Exact cube-root fluctuations in an area-constrained random walk model

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Joint work with Romain Panis

The phase separation problem

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- Archetypical example: supercritical Ising model with coexistence conditioning (Wulff conditioning).
- Formally, for $\beta > \beta_c$, geometry of a typical configuration sampled under

$$\mu^+_{\Lambda_N}\left[\ \cdot\ \middle|\ \#\{-\ \text{spins}\ \} = \left(\frac{1-m(\beta)}{2}+\varepsilon\right)N^d\right].$$

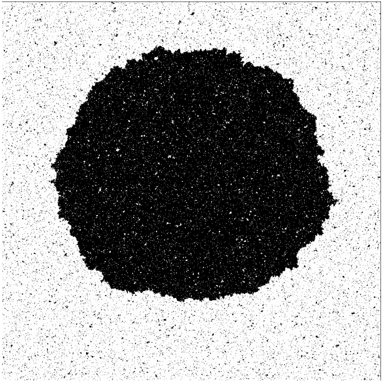


Figure: The Ising droplet at $\beta = 1/2$. Simulation by R. Cerf.

The phase separation problem

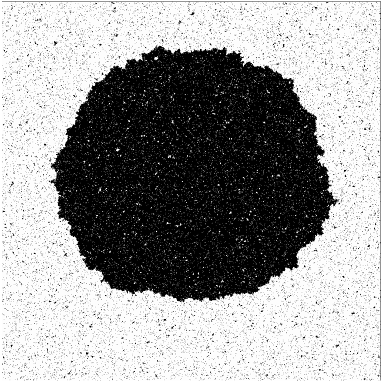
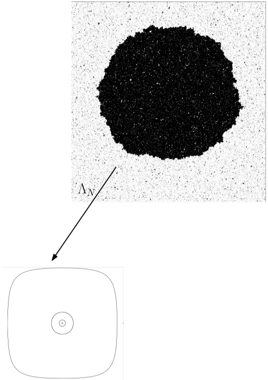


Figure: The Ising droplet at $\beta = 1/2$. Simulation by R. Cerf.

- In 2D, the object of interest is the *interface* formed by the droplet.

Several scales of interest



- First scale of interest: hydrodynamic scale

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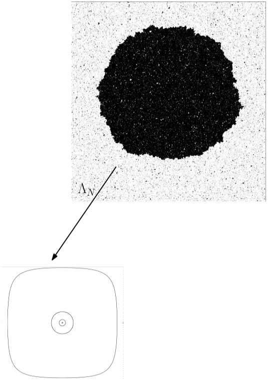

- First scale of interest: hydrodynamic scale
- Appearance of a *deterministic* Wulff shape (Dobrushin, Kotecky, Shlosman '92; Cerf '06).

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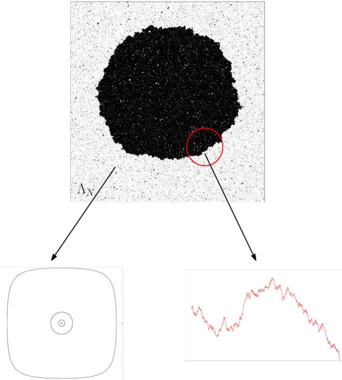

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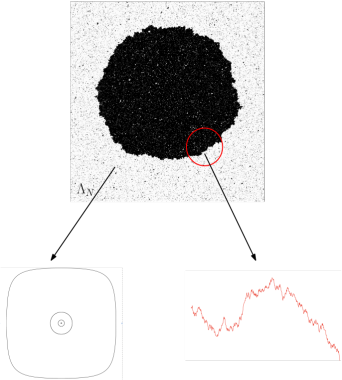

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- Brownian fluctuations (Dobrushin, Hryniv '97, low temperature regime)

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Can we identify the scale at which the conditioning and the Gaussian fluctuations of the interface have the same order of magnitude ?

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Observation (very vague version), Alexander '01; Hammond '12

The competition between Gaussian randomness and global curvature induced by the Wulff conditioning should occur at scale $N^{2/3}$

A simple toy model for the phase separation interface

- We fix $0 < \lambda < 1/2$. Background measure on the set of downright oriented paths :

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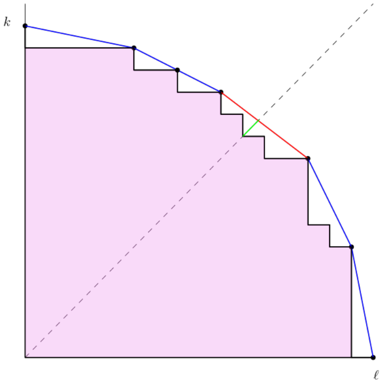
- This measure experiences a competition between Gaussian fluctuations and global curvature due to the conditioning.

Two observables of interest



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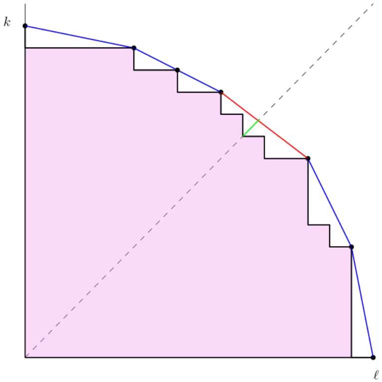

- The mean facet length: $\mathrm{MeanFL}(\Gamma)$.
- The mean local roughness: $\mathrm{MeanLR}(\Gamma)$.

The result: sharp scaling for MeanFL and MeanLR.

Question

What is the typical behaviour of MeanLR, MeanFL when $N \to \infty$?

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What is the typical behaviour of MeanLR, MeanFL when $N \to \infty$?

Theorem (D'A., Panis '23+)

For any $\varepsilon > 0$, there exist $c(\varepsilon), C(\varepsilon) > 0$ and $N_0 \geq 0$ such that for any $N \geq N_0$,

$$\mathbb{P}_\lambda^{N^2}\left[c(\varepsilon)N^{2/3} \leq \mathrm{MeanFL}(\Gamma) \leq C(\varepsilon)N^{2/3}\right] > 1 - \varepsilon$$

and

$$\mathbb{P}_\lambda^{N^2}\left[c(\varepsilon)N^{1/3} \leq \mathrm{MeanLR}(\Gamma) \leq C(\varepsilon)N^{1/3}\right] > 1 - \varepsilon$$

A second result: polylogarithmic correction for the *maximal* facet lengths and roughness

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Theorem (Hammond '12, D'A., Panis '23+)

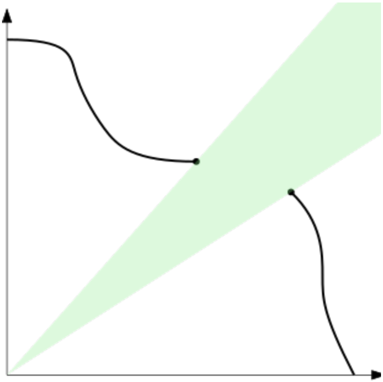
There exist $c, C > 0$ such that when $N \to \infty$,

$$\mathbb{P}_\lambda^{N^2}\left[c < \frac{\mathrm{MaxFL}(\Gamma)}{N^{2/3}(\log N)^{1/3}} < C\right] \to 1$$

and

$$\mathbb{P}_\lambda^{N^2}\left[c < \frac{\mathrm{MaxLR}(\Gamma)}{N^{1/3}(\log N)^{2/3}} < C\right] \to 1$$

A *resampling* interpretation of the Spatial Markov property



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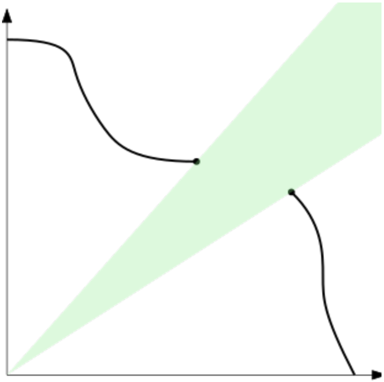
A *resampling* interpretation of the Spatial Markov property



- What is the distribution of the erased portion conditionally on the remaining part ?
- Answer: uniform amongst paths linking the endpoints and satisfying the area condition.

Two elementary inputs

Lemma 1 (Containment)

With very high probability,

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Lemma 2 (Scaling of the excess of area)

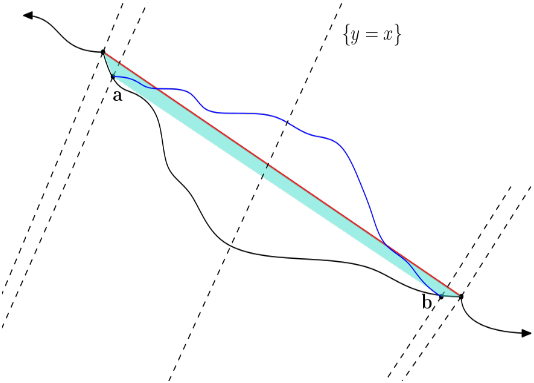
$$\mathbb{P}_{\lambda}^{N^2}\left[\text{ExcessArea} > tN\right] \leq C\mathrm{e}^{-ct}.$$

Idea of the proof : a naive upper bound

A non-sharp estimate

$$\mathbb{P}_{\lambda}^{N^2}\left[\text{MeanFL} > N^{2/3+\epsilon}\right] = \exp(-O(N^{3\epsilon/2}))$$

A refinement: a better upper bound



- We lost too much in the naive exploration !

- Question (important): what is the area contribution of a sector of angular opening $N^{-1/3}$?

Lower bounds: the Brownian realm

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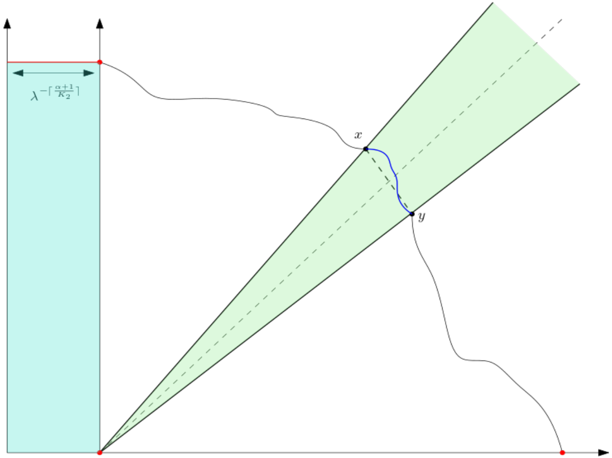
- Answer:

$$\mathbb{P}_{\lambda}^{N^2}\left[\mathcal{A}(\text{MidSector}) > \beta N\right] < C\mathrm{e}^{-\beta^2}.$$

Lower bounds: the Brownian realm



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- We conclude using Brownian inputs (Groeneboom '83; Suidan '02; Baladbaoui, Pitman '11).

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  - Interface locally being a random walk, Donsker's invariance principle for an unconditionned interface at scale $(N^{2/3}, N^{1/3})$: known by the celebrated *Ornstein-Zernike theory* Campanino, Ioffe, Velenik '01, very robust.

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  - Spatial Markov property and exponential mixing give an "almost Brownian Gibbs property" : sufficient at scale $(N^{2/3}, N^{1/3})$, Hammond '12.

Perspectives

Natural question : scaling limit along the mean facet ? Full scaling limit ?

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After rescaling by $N^{2/3}, N^{1/3}$, the random walk excursion under the mean facet converges towards a *Ferrari-Spohn* excursion.

We (and many others !) believe that this scaling limit should be universal in most of 2D phase separation interfaces.

The end

Thank you for your attention !