Phase diagram of the Ashkin–Teller model

Moritz Dober

joint with Yacine Aoun and Alexander Glazman

Faculty of Mathematics
University of Vienna

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Mathematical Physics In Les Diablerets

Outline

1. The Ashkin–Teller model:
   - Definition
   - Infinite-volume measures and phase transition(s)
   - Duality
2. Results: previous and new
   - Two distinct transitions in one ‘half’ of the phase diagram
   - Unique phase transition in the other ‘half’
3. Tools and sketch of proof of main result
   - Connection to six-vertex
   - Graphical representation
   - Sketch of proof of main result
4. What next?

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Introduced in '43 by Ashkin and Teller as generalisation of the Ising model. We introduce a representation due to Fan '72:

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- given $J, U \geq 0$ (*coupling constants*), assign the energy

$$H(\tau, \tau') := - \sum_{xy \in E} J(\tau_x \tau_y + \tau'_x \tau'_y) + U \tau_x \tau'_x \tau_y \tau'_y,$$

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- and define probability measures

$$\mathrm{AT}_G[\tau, \tau'] := \frac{1}{Z} \cdot \exp\left(-H(\tau, \tau')\right),$$

and

$$\mathrm{AT}_G^{+,+} := \mathrm{AT}_G\left[\, \cdot \,\middle|\, \tau = \tau' = +1 \text{ on } \partial G\right].$$

Infinite-volume measures and phase transitions I

Correlation inequalities guarantee existence of weak limits:

$$\mathrm{AT}_G \Rightarrow \mathrm{AT} \quad \text{and} \quad \mathrm{AT}_G^{+,+} \Rightarrow \mathrm{AT}^{+,+} \quad \text{as } G \nearrow \mathbb{Z}^d.$$

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In dimension $d \geq 2$, there exist $\beta_c^\tau, \beta_c^{\tau\tau'} \in (0, \infty)$ such that

$$\mathrm{AT}_\beta(\tau_0 \tau_x) \begin{cases} \xrightarrow{|x| \to \infty} 0 & \text{if } \beta < \beta_c^\tau, \\ \geq C_\beta > 0 & \text{if } \beta > \beta_c^\tau, \end{cases}$$

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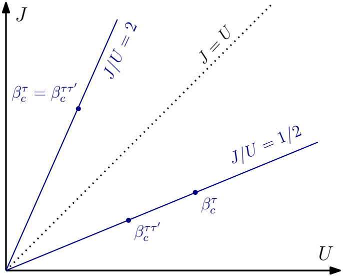

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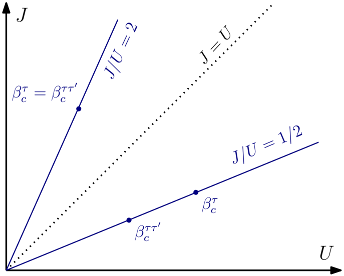

Sharpness via the OSSS approach (Duminil-Copin–Raoufi–Tassion '19).

Infinite-volume measures and phase transitions II

Heuristically: order in τ and in τ' $\Rightarrow$ order in $\tau\tau'$.
This suggests

$$\beta_c^{\tau} \geq \beta_c^{\tau\tau'},$$

which follows from correlation inequalities (Kelly–Sherman '68).

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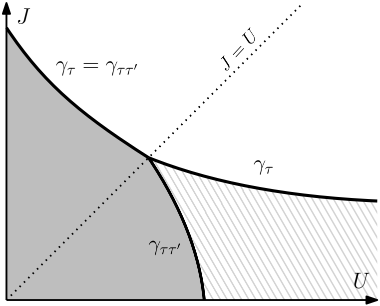
Define the transition curves γ_τ and $\gamma_{\tau\tau'}$ by

$$\gamma_\tau := \{(J,U) : J,U > 0 \text{ and } \beta_c^{\tau} = 1\},$$
$$\gamma_{\tau\tau'} := \{(J,U) : J,U > 0 \text{ and } \beta_c^{\tau\tau'} = 1\}.$$



Duality

As for Ising and Potts:

Relate AT on some planar graph G to AT on its dual G^* with (possibly) different J, U and different *boundary conditions*.

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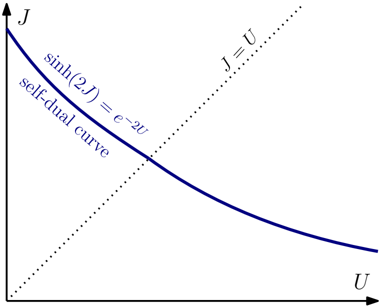
The *self-dual curve*

$$\sinh(2J) = e^{-2U}$$

describes the invariant parameters.

For each $J, U > 0$, there exists a unique $\beta_{\mathrm{sd}} > 0$ such that $\beta_{\mathrm{sd}} \cdot (J, U)$ lies on this curve.



Results for $J < U$

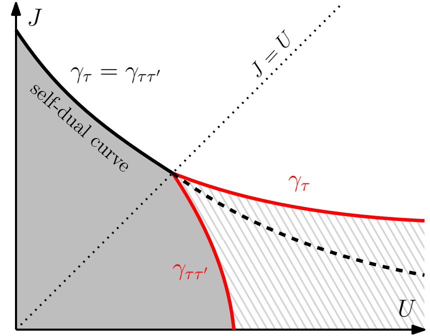
Main result: two distinct phase transitions for $J < U$ in $d = 2$.

Theorem [Aoun–D.–Glazman '22]

In dimension $d = 2$:

(i) For any $0 < J < U$, we have $\beta_c^\tau > \beta_{\text{sd}} > \beta_c^{\tau\tau'}$,

(ii) γ_τ and $\gamma_{\tau\tau'}$ are dual to each other.



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Previous results:

1. $\beta_c^{\tau} > \beta_c^{\tau\tau'}$:
   - conjectured by Wegner '72, Fan '72, and by Wu–Lin '74,
   - Pfister '82: for $2J < U$ using correlation inequalities only,
   - Pfister–Velenik '97 and Häggström '97: for $J << U$,
2. Intermediate behaviour on self-dual line $(J < U)$: Glazman–Peled '19
   $\rightarrow$ starting point of our work; details later

Results for $J \geq U$

- The graphical representation directly implies
$$\beta_c^\tau = \beta_c^{\tau\tau'} =: \beta_c$$
in any dimension $d \geq 2$.

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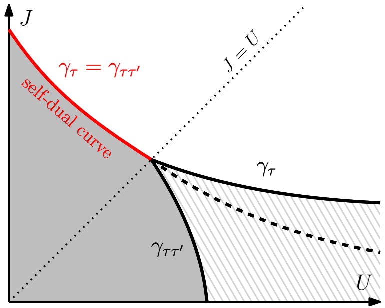
$$\beta_c^{\tau} = \beta_c^{\tau\tau'} =: \beta_c$$
in any dimension $d \geq 2$.
- In dimension $d = 2$,
$$\beta_c = \beta_{\mathrm{sd}}$$
follows from duality, sharpness and the following lemma.

Lemma [Aoun–D.–Glazman '22]

The set of $J, U > 0$ for which
$$\mathrm{AT} \neq \mathrm{AT}^{+,+}$$
has Lebesgue measure 0.



From AT to the six-vertex model I

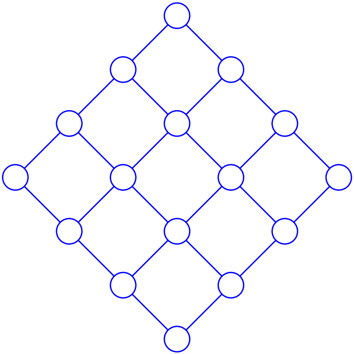
Relation to eight-vertex already noticed by Fan '72 and Wegner '72.

Intuition:

Fix the product $\tau\tau'$ and apply duality to τ.

Fix a finite subgraph $G = (V, E)$ of $\mathbb{Z}^2$ and $J, U > 0$, and take

$$(\tau, \tau') \sim \mathrm{AT}_G \left[\cdot \,\middle|\, \tau = \tau' \text{ on } \partial G \right].$$



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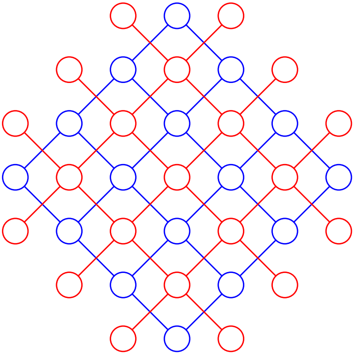

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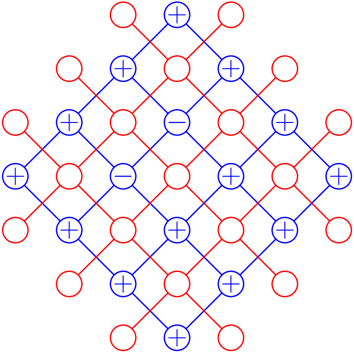

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- Define $\sigma = (\sigma^\bullet, \sigma^\bullet) \in \{\pm 1\}^{V \cup V^*}$:
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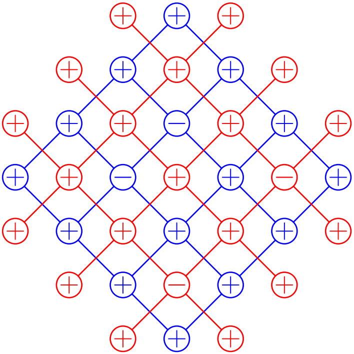

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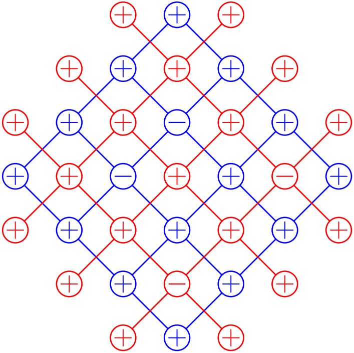

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- For any edge $e \in E$: $\sigma^{\bullet}$ is constant on e, or $\sigma^{\bullet}$ is constant on e^* (→ *ice rule*).

From AT to the six-vertex model II

The law of σ satisfies

$$\mathbb{P}[\sigma] \propto \left(\frac{e^{-2U}}{\cosh 2J}\right)^{|E_{\sigma^\bullet}|} \left(\frac{\sinh 2J}{\cosh 2J}\right)^{|E_{\sigma^\bullet}|} \mathbb{1}_{\{\text{ice-rule}\}} \mathbb{1}_{\{\text{boundary conditions}\}},$$

where $E_{\sigma^\bullet} = \{xy \in E : \sigma^\bullet(x) \neq \sigma^\bullet(y)\}$, and similarly for $E_{\sigma^\bullet}$.

From AT to the six-vertex model II

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$$\mathbb{P}[\sigma] \propto \left(\frac{e^{-2U}}{\cosh 2J}\right)^{|E_{\sigma^{\color{blue}\bullet}}|} \left(\frac{\sinh 2J}{\cosh 2J}\right)^{|E_{\sigma^{\color{red}\bullet}}|} \mathbb{1}_{\{\text{ice-rule}\}} \mathbb{1}_{\{\text{boundary conditions}\}},$$

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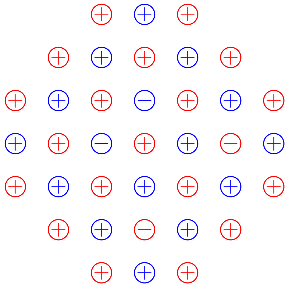
Six-vertex spins induce *edge orientations* and a *height function*:



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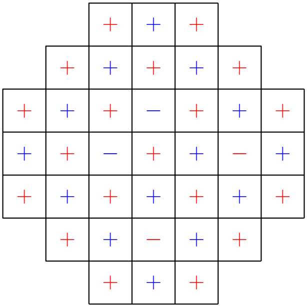

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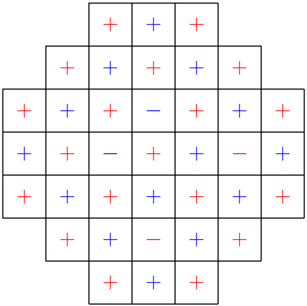

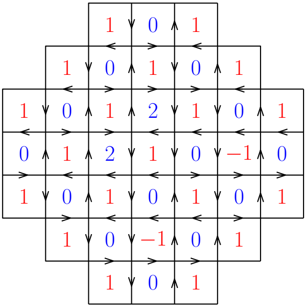

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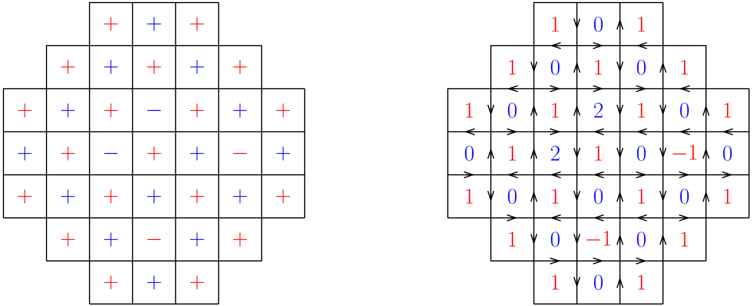

Note: the boundary conditions $\tau = \tau'$ in AT impose both a layer of 1s and a layer of 0s on the boundary for the height function.

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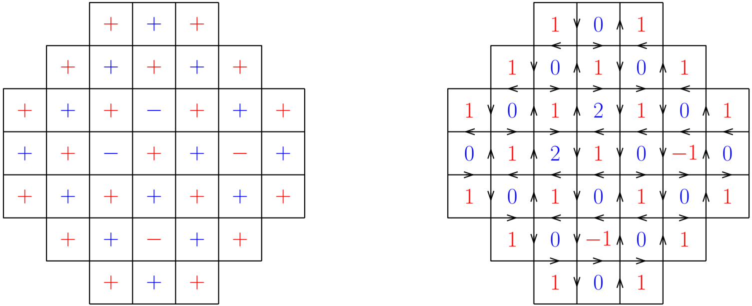

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For J, U self-dual, the corresponding six-vertex model is not staggered and reduces to the usual six-vertex model with weights

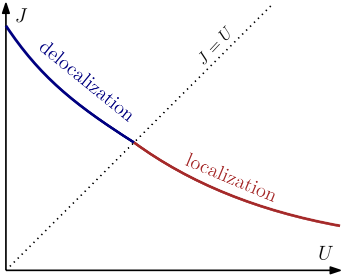
$$a = b \in (0,1),\ c = 1,$$

and $a < \frac{1}{2}$ if and only if $J < U$.

Input from six-vertex

On the self-dual line, the corresponding six-vertex height function

- delocalizes for $J \geq U$ (Duminil-Copin–Karrila–Manolescu–Oulamara '20, Glazman–Lammers '22+),
- localizes for $J < U$ (Duminil-Copin–Gagnebin–Harel–Manolescu–Tassion '16, Glazman–Peled '19).



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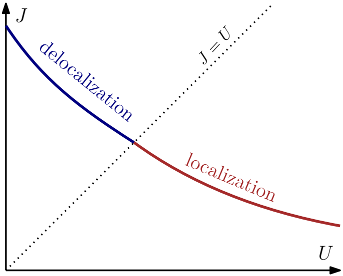

Localization of the height function implies long-range order in

$$\sigma^{\textcolor{blue}{\bullet}} = \tau\tau' \quad \text{and} \quad \sigma^{\textcolor{red}{\bullet}} = \tau^*.$$

Intermediate behaviour on self-dual line

Building on that, the following was shown.

Theorem [Glazman–Peled '19]

For $d = 2$ and any self-dual $J < U$,

$$\mathrm{AT}_G\left[\,\cdot\,\middle|\,\tau = \tau' \text{ on } \partial G\right] \Rightarrow \mathrm{AT}^{\mathrm{f},+} \quad \text{as } G \nearrow \mathbb{Z}^2,$$

and the limit satisfies

$$\mathrm{AT}^{\mathrm{f},+}[\tau_0\tau_x] \le e^{-c\cdot|x|} \quad \text{while} \quad \mathrm{AT}^{\mathrm{f},+}[\tau_0\tau_0'\tau_x\tau_x'] \ge C > 0.$$

→ starting point of our work.

Sketch of proof of main result

Key result:

Proposition [Aoun–D.–Glazman '22]

For $J < U$ self-dual, there exists $c > 0$ such that, for any $n \geq 1$,

$$\mathrm{AT}^{+,+}_{\Lambda_n}[\tau_0] \leq e^{-cn},$$

where Λ_n is induced by $[-n, n]^2 \cap \mathbb{Z}^2$.

- classical arguments in the graphical representation show that

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathrm{AT}^{+,+}_{\Lambda_n,\beta}[\tau_0]$$

  exists and is right-continuous in β; alternatively: $\varphi_\beta(S)$-argument
- duality, sharpness and the lemma ($\mathrm{AT} = \mathrm{AT}^{+,+}$ a.e.) imply the theorem ($\beta_c^\tau > \beta_{\mathrm{sd}} > \beta_c^{\tau\tau'}$ and duality of γ_τ and $\gamma_{\tau\tau'}$)

Graphical representation

Introduced by Chayes–Machta '97 and Pfister–Velenik '97.

Express correlations in AT in terms of connection probabilities in some edge model:

- $G = (V, E)$ finite subgraph of $\mathbb{Z}^d$ and $J, U > 0$,
- configuration space: $\Omega = \{0, 1\}^E \times \{0, 1\}^E$
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$\mathbf{J} < \mathbf{U}$: pairs $(\omega_\tau, \omega_{\tau\tau'})$ with $\omega_\tau \subseteq \omega_{\tau\tau'}$ and

$$\mathrm{AT}_G[\tau_x \tau_y] = \mathrm{ATRC}_G[x \xleftrightarrow{\omega_\tau} y],$$

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Boundary conditions relevant for the proof:

$$\mathrm{AT}_G^{+,+}[\tau_x] = \mathrm{ATRC}_G^{\mathrm{w},\mathrm{w}}[x \xleftrightarrow{\omega_\tau} \partial G],$$
$$\mathrm{AT}_G\left[\,\tau_x \,\middle|\, \tau = \tau' \text{ on } \partial G\right] = \mathrm{ATRC}_G^{\mathrm{f},\mathrm{w}}[x \xleftrightarrow{\omega_\tau} \partial G].$$

Steps and ingredients for proof of key result

Fix $J < U$ on the self-dual curve.
Step 1. Prove the exponential decay in Glazman–Peled '19 in finite volume: for any $G \supseteq \Lambda_n$,

$$\mathrm{ATRC}_G^{\mathrm{f,w}}\left[0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n\right] \leq e^{-cn}.$$

BKW coupling (Baxter–Kelland–Wu '73), and exponential relaxation for the critical random-cluster model with cluster-weight $q > 4$ and free boundary conditions (Duminil-Copin–Gagnebin–Harel–Manolescu–Tassion '16).

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Step 2. Show that $\mathrm{ATRC}^{\mathrm{w,w}}[0 \xleftrightarrow{\omega_\tau} \infty] = 0$ (infinite volume).

Exploration argument in the corresponding six-vertex model (*semi-free*), and non-coextistence thm. (Sheffield '05, Duminil-Copin–Raoufi–Tassion '19).

Steps and ingredients for proof of key result

Fix $J < U$ on the self-dual curve.

Step 1. Prove the exponential decay in Glazman–Peled '19 in finite volume: for any $G \supseteq \Lambda_n$,

$$\mathrm{ATRC}_G^{\mathrm{f,w}}\left[0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n\right] \le e^{-cn}.$$

BKW coupling (Baxter–Kelland–Wu '73), and exponential relaxation for the critical random-cluster model with cluster-weight $q > 4$ and free boundary conditions (Duminil-Copin–Gagnebin–Harel–Manolescu–Tassion '16).

Step 2. Show that $\mathrm{ATRC}^{\mathrm{w,w}}[0 \xleftrightarrow{\omega_\tau} \infty] = 0$ (infinite volume).

Exploration argument in the corresponding six-vertex model (*semi-free*), and non-coextistence thm. (Sheffield '05, Duminil-Copin–Raoufi–Tassion '19).

Step 3. Given step 2, show that, for any $\delta > 0$, there exists $\alpha > 0$ with

$$\mathrm{ATRC}_{\Lambda_{2n}}^{\mathrm{w,w}}[0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n] \le e^{-\alpha n} + e^{\delta n} \sum_{\Lambda_n \subseteq G \subseteq \Lambda_{2n}} p_G \cdot \mathrm{ATRC}_G^{\mathrm{f,w}}[0 \xleftrightarrow{\omega_\tau} \partial\Lambda_n].$$

Finding almost free ω_τ-circuits (Alexander '04, Campanino–Ioffe–Velenik '08).

What next?

- Delevop Ornstein–Zernike theory in AT
  $\to$ transport via couplings to six-vertex and further to critical random-cluster model with $q > 4$
  (with Alexander Glazman and Sébastien Ott)
- Sharpness when $U < 0$ (graphical representation partially available)
- Continuity of the phase transitions when $J < U$

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Thank you for your attention!