Fock Weight Dimers, Their Surface Tension And Some Computation

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Based on joint work with Alexander Bobenko and Yuri Suris [BBS '23+].

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Contents

The Dimer Model

- Planar finite bipartite graph G with weights K_{wb} on edges. Pick random perfect matching M with

$$\mathbb{P}(M) = \frac{1}{Z} \prod_{wb \in M} K_{wb}.$$

- Gauge transformation: Multiplying all weights incident to one vertex with C does not change the measure. Face weights

$$W_f = \prod_{i=1}^{n} \frac{K_{w_i b_i}}{K_{w_{i+1} b_i}}$$

unchanged.

- If $\mathrm{sign}(W_f) = (-1)^{n+1}\ \forall f$ (Kasteleyn condition) then [Kasteleyn'67]

$$Z = |det(K)|.$$

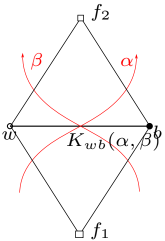
- Determinantal process: [Kenyon '97]

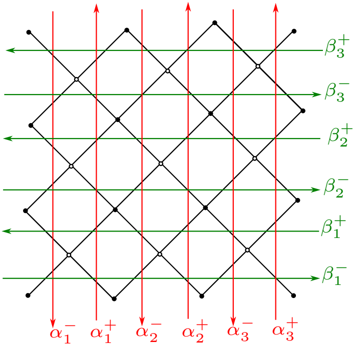
$$\mathbb{P}(w_1 b_1, \dots, w_n b_n) = \prod K_{w_i b_i} \det\left(K^{-1}_{b_i w_j}\right)_{1 \le i,j \le n}$$

The Isoradial Case

- We will consider fundamental domain with repeating train tracks.
- Kenyon's critical weights: $\alpha, \beta \in S^1$ and

$$K_{bw} = f_2 - f_1.$$

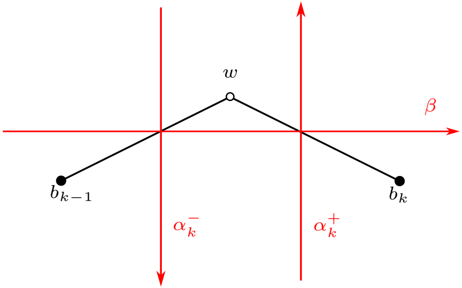




The Isoradial Case

One way to get them through an inverse approach:
Define functions $\psi_b(P)$ with $P \in \hat{\mathbb{C}}$ on black vertices such that

$$\frac{\psi_{b_k}(P)}{\psi_{b_{k-1}}(P)} = \frac{P - \alpha_k^-}{P - \alpha_k^+}$$

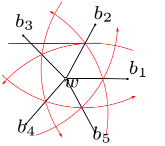


The Isoradial Case

Theorem (Kenyon '02)

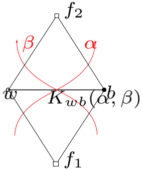
ψ_{b_k} around a white vertex are linearly dependent with

$$\sum_{k=1}^{n}(\alpha_{k-1}-\alpha_k)\psi_{b_k}(P)=0.$$

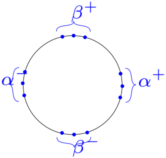


Thus $K_{wb}=\alpha_{k-1}-\alpha_k=f_2-f_1$ isoradial weights.
Face weights are then cross ratios

$$W_f=\frac{\alpha_1-\alpha_2}{\alpha_3-\alpha_1}\frac{\alpha_3-\alpha_4}{\alpha_1-\alpha_4}=(-1)^n\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$$



K Kasteleyn if $\alpha_1<\alpha_2<\alpha_3<\alpha_4<\alpha_1$ around each face.



Thus we have a set of functions $\psi : \mathbb{C}^B$ in the kernel of K parametrized by $P \in \mathbb{C}$. Their monodromies

$$z(P) = \frac{\psi_{(1,0)}(P)}{\psi_{(0,0)}(P)} = \prod_{k=1}^{d} \frac{P - \alpha_k^-}{P - \alpha_k^+}, \quad w(P) = \frac{\psi_{(0,1)}(P)}{\psi_{(0,0)}(P)} = \prod_{k=1}^{d} \frac{P - \beta_k^-}{P - \beta_k^+}$$

lie on the spectral curve $\mathcal{C} = \{\mathcal{P}(z, w) = \det(K(z(P), w(P))) = 0\}$ which is a Harnack curve [Kenyon-Okounkov-Sheffield '03] of geometric genus 0 [Kenyon-Okounkov '03].

Periodic Graphs And Weights

$\mathcal{P}$ encodes a lot of information about the scaling limit.
The Ronkin function

$$R(X,Y) = \frac{1}{(2\pi i)^2} \int_{|z|=e^X} \int_{|w|=e^Y} \log(\mathcal{P}(z,w)) \frac{dz}{z} \frac{dw}{w}$$

gives us the free energy

$$\log(Z) := \lim_{n\to\infty} \frac{1}{n^2} \log(Z(G_n)) = R(0,0),$$

the Amoeba map

$$\mathcal{A}(z,w) = (\log(|z|), \log(|w|))$$

gives the phase diagram, and the Legendre dual

$$\sigma(s,t) = R(X,Y) - sX - tY$$

is the surface tension.

Higher Genus

Now we want to generalize this to some compact Riemann surface Γ.
Weights first defined by Fock '15. Construction and connection to dimers based on Boutillier, Cimasoni, de Tilière 2020, 2022.

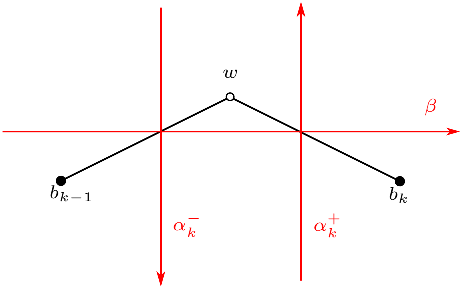
- Let Γ be a compact Riemann surface with a fixed homology basis $a_1, \dots, a_g$, $b_1, \dots, b_g$.
- $\omega = (\omega_1, \dots, \omega_g)$ the set of dual holomorphic differentials normalized to periods $\int_{a_j} \omega_k = \delta_{jk}$, $\int_{b_j} \omega_k = B_{jk}$ symmetric with $\mathrm{Im}(B)$ positive definite.
- $J(\Gamma) = \mathbb{C}^g / (\mathbb{Z}^g + B\mathbb{Z}^g)$ the Jacobi variety of Γ;
- $A : \Gamma \to J(\Gamma)$, $P \mapsto A(P) = \int_{P_0}^{P} \boldsymbol{\omega} \pmod{\mathbb{Z}^g + B\mathbb{Z}^g}$ the Abel map.
- The theta function is $\theta(z|B) = \sum_{m \in \mathbb{Z}^g} \exp\left(\pi i \langle Bm, m\rangle + 2\pi i \langle z, m\rangle\right)$.
- Discrete Abel map on vertices: $\eta(v)$ picks up $-A(\alpha)$ whenever crossing a train track with α parameter pointing to the left and $+A(\alpha)$ if to the right.

Higher Genus

Proposition (Classical)

$$\frac{\psi_{b_k}(P)}{\psi_{b_{k-1}}(P)} = \frac{\theta(A(P)+\eta(b_k)+Z)}{\theta(A(P)+\eta(b_{k-1})+Z)} \frac{E(P,\alpha_k^-)}{E(P,\alpha_k^+)}$$

is a meromorphic single valued function with a zero in α_k^-*, pole in* α_k^+*.*



Theorem (Fock'15)

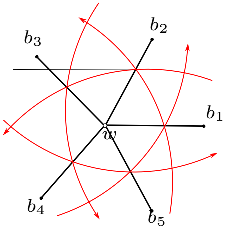
Around a white star we have

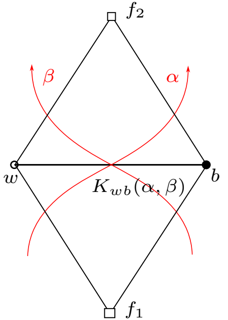
$$\sum_{k=1}^{n} K_{wb_k}(\alpha_{k-1}, \alpha_k)\psi_k(P) = 0,$$

where

$$K_{wb_k}(\alpha_{k-1}, \alpha_k) = \frac{E(\alpha_{k-1}, \alpha_k)}{\theta\big(\eta(f_{k-1}) + Z\big)\theta\big(\eta(f_k) + Z\big)}$$

- Proof via residues
- Case of $\deg(w) = 3$ is equivalent to *Fay's trisecant identity*.





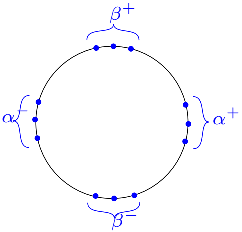
Kasteleyn condition from Harnack data

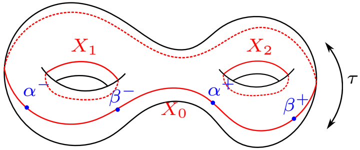
We call $(\Gamma, \{\alpha\})$ Harnack data if Γ M-curve and angles lie on a single real oval X_0 and are in cyclic order around every face.

Theorem (B,C,dT '20,'22)

For Harnack data $(\Gamma, \{\alpha\})$ the weights K_{bw} are Kasteleyn.

$$W_f = \frac{\theta(\eta(f_e)+Z)}{\theta(\eta(f_n)+Z)} \frac{\theta(\eta(f_w)+Z)}{\theta(\eta(f_s)+Z)} \frac{E(\alpha_1,\alpha_2)}{E(\alpha_3,\alpha_2)} \frac{E(\alpha_3,\alpha_4)}{E(\alpha_1,\alpha_4)}$$

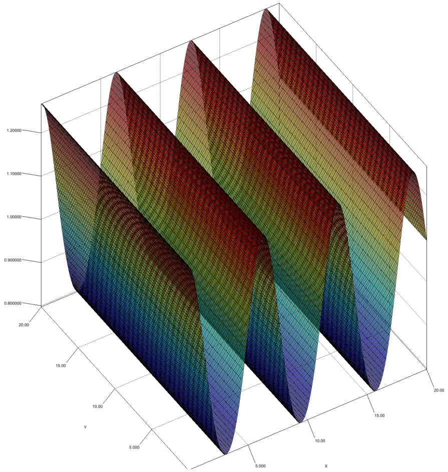


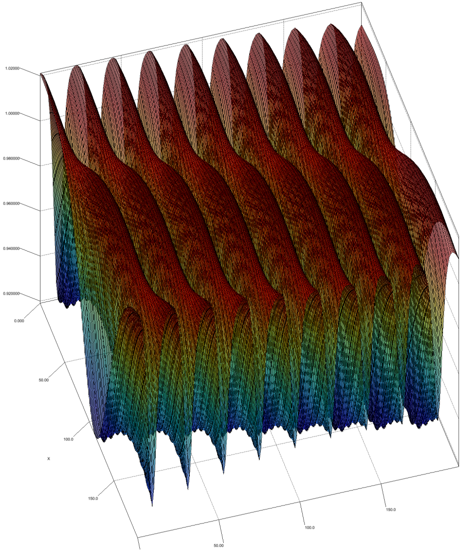


Properties

- Compatible with spider move. Fock
- Local inverse formulas for inverses: K_{bw}^{-1,P_0} depends only on the weights of a path between b and w. [B,C,dT]
- No obvious analogy to isoradial embedding encoding the weights.
- Weights are periodic if $\sum_{i=1}^{d} A(\alpha_i^+) - A(\alpha_i^-) = \sum_{i=1}^{d} A(\beta_i^+) - A(\beta_i^-) = 0 \in J(\Gamma)$.

What do these weights look like?





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Two holomorphic differentials

[Krichiver 2013]

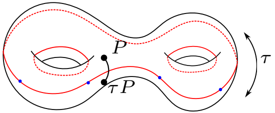
Let $d\xi_1$ be the Abelian differential with $\text{res}_{\alpha_i^+} d\xi_1 = 1$, $\text{res}_{\alpha_i^-} d\xi_1 = -1$, holomorphic otherwise and with purely imaginary periods. Then $\text{Re}(\xi_1)$ is well defined. $d\xi_2$ similar with β. Normalize such that $\xi_i(X_0) \subset \mathbb{R}$.

$$\xi_1 = x_1 + iy_1(x_1, x_2)$$
$$\xi_2 = x_2 + iy_2(x_1, x_2)$$

Define Amoeba map $\mathcal{A}(P) = (x_1, x_2)$ same as algebraic Amoeba map in periodic case.

Krichiver's Ronkin Function

Consider $P \in \Gamma^+$, and a path l between $\tau P, P$.

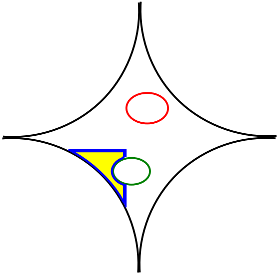


$$h(P) := \frac{1}{2\pi i} \int_l \xi_2 d\xi_1$$

Then

$$\rho(x_1, x_2) = -h(P) + \frac{1}{\pi} x_2 y_1$$

$$\sigma(y_1, y_2) = h(P) - \frac{1}{\pi} y_2 x_1 = -\rho(x_1, x_2) + \frac{1}{\pi}(x_2 y_1 - y_2 x_1)$$



define generalizations of the Ronkin function and its Legendre dual. Both are convex. In the periodic case they agree with the classical Ronkin function and surface tension.

Proposition (BBS)

This agrees with Krichiver's construction

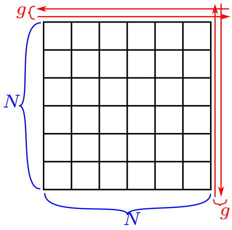
Regularized Thermodynamic Limit

Want to make sense of the dimer connection in the quasiperiodic case. Regularize NxN region to get periodic weights. R_N Ronkin function of this regularized region.

Theorem (BBS)

$$\frac{1}{N^2} R_N \to \rho.$$

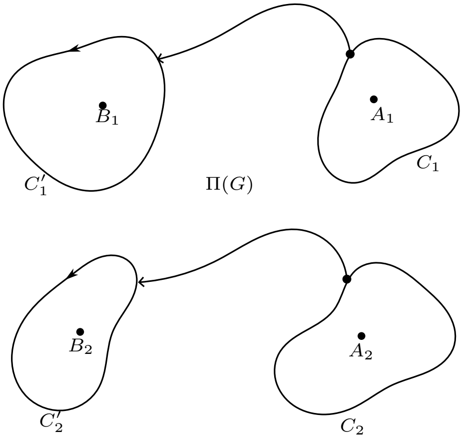
In particular $\frac{1}{N^2}\log(Z_N) \to \log(Z) =:$ *definition of regularized thermodynamic free energy per fundamental domain.*



Contents

Schottky Uniformization

- σ_n Moebius transformation such that
$$\frac{\sigma_n z - B_n}{\sigma_n z - A_n} = \mu_n \frac{z - B_n}{z - A_n}, \ |\mu_n| < 1.$$
- disjoint disks $C_n \xrightarrow{\sigma_n} C_n'$ then Schottky group G free group generated by $\{\sigma_n\}$
- All C_n circles $\implies$ G *classical* Schottky group.
- discontinuity set $\Omega(G) = \hat{\mathbb{C}} \setminus \overline{\{\text{fixed points}\}}$
- $\Pi(G) \overset{1:1}{\approx} \Omega(G)/G = \Gamma$ Riemann surface.

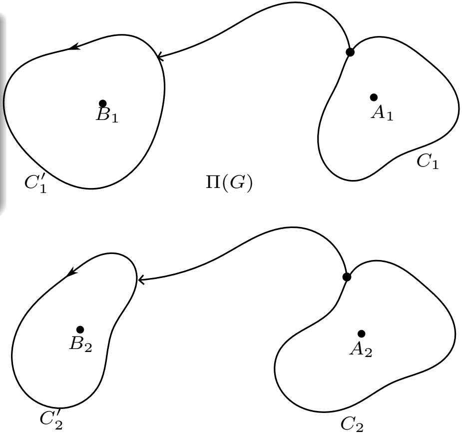


Schottky Uniformization

Theorem (Schottky Uniformization Theorem, Ford '1929)

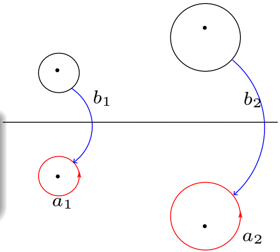
Any Γ *with a choice of homologically independent simple disjoint loops* $\nu_1, \ldots, \nu_g$ *can be realized as* $\Omega(G)/G$ *for* G *Schottky.*

- number of parameters $= 3g - 3$.
- Open: Can arbitrary Γ be uniformized by *classical* Schottky group?



Schottky Uniformization: Differentials

$$\omega_n(z) = \sum_{\omega \in G/G_n} \left(\frac{1}{z - \sigma B_n} - \frac{1}{z - \sigma A_n}\right) dz$$



Theorem (Classical)

If $\{\omega_n\}$ converges, then $\omega_1, \ldots, \omega_g$ holomorphic differentials dual to cycles $\{a_n\}, \{b_n\}$.

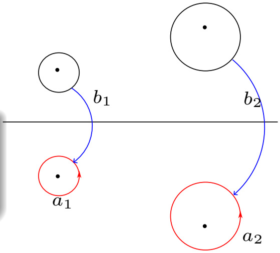
Proof.

- $\omega_n(\sigma z) = \omega_n(z)$
- Poles outside $\Pi(G) \implies$ holomorphic.
- $\int_{a_m} \omega_n = 2\pi i \delta_{nm}$ by residue thm.

□

Schottky Uniformization: Differentials

$$\omega_n(z) = \sum_{\omega \in G/G_n} \left(\frac{1}{z - \sigma B_n} - \frac{1}{z - \sigma A_n}\right) dz$$



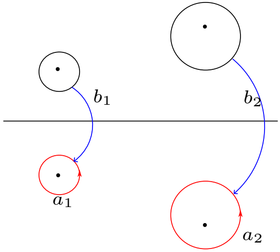
Theorem (Classical)

If $\{\omega_n\}$ converges, then $\omega_1, \ldots, \omega_g$ holomorphic differentials dual to cycles $\{a_n\}, \{b_n\}$.

- If pairs a_n, b_n decomposable into pairs of pants bounded by circles, then convergence known.
- In particular true for M-curves.

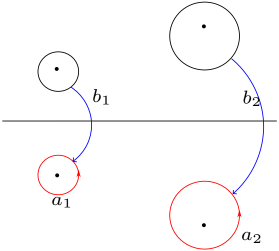
Schottky Uniformization: M-curves

- $B_i = \bar{A}_i, \mu < 1$ gives an M-curve. Not always decomposable.
- $A_i < B_i \in \mathbb{R}, \mu < 1$ and $[A_j, B_j] \cap [A_i, B_i] = \varnothing$ also gives M-curve. Decomposable by vertical lines.
- Useful for different limits.



Schottky Uniformization: Differentials

- $\omega_n(z) = \sum_{\omega \in G/G_n} (\frac{1}{z-\sigma B_n} - \frac{1}{z-\sigma A_n}) dz$
- $B_{nm} = \sum_{\sigma \in G_m \backslash G/G_n} \log\{B_m, \sigma B_n, A_m, \sigma A_n\}$
- B matrix known $\implies$ can compute Theta functions $\theta(z|B)$.



Can write our differentials $d\xi_1, d\xi_2$:

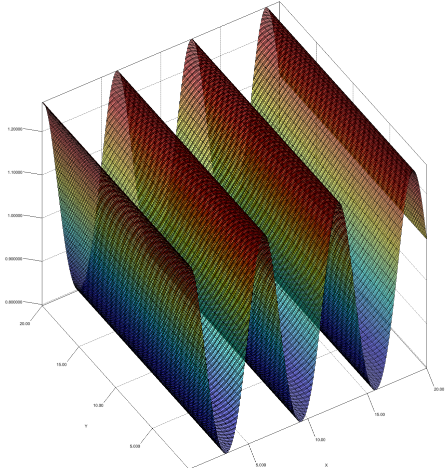
$$d\xi_1 = \sum_{\sigma \in G} \left(\frac{1}{\sigma z - \alpha^-} - \frac{1}{\sigma z - \alpha^+} \right) \sigma'(z) dz$$

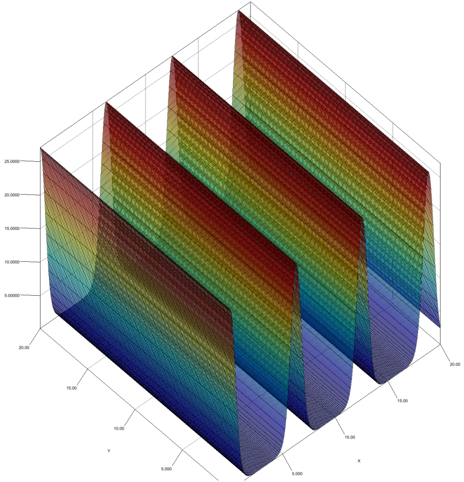
Thus

$$\mathrm{Re}\xi_1(z) = \sum_{\sigma \in G} \log |\{\sigma z - \alpha^-, \sigma z - \alpha^+, \sigma z_0 - \alpha^+, \sigma z_0 - \alpha^-\}|.$$

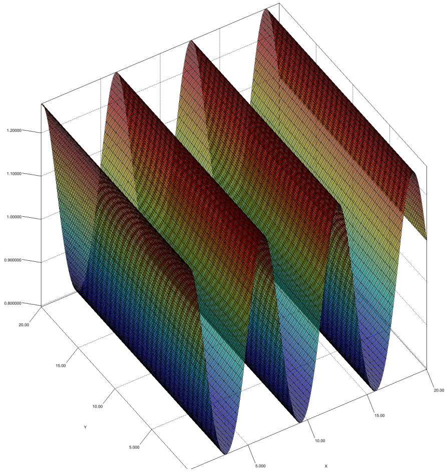
Weights

Numerical approximations use the jtem library by Schmies '2005.





Weights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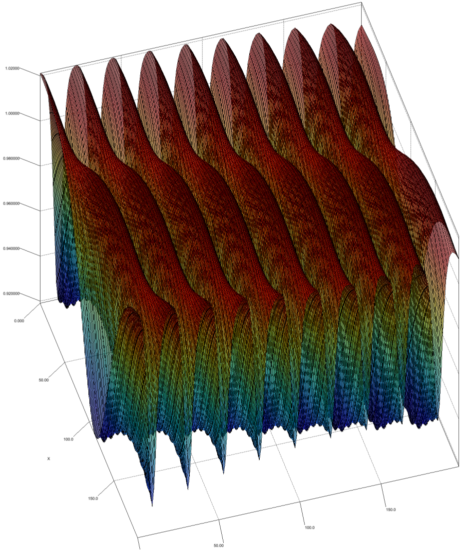
1.02000
1.00000
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0.960000
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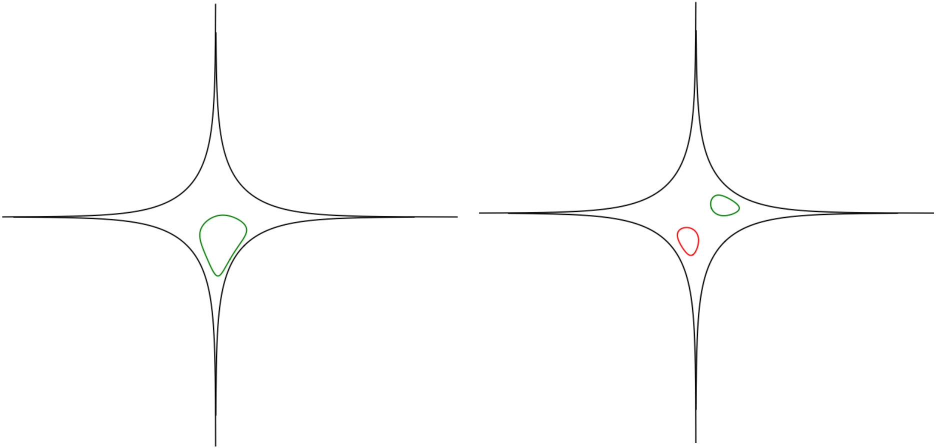
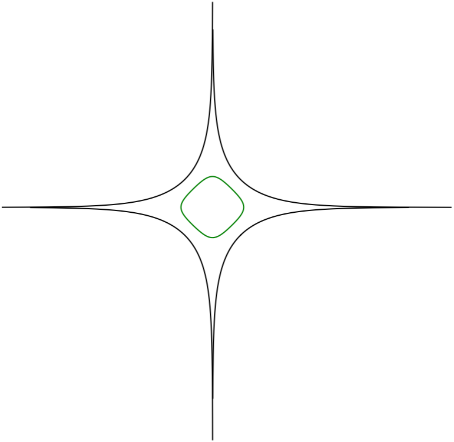
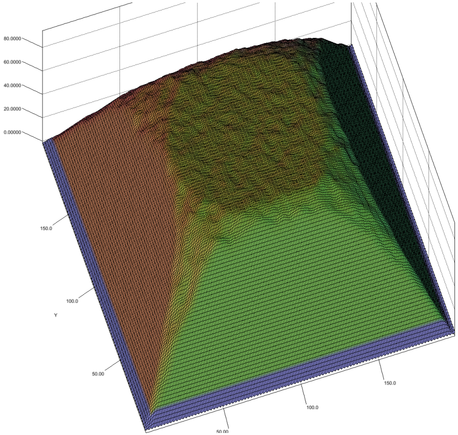
Amoebas

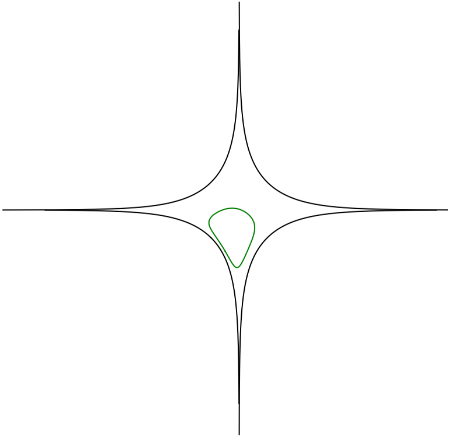


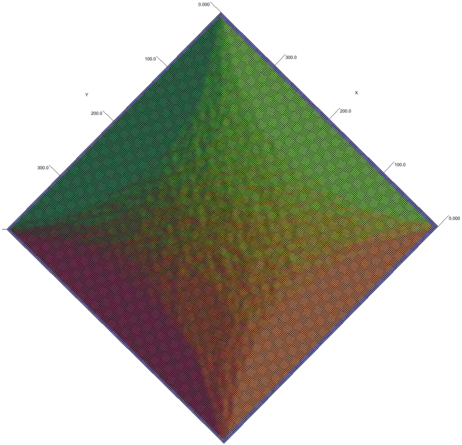
Amoeba And Height



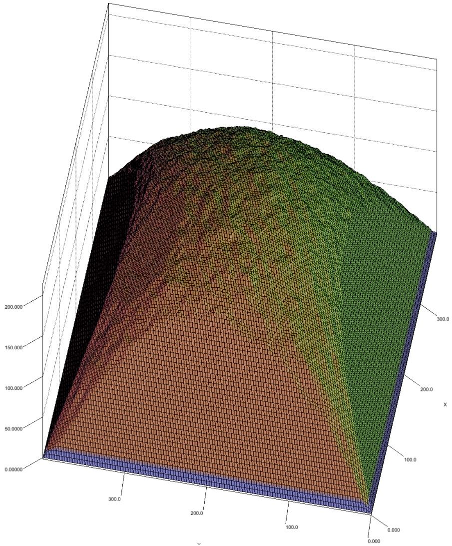


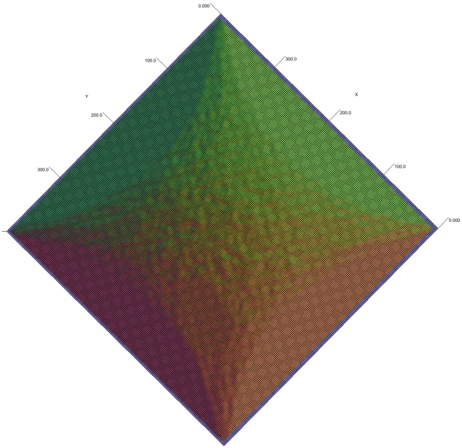
Amoeba And Height





Random Height Function





Prospects

- More robust limiting procedure.
- Hope for a formula of surface tension σ as generalization of Kenyon's formula averaging it over the Schottky group.
- K^{-1} and Gibbs measures in terms of Schottky.

Thank you!