sqrt{\omega^2-k_{\min}^2},p) < G(\omega,p) = G(0,p)$, to conclude that it holds again $\max_{[0,\sqrt{\omega^2-k_{\min}^2}]} G(x,p) = G(0,p)$. Next we focus on the second interval, considering the function $F(x,p)$. The zeros of the derivative $\frac{\partial F}{\partial x}$ are given by the zeros of the equation

$$x(4p^2+x^4)(2^2+x^2-2p^2) = (2p^2-x^2)((\omega^2+x^2)^2+4p^2)\sqrt{\omega^2+x^2}.$$

Repeating an argument similar to the one above, we find that again there is a unique zero x_2, in this case $\forall p > 0$, which again might or might not belong to the interval $[0,\sqrt{k_{\max}^2-\omega^2}]$. If x_2 is outside the interval or $F(\sqrt{k_{\max}^2-\omega^2},\bar{p}) \leq F(0,\bar{p})$, then we can conclude that the optimal value p^* is given by $p^* = \bar{p}$, i.e. the value which minimizes the convergence factor for the frequency $k = \omega$. Otherwise the local maxima are located at $x = 0$ and $x = \sqrt{k_{\max}^2-\omega^2}$. We compute the partial derivative w.r.t the variable p, which satisfies $\frac{\partial F}{\partial p}|_{x=\sqrt{k_{\max}^2-\omega^2}} < 0$ for $p \in \mathbb{I} = [0,\sqrt{\frac{k_{\max}^2-\omega^2}{2}}]$, and under the non restrictive hypothesis $k_{\max} > 2\omega$, we have that $\bar{p} \in \mathbb{I}$. Analyzing the sign of the derivative shows that it is not useful to look for p^* in $[0,\frac{\omega}{\sqrt{2}}]$, since both local maxima would increase. This justifies why we studied G only for $p \geq \bar{p}$. Since $\frac{\partial F}{\partial p}|_{x=0} > 0$ for $p > \frac{\omega}{\sqrt{2}}$ and because

$$F(\sqrt{k_{\max}^2-\omega^2},\sqrt{\frac{k_{\max}^2-\omega^2}{2}}) = \left(\frac{(\sqrt{2}-1)^2+1}{(\sqrt{2}+1)^2+1}\right)^2 < F(0,\frac{\omega}{\sqrt{2}}) < F(0,\sqrt{\frac{k_{\max}^2-\omega^2}{2}}), \qquad (18)$$

we conclude that there exists a unique value p^* such that $F(0,p^*) = F(\sqrt{k_{\max}^2-\omega^2},p^*)$, which concludes the proof.

Remark 1. It is interesting to note that this problem is different from the ones already studied in the literature, because the convergence factor is immediately bounded from below: it is not possible to get a better convergence factor than $\rho^2(k,p) = \frac{(\sqrt{2}-1)^2+1}{(\sqrt{2}+1)^2+1}$. We also did not have to exclude the resonnance frequency $k=\omega$ by introducing ω_- and ω_+, as in the Helmholtz case [8]; the optimized Schwarz method can benefit from the heterogeneity, leading to $|\rho(k=\omega,p)|^2 < 1$.

We now present two asymptotic results. First we want to study how our algorithm behaves when we take finer and finer meshes. Let $h \to 0$, h being the mesh size, and suppose that the maximum frequency supported by the numerical grid scales like $k_{\max} = \pi/h \to \infty$.

Theorem 3. *When the physical parameters ω and $k_{\min}$ are fixed, $k_{\max} = \frac{\pi}{h}$ and $h \to 0$, then the solution of problem (15) is given by*

$$p^* = \frac{\sqrt{\omega\pi}}{2} \cdot h^{-1/2} + o(h^{-1/2}), \quad |\rho(k,p^*)|^2 = 1 - \frac{4\sqrt{\omega}}{\sqrt{\pi}} h^{\frac{1}{2}} + o(h^{1/2}). \tag{19}$$

Proof. For $k_{\max} \to \infty$, $\rho(k_{\max},p) \to 1$, and hence the solution of the minimax problem is given by equioscillation. Inserting the ansatz $p \approx C_p h^{-\alpha}$ into $|\rho(k=\omega,p)|^2 = |\rho(k=k_{\max},p)|^2$ and comparing the leading order terms then gives the result.

The second result is typical of the Helmholtz equation. As ω increases, in order to control the so called pollution effect, we need to decrease significantly h in order to have a good approximation of the solution. Generally, the scaling relation used is $h = \frac{C_h}{\omega^\gamma}$, with $\gamma > 1$. Common values are $\gamma = \frac{3}{2}$, or $\gamma = 2$.

Theorem 4. *If $k_{\min}$ is fixed, $k_{\max} = \frac{\pi}{h}$, ω goes to infinity and $h = \frac{C_h}{\omega^\gamma}$, with $\gamma > 1$, then the solution of problem (15) is given by*

$$p^* = \frac{\sqrt{\pi}}{2\sqrt{C_h}} \cdot \omega^{\frac{1+\gamma}{2}} + o(\omega^{\frac{1+\gamma}{2}}), \quad |\rho(k,p^*)|^2 = 1 - \frac{4\sqrt{C_h}}{\sqrt{\pi}} \omega^{\frac{1-\gamma}{2}} + o(\omega^{\frac{1-\gamma}{2}}).$$

Proof. A direct calculation shows that $|\rho(k=k_{\max},\frac{\omega}{\sqrt{2}})|^2 \to 1$ for $\omega \to \infty$, and thus again the solution is given by equioscillation. Expanding equation $|\rho(k=\omega,p)|^2 = |\rho(k=k_{\max},p)|^2$, with the ansatz $p = C_p \omega^\alpha$ then leads to the desired result.

3 Numerical experiments

We implemented our heterogeneous optimized Schwarz method on a square domain $\Omega := (-1,1)\times(-1,1)$, with $\Omega_1 := (-1,0)\times(-1,1)$ and $\Omega_2 := (0,1)\times(-1,1)$. We used second order centered finite differences for the interior points and first order approximations for the boundary terms. In Figure 1 on the left, we show the modulus of the solution of problem (1) for $\omega^2 = 50$ and $f = 1$. On the right in Figure 1, we show a comparison between the optimal numerical value p and the theoretical estimation provided by Theorem 2. We see that our simplified analysis

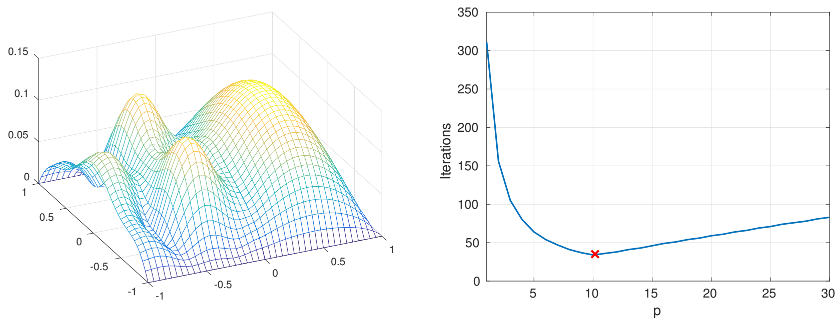

Fig. 1 Parameters $\omega^2 = 50$, $h = 0.05$. Left: Modulus of $u(x,y)$. Right: Parameter p vs number of iterations. The optimal p given by equioscillation is indicated by a star.

on unbounded domains is able to give quite a good approximation of the optimal parameter in the bounded domain context. Finally, we show in Table 1 the behavior of the algorithm when the mesh size h decreases and for large values of ω, with $h\omega^{\frac{3}{2}} =$ const. In brackets, we show the number of iterations required for a non-

h	Optimal p^*	$\max_k \|\rho^2(p^*,k)\|$	iterations
$\frac{1}{50}$	16.52	0.4225	53 (810)
$\frac{1}{100}$	23.53	0.55043	73 (1614)
$\frac{1}{200}$	33.37	0.6543	104 (3284)
$\frac{1}{400}$	47.27	0.7403	148 (6554)

ω	Optimal p^*	$\max_k \|\rho^2(p^*,k)\|$	iterations
10π	34.8451	0.2119	31 (839)
20π	84.7084	0.2622	38 (2954)
40π	205.0570	0.3167	46 (8096)
60π	342.6739	0.3506	48 (>10000)

Table 1 The two tables show the behaviour of the heterogeneous optimized Schwarz method under mesh refinement and when ω increases with $h\omega^{\frac{3}{2}}$ held constant.

optimized case, i.e. using $p = 1$. We clearly see that the optimization leads to a much better algorithm, which deteriorates much more slowly when the mesh is refined, and ω increases.

4 Conclusions

We presented and analysed a heterogeneous optimized Schwarz method for the coupling of Helmholtz and Laplace equations. We proved the well-possedness of the coupled problem, and then introduced optimized Robin transmission conditions, giving asymptotic formulas for the optimized parameters and associated convergence factor. Our results indicate that a much weaker dependence on the mesh parameter can be achieved with optimized transmission conditions, and we are currently working on further improvement by studying second order optimized transmission conditions.

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References

1. Ciarlet, P.: Linear and Nonlinear Functional Analysis with Applications:. Applied mathematics. Society for Industrial and Applied Mathematics (2013)
2. Després, B.: Méthodes de décomposition de domaine pour les problèmes de propagation d'ondes en régimes harmoniques. Ph.D. thesis, Université Dauphine-Parix IX (1991)
3. Dolean, V., Gander, M.J., Gerardo-Giorda, L.: Optimized Schwarz methods for Maxwell's equations. SIAM Journal on Scientific Computing **31**(3), 2193–2213 (2009)
4. El Bouajaji, M., Dolean, V., Gander, M.J., Lanteri, S.: Comparison of a one and two parameter family of transmission conditions for maxwells equations with damping. In: Domain Decomposition Methods in Science and Engineering XX, pp. 271–278. Springer (2013)
5. Ernst, O.G., Gander, M.J.: Why it is difficult to solve Helmholtz problems with classical iterative methods. In: Numerical analysis of multiscale problems, pp. 325–363. Springer Berlin Heidelberg (2012)
6. Gander, M.J.: Optimized Schwarz methods. SIAM Journal on Numerical Analysis **44**(2), 699–731 (2006)
7. Gander, M.J., Halpern, L., Magoules, F.: An optimized Schwarz method with two-sided Robin transmission conditions for the Helmholtz equation. International journal for numerical methods in fluids **55**(2), 163–175 (2007)
8. Gander, M.J., Magoules, F., Nataf, F.: Optimized Schwarz methods without overlap for the Helmholtz equation. SIAM Journal on Scientific Computing **24**(1), 38–60 (2002)
9. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (1985)