What is fractional averaging?

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Multi-scale

A dynamical system such as from engineering, science, economics, and ecology consists of many different variables interacting with each other. Given an object of study, the interacting elements are classified as into two categories: influential or negligible, the negligible is either neglected or included in the model through the CLT theorem and modelled by randomness.

Some influential interacting variables evolve at the same time scale as our objects, some evolves at a slower scale (so can be treated to be constant in time), others at faster scales. If we model the evolution of the slow variables by a random differential equations, the fast variable entered into the expression for the vector fields. On the time scale of the object of interest, the precise positions of fast variables are not tractable but often not needed. Instead, one focuses on the persistent effects of the fast variables.

Two time scale stochastic equations

Stochastic equations with slow and fast variables already separated:

$$\dot{x}_t^\varepsilon = F_0(x_t^\varepsilon, y_t^\varepsilon) + F(x_t^\varepsilon, y_t^\varepsilon)\dot{\xi}_t$$

where $y_t^\varepsilon = y_{\frac{t}{\varepsilon}}$ for a suitable process y or

$$\dot{y}_t^\varepsilon = \frac{1}{\varepsilon}\sigma_0(x_t^\varepsilon, y_t) + \varepsilon^{-\alpha}\sigma(x_t^\varepsilon, y_t)\dot{\eta}_t$$

As the time separation parameter $\varepsilon \to 0$, the position of y_t^ε is not tractable and irrelevant. The aim is to track down its persistent effect and deduce an autonomous equation for the variable of interests.

Examples of such fast motions are for example periodic functions or ergodic (stationary) Markovian process.

What noise?

According to J. Jona-Lasino, 'the critical point of a second order phase transitions so far represents in physics the most important instance where the central limit theorem breaks down'. By this he refers to the convergence of the rescaled sum of an infinite number of mean zero random variables $\frac{1}{n^{-\alpha}} \sum_{i=1}^{n} X_i$ to a Gaussian random variable.

This breaks down precisely when there is a strong correlation of the said random variables. In deed Rosenblatt showed that if Y_k is a Gaussian sequence with mean zero and variance $E(Y_0 Y_n) \sim n^{-a}$, with $a \in (0, \frac{1}{2})$ then $\frac{1}{n^{1-a}} \sum_{i=1}^{n} H_2(X_i)$ converges to a non-Gaussian random variable.

Functional Limit Theorems

In fact even when the limits are Gausssian distributed, Donsker's invariance principle may fail:

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{i=1}^{[nt]} X_i$$

may not be a Brownian motion.

The scale α would actually yield the self-similarity exponent of the process in the domain of attraction.
One of these processes are fractional Brownian motions, the others are Hermit processes $Z^{H,m}$.

The Nile

Each summer, the river Nile overflows and floods the surrounding areas, leaving behind rich fertile silt for agriculture.
If the inundation was inadequate, only a small area would be covered with the life-giving silt, famine follows.

During the Pharaonic Period, forecast for the water flow was used to compute taxes.

Time series data

Records on the height of the annual flow has been kept for 3 millennia, with numerous Nilometers.

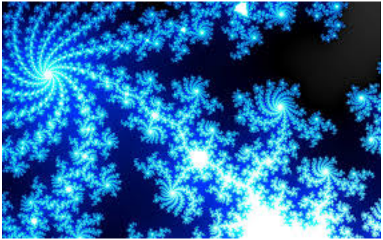
In 1906, Harold Hurst started to work in the Survey Department of Egypt in October 1906, which was responsible for collecting data throughout the Nile basin.

Harold Hurst

Hurst worked in Egypt from 1906-1968, studying the annual Nile overflow series data he discovered a heavier flood year is followed by a heavier than average flood year and a draught year flow is followed by a lighter than average river flow.

The Hurst phenomenon is modelled by Mandelbrot and van Ness with fractional BM (1968).

Mandelbrot studied fractals to capture the roughness persistent at all levels.



$$B_t = \int_0^t (t-u)^{H-\frac{1}{2}}\, dW_u + \int_{-\infty}^0 \left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right] dW_u.$$

Self-similar: $B_{at} = a^H B_t$.

Correlated noise

The 'derivative' of a fractional Brownian motion is the simplest noise with correlation. A fBM is a Gaussian process with stationary increment and $\mathbf{E}(B_{t+s} - B_s)^2 = t^{2H}$.
For t large, $H \neq \frac{1}{2}$:

$$\mathbf{E}(B_{t+s+1} - B_{t+s})(B_{s+1} - B_s) \sim t^{2H-2}.$$

The dynamics of an equation driven by a long range fractional Brownian motion ($H > \frac{1}{2}$) is quite different from that of a Brownian motion. Very little is know of its invariant measure, no formula exists so far except for the fractional Ornstein-Uhlenbeck equation. Even little is know of its densities and lower and upper bounds.

This talk draws from

- Fast variable non-Markovian, driven by fBM. Time homogenisation problem (multiple scaling constants, the effect dynamics is much richer than that given by the Markovian dynamics), Johann Gehringer +L.
- Slow variable driven by fBM. The analysis is a puzzle. Hairer+L.
- Fractional averaging with fast fractional dynamics, Sieber+L.
- non-product form for Volterra kernels, Gehringer +L+Sieber - for SPDE

$$dX_t^\varepsilon = AX_t^\varepsilon \, dt + f\big(X_t^\varepsilon, Y_{\frac{t}{\varepsilon}}\big) \, dt + g\big(X_t^\varepsilon, Y_{\frac{t}{\varepsilon}}\big) \, dB_t.$$

  L.+Sieber
- L. + Planloup+Sieber: smooth dependence of invariant measures for SDEs driven by fBMs depending on a parameter.

Ergodicity of Fast Motion

Let y_t be a stationary process with invariant measure μ, define

$$\bar{f} := \int_{\mathcal{Y}} f(y)\,\mu(dy).$$

Functional LLN and Averaging Principle

$$\dot{x}_t^\varepsilon = F_0(x_t^\varepsilon, y_{t/\varepsilon}) + F(x_t^\varepsilon, y_{t/\varepsilon})\dot{\xi}_t$$

Definition

y_t satisfies a **Functional LLN** if for any f regular

$$\left|\varepsilon \int_0^t f(y_{r/\varepsilon})dr - t\bar{f}\right| = o(\varepsilon)$$

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Let Y_s be an independent stationary ergodic Markov process with generator $\mathcal{L}$ which one assumes nice. Functional LLN implies that

$$\int_0^t f(Y_{s/\varepsilon})dW_s \to \hat{W}_t$$

a Wiener process with covariance $(\overline{f \otimes f})^{\frac{1}{2}}$.

$$dx_t^\varepsilon = f(x_t^\varepsilon, Y_{s/\varepsilon})dW_s$$

Diffusion Creation /Homogenesation

The functional CLT for the Markov process
If $\bar{f} = 0$ and regular

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t f(Y_{s/\varepsilon}) ds \to \left(\overline{f \mathcal{L}^{-1} f}\right)^{\frac{1}{2}} W_t.$$

Diffusion creation problem:

$$\dot{x}_t^\varepsilon = \sqrt{\varepsilon}\, f(x_t^\varepsilon, y_{\frac{t}{\varepsilon}}).$$

Integration and enhanced processes

$$\int_0^t f(B_s)dB_s \sim \sum \int_u^v f(B_u) + f'(B_u)\delta B_{ur} + \dots)dB_r$$

$$\sim \sum \overbrace{f(B_u)\delta B_{uv} + f'(B_u)\int_u^v \delta B_{ur} dB_r}^{A_{uv}} + \dots .$$

For $H > \frac{1}{2}$, this is Riemann-Stieljes, for $H \in (\frac{1}{3}, \frac{1}{2})$, we need the second order term, the iterated integrals of B.
Enhanced process $(B, \mathbb{B})$ where

$$\mathbb{B}_{uv} = \int_u^v (B_u - B_r)dB_r.$$

For $H \in (\frac{1}{4}, \frac{1}{3})$, need to Taylor expand to order 3.

To define $\int f(x_s)dB_s$ we assume that x_s is controlled by B: it is similar to B plus smooth order terms.

Sewing Lemma

Given a two parameter object $A_{u,v}$ define $\delta A_{s,u,t} = A_{st} - A_{su} - A_{ut}$. If

$$|A_{st}| \leq K'|t-s|^{\eta}, \qquad |\delta A_{sut}| \leq K|t-s|^{\bar{\eta}}$$

for $\eta > 0, \bar{\eta} > 1$ then $\lim \sum A_{uv}$ exists and defines an integral I

$$|I_{uv} - A_{uv}| \leq K|t-s|^{\bar{\eta}}.$$

Sewing is used to take away probability from Itô integral.

It turns out that there is a stochastic sewing lemma pumping probability back into action when sewing lemma fails.

Rough functional limit theorem

Y_s is a fractional Ornstein-Uhlenbeck process,
$H^*(m) = m(H-1)+1$, G function with Hermite rank m.

- $$\frac{1}{\sqrt{\varepsilon}}\int_0^t G(Y_{s/\varepsilon})ds \to cW_t, \qquad \text{if} \quad H^*(m) < \frac{1}{2},$$
- $$\frac{1}{\sqrt{\varepsilon}\sqrt{|\ln \varepsilon|}}\int_0^t G(Y_{s/\varepsilon})ds \to cW_t, \qquad \text{if} \quad H^*(m) = \frac{1}{2},$$
- $$\varepsilon^{H^*(m)-1}\int_0^t G(y_{s/\varepsilon})ds \to c\bar{Z}_t^{H*(m),m}, \qquad \text{if} \quad H^*(m) > \frac{1}{2}, .$$

These are essentially known, as for all previous FCL, they converge weakly.
Theorem (Gehringer-L.) Functional limit theorem holds in rough path topology, so limit of $\dot{x}_t^\varepsilon = f(x_t^\varepsilon)G(y_{\frac{t}{\varepsilon}})$ converge.

Young bounds fails

For $H > \frac{1}{2}$, the following is the essence for defining integrals:

$$\left| \int_u^v f_s dB_s - f_v \delta B_{u,v} \right| \le |f|_\alpha |B|_\beta |t-s|^{\alpha+\beta}.$$

$$dx_t^\varepsilon = F_0(x_t^\varepsilon, y_{t/\varepsilon}) + F(x_t^\varepsilon, y_{t/\varepsilon})\, dB_t$$

One difficulty is that the uniform estimates for x_t^ε are difficulty obtain, while for diffusions or ODEs they come for free. The usual young bound, in case $H > \frac{1}{2}$ blows up as $\varepsilon \to 0$.

The other difficulty is that we have no clue as to what limit to expect. Martingale problem technique is used for averaging principle.

Functional LLN for stochastic integrals

Fractional Averaging

$$dx_t^\varepsilon = f(x_t^\varepsilon, y_t^\varepsilon)\, dB_t + g(x_t^\varepsilon, y_t^\varepsilon)\, dt\ ,$$

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Proposition. Let (B_t) be a fBM of Hurst parameter $H > \frac{1}{2}$ and y_t is a uniformly elliptic Markov process on a compact manifold solving a feedback SDE. Then x_t^ε converges to $\bar{x}_t$ in probability

$$d\bar{x}_t = \bar{f}(\bar{x}_t)\, dB_t + \bar{g}(\bar{x}_t)\, dt\ .$$

In particular for any $\beta < H$, there exists $\kappa > 0$ such that

$$\lim_{\varepsilon \to 0} \mathbf{P}\Big(\Big| \int_s^t f(y_{\frac{r}{\varepsilon}}) dB_r - \bar{f}(B_t - B_s) \Big|_\beta > \varepsilon^\kappa \Big) = 0\ .$$

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Notes

Annealed limit. $x_t^\varepsilon \to \bar{x}$ in probability.

Technical difficulty: Tightness

Quenched Problem. If we fix a path $h = B(\omega)$, does the convergence hold for each fixed fBM path?

Feedback model.

Toy Model : Averaging with rough path tools
based on Hairer-L'20

Strong Mixing conditions and Stochastic LLN

One useful ergodic condition on a stationary process y is the strong mixing condition, in particular for f, g bounded measurable,

$$\left|\mathbf{E} f(y_t) g(y_s)\right| \leq 4|t-s|^{-\delta}|f|_\infty |g|_\infty \ ,$$

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Then if f is bounded and $\bar{f} = 0$,

$$\begin{aligned} \mathbf{E}\Big(\int_s^t f(y_{r/\varepsilon})\,dr\Big)^2 &= \varepsilon^2 \mathbf{E} \int_{s/\varepsilon}^{t/\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} f(y_r) f(y_{\bar{r}})\,dr\,d\bar{r} \\ &\lesssim |f|_\infty^2\, \varepsilon^2 \int_{s/\varepsilon}^{t/\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} |r-\bar{r}|^{-\delta}\,dr\,d\bar{r} \\ &\lesssim \varepsilon^\delta |t-s|^{2-\delta} \ . \end{aligned}$$

Using time average in topology

$$\left\| \int_s^t f(y_{r/\varepsilon})\, dr \right\|_p \lesssim \varepsilon^{\frac{\delta}{p}} |t-s|^{1-\frac{\delta}{p}} \,.$$

$$|h|_{-\kappa} = \sup_{0\le s,t\le T} |t-s|^{\kappa-1} \left| \int_s^t h(r)\, dr \right| \,.$$

Lemma (Hairer-L'20)

If y is strong mixing with rate δ, $F : \mathbf{R}^d \times \mathcal{Y} \to \mathbf{R}$ bounded measurable, uniformly Lipschitz continuous in the first variable ($\mathcal{Y}$ compact), then $\|F(x, y_{\cdot/\varepsilon}) - \bar{F}\|_{-\kappa,\gamma} \to 0$.

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$$\Big\| \int_s^t (F(x, y_{r/\varepsilon}) - F(z, y_{r/\varepsilon}))\, dr \Big\|_p \lesssim \varepsilon^{\frac{\delta}{p}} |x-z| |t-s|^{1-\frac{\delta}{p}}\;.$$

$F(x, y_{\cdot/\varepsilon})$ is of class $\mathbf{C}^{-\kappa,\gamma}$ a.s.

Lemma (Hairer-L'20)

Let $H > \frac{1}{2}$. Let $f_n, \bar{f} : \mathbf{R}_+ \times \mathbf{R}^d \to L(\mathbf{R}^m, \mathbf{R}^d)$ be in $\mathcal{C}^{\zeta,2}$. Let x^n and x be the $\mathcal{C}^\alpha$ solutions to the equations

$$dx_t^n = f_n(t, x_t^n)\, dB_t, + f_0(t, x_t^n) dt, \quad dx_t = \bar{f}(x_t)\, dB_t + \bar{f}_0(x_t) dt\ ,$$

with $x_0^n = x_0$. Suppose that $\kappa, \gamma \geq 0$,

$$\lim_{n\to\infty} |f_n - \bar{f}|_{-\kappa,\gamma} = 0\ .$$

Then, $x^n \to x$ in probability in $\mathcal{C}^\alpha$, for $\alpha \in (\frac{1}{2}, H - \kappa)$, $\zeta + \alpha > 1$ and $H - \kappa + \gamma\alpha > 1$.

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Corollary (Fractional Averaging Theorem)

Let f, g be bounded measurable and of class BC^2 in their first arguments. Then, any solutions x^ε converges to $\bar{x}$.

$$dx_t^\varepsilon = f(x_t^\varepsilon, y_t^\varepsilon)\, dB_t + g(x_t^\varepsilon, y_t^\varepsilon)\, dt \ ,$$

$$d\bar{x}_t = \bar{f}(\bar{x}_t)\, dB_t + \bar{g}(\bar{x}_t)\, dt \ .$$

Ingredients of the proof

Let $\mathcal{B}_{\alpha,p}$ denote adapted stochastic process with $\delta x \in H_p^\alpha$, i.e.

$$\sup_{s,t} |t-s|^{-\alpha} \|x_t - x_s\|_{L^p} < \infty \, .$$

Lemma (Hairer-L'20)

Let $p \geq 2$ and $\alpha > \frac{1}{2}$, $x_\cdot \in \mathcal{B}_{\alpha,p}$, $f : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}$ with $f \in \mathcal{C}^{-\kappa,\gamma}$ (deterministic) for some $\kappa, \gamma \geq 0$ such that $\eta = H - \kappa > \frac{1}{2}$ and $\bar{\eta} = H - \kappa + \gamma\alpha > 1$, we may define

$$\int_s^t f(r, x_r) \, dB_r$$

by sewing up $A_{s,t} = \int_s^t f(r, x_s) \, dB_r$ (the integral is interpreted as the sum of a Wiener integral and Riemann-Stieltjes integral).

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by sewing up $A_{s,t} = \int_s^t f(r, x_s)\, dB_r$ (the integral is interpreted as the sum of a Wiener integral and Riemann-Stieltjes integral). Moreover, uniformly over $s, t \in [0,T]$.

$$\Big\| \int_s^t f(r, x_r)\, dB_r \Big\|_p \lesssim |f|_{-\kappa,\gamma} \Big(|t-s|^{H-\kappa} + \|x\|_{\alpha,p}^\gamma |t-s|^{\bar\eta} \Big) \,.$$

A unification of fractional averaging, averaging and homogenisation

Hairer-Li 21

Unusual Limit Theorems

Y_s stationary ergodic Markov process with all nice properties and generator $\mathcal{L}$.

Lemma (Hairer+Li 21)

- If $H > \frac{1}{2}$, the following converges in probability

$$\int_0^t f(Y_{s/\varepsilon})dB_s \to \bar{f}B_t.$$

- If $H \in (\frac{1}{3}, \frac{1}{2})$ (and $\bar{f} = 0$ incase $H > \frac{1}{2}$)

$$\varepsilon^{\frac{1}{2}-H} \int_0^t f(Y_{s/\varepsilon})dB_s \to \Sigma W_t.$$

$$\Sigma = \frac{1}{2}\Gamma(2H+1)\overline{F \otimes \mathcal{L}^{1-2H}F}.$$

Concluding Remark

These limit theorems underlying the effective dynamics for stochastic systems whose slow motion is driven by fractional Brownian motions.

For low H we have to slow it down to see an effective motion. These results, best described as factional averaging, are given in Hairer + Li 21 for $H \in (\frac{1}{3}, \frac{1}{2})$, which also surprisingly connects the two classic topics ‘stochastic averaging’ and ‘homogenisation’. It is surprising because the first is a LLN ($H = \frac{1}{2}$) and the second is a fluctuation theorem ($H = 1$), one did not previously connecting the dots on the form of the effective limit. Here there is a unified formula for $H \in (\frac{1}{3}, 1]$, and our results resembles an averaging for small H and fluctuation for $H > \frac{1}{2}$.