Wave propagation in inhomogeneous media: An introduction to Generalized Plane Waves

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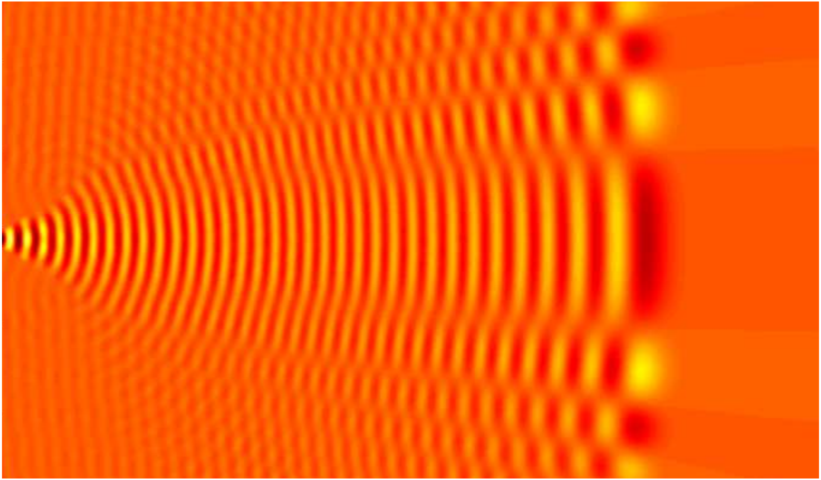


- Linear wave propagation in inhomogeneous media, BVP
- Continuous transition from propagating to evanescent medium
- Accurate numerical simulation of this transition

Time-harmonic waves through homogeneous media
Frequency domain

Low-order Galerkin methods

✗ Large number of points per wavelength

High-order Galerkin methods

- Finite Element Methods $\sharp BF = O(n^2)$
  - h and p convergence
- Wave-based methods $\sharp BF = O(n)$
  IDEA incorporate oscillations in the Basis Functions
  - Partition of Unity Method, Discontinuous Enrichment method
  - ✓ **Focus** Trefftz methods

Integral equation based methods

- Dense linear algebra

A comparison of high-order polynomial and wave-based methods for Helmholtz problems. Lieu A., Gabard G., Bériot H. JCP 2016

Introduction to Trefftz methods

Some weak formulations

Galerkin method

$$\text{Find } u \in \mathbb{H} \ s.t. \ \mathcal{A}(u,v) = \ell(v), \forall v \in \mathbb{H}$$

Discontinuous Galerkin method

Mesh dependent formulation : Mesh $\mathcal{T}_h = \{K\}$

$$\text{Find } u \in \mathbb{H} := \prod_K \mathbb{H}_K \ s.t. \ \mathcal{A}_h(u,v) = \ell_h(v), \forall v \in \mathbb{H}$$

Trefftz method

Mesh dependent formulation : Mesh $\mathcal{T}_h = \{K\}$

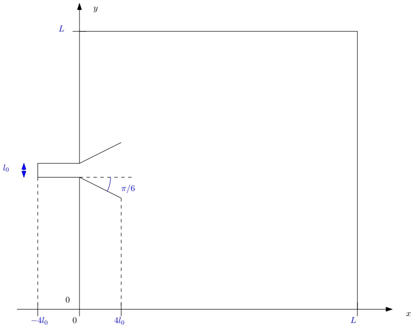
Equation-based space : $\mathcal{L}u = 0$

$$\mathbb{H}_K = \{v \in L^2(K) : v \text{ smooth}, \mathcal{L}v = 0\}$$

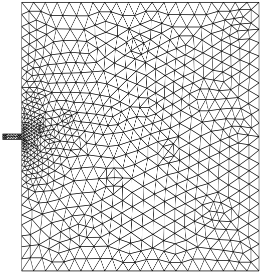
$$\text{Find } u \in \mathbb{H} := \prod_K \mathbb{H}_K \ s.t. \ \mathcal{A}_h(u,v) = \ell_h(v), \forall v \in \mathbb{H}$$

Trefftz framework

Computational domain



Mesh $\mathcal{T}_h = \{K\}$



Trefftz methods

- Local basis functions
- Mesh dependent formulation
- Lower dimensional integration
- Exact solutions of the equation

Novelty

- Variable coefficients

Challenges

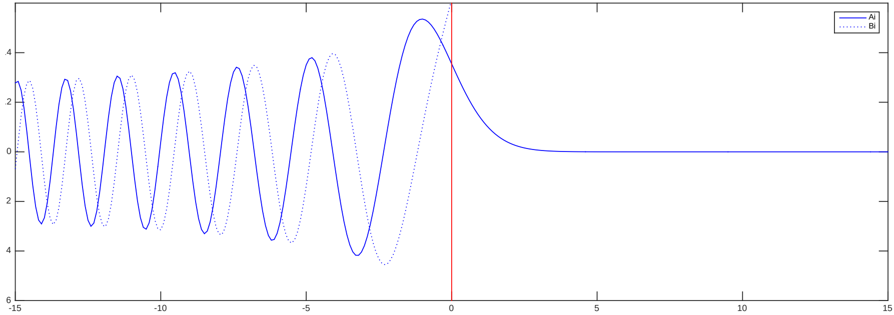
- Design basis functions
- Convergence of the method

The PDE model

2D Helmholtz equation for the total field

- $-\Delta u - \frac{\omega^2}{c^2}\epsilon(\mathbf{x})u = 0$
- smooth variable coefficient : Cutoff $\Leftrightarrow \epsilon(\mathbf{x}) = 0$

Airy function in 1D : $-u'' + xu = 0$



✓ $\epsilon < 0$ $\Rightarrow$ Propagating waves

▸ $\epsilon = 0$ $\Rightarrow$ Cutoff

✗ $\epsilon > 0$ $\Rightarrow$ Evanescent waves

The boundary value problem

2D Helmholtz equation for the total field

- $-\Delta u - \frac{\omega^2}{c^2}\epsilon(\mathbf{x})u = 0$
- smooth variable coefficient
  $sign = \pm 1$, Cutoff $\Leftrightarrow \epsilon(\mathbf{x}) = 0$

Bounded domain and boundary conditions

$\Omega \subset \mathbb{R}^2$ bounded domain

$$-\Delta u(\mathbf{x}) - \kappa^2 \epsilon(\mathbf{x}) u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega$$

Boundary conditions : metallic, absorbing, incoming

$$\begin{array}{ll} \partial_n u + \imath \kappa \vartheta u = g, & \mathbf{x} \in \partial\Omega^R \\ u = 0, & \mathbf{x} \in \partial\Omega^D \end{array}$$

Trefftz weak formulation for Helmholtz BVP

Weak formulation

Notation : Local operators Δ_h, ∇_h

$$\int_K u\overline{(-\Delta_h v - \kappa^2 \epsilon v)} + \int_{\partial K} \hat{u}\overline{\nabla_h v \cdot \mathbf{n}} - \int_{\partial K} \widehat{\nabla_h u} \cdot \mathbf{n}\overline{v} = 0$$

TG space $\mathbb{H} = \{v \in L^2(\Omega) : v$ smooth and $-\Delta_h v - \kappa^2 \epsilon v = 0, \forall K \in \mathcal{T}_h\}$

Trefftz weak formulation for Helmholtz BVP

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TG space $\mathbb{H} = \{v \in L^2(\Omega) : v$ smooth
and $-\Delta_h v - \kappa^2 \epsilon v = 0, \forall K \in \mathcal{T}_h\}$

Numerical fluxes

$$\imath\kappa\hat{\sigma} = \begin{cases} \{\{\nabla_h u\}\} - \alpha \imath\kappa [\![u]\!] & \text{on } \mathcal{E}^I \\ \nabla_h u - (1-\delta)(\nabla_h u + \imath\kappa\vartheta u\mathbf{n} - g\mathbf{n}) & \text{on } \mathcal{E}^R \\ \nabla_h u - \alpha\imath\kappa u\mathbf{n} & \text{on } \mathcal{E}^D \end{cases}$$

$$\hat{u} = \begin{cases} \{\{u\}\} - \beta(\imath\kappa)^{-1} [\![\nabla_h u]\!] & \text{on } \mathcal{E}^I \\ u - \delta((\imath\kappa\vartheta)^{-1}\nabla_h u \cdot \mathbf{n} + u - (\imath\kappa\vartheta)^{-1} g) & \text{on } \mathcal{E}^R \\ 0 & \text{on } \mathcal{E}^D \end{cases}$$

α, β, δ piece-wise positive constants

Bilinear and antilinear forms on $\mathbb{H}$

$$\begin{aligned}
\mathcal{A}_h(u,v) &= \int_{\mathcal{E}^I} \Big(\{\{u\}\} [\![\nabla_h \overline{v}]\!] - \beta(\imath\kappa)^{-1} [\![\nabla_h u]\!][\![\nabla_h \overline{v}]\!] \\
&\qquad - \{\{\nabla_h u\}\} \cdot [\![\overline{v}]\!] + \alpha(\imath\kappa)[\![u]\!] \cdot [\![\overline{v}]\!] \Big) dS \\
&\quad + \int_{\mathcal{E}^R} \Big((1-\delta) u \nabla_h \overline{v} \cdot \mathbf{n} - \delta(\imath\kappa\vartheta)^{-1} (\nabla_h u \cdot \mathbf{n})(\nabla_h \overline{v} \cdot \mathbf{n}) \\
&\qquad - \delta \nabla_h u \cdot \mathbf{n} \overline{v} + (1-\delta)\imath\kappa\vartheta u \overline{v} \Big) dS \\
&\quad + \int_{\mathcal{E}^D} \Big(\nabla_h u \cdot \mathbf{n} \overline{v} + \alpha \imath\kappa u \overline{v} \Big) dS
\end{aligned}$$

$$\ell_h(v) = \int_{\mathcal{E}^R} \Big(-\delta(\imath\kappa\vartheta)^{-1} g (\nabla_h \overline{v} \cdot \mathbf{n}) + (1-\delta) g \overline{v} \Big) dS$$

Find $u \in \mathbb{H} = \prod_K \mathbb{H}_K$ $s.t.$ $\mathcal{A}_h(u,v) = \ell_h(v), \forall v \in \mathbb{H}$

Next step : Discretization

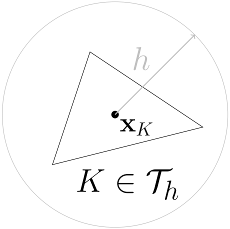
Sequence of finite dimensional space $\mathbb{V}_h = \prod_K \mathbb{V}_K$
with $\mathbb{V}_K \subset \mathbb{H}_K$?

Generalized Plane Waves (GPWs) at a glance

- Smooth functions
- Associated with partial differential equation $\mathcal{L}u = 0$
- Introduced for variable coefficient operators
  - Local approximation
- *Generalization of classical PW*

Goal

- High order approximation $u \approx u_a$



Challenges to find u_a

- Design

$$\mathcal{L}u_a \approx 0$$

- Best approximation properties

$$\forall\ u \text{ s.t. } \mathcal{L}u = 0, \exists\ u_a \text{ satisfying } \|u - u_a\| \leq Ch^n$$

Construction of GPWs
The Helmholtz example

$$\mathcal{L}_H u_a \approx 0$$

Local definition of a GPW

Constant coefficient

$$\mathcal{L}_I = -\Delta - \kappa^2 \epsilon_K$$

$$\varphi(\mathbf{x}) = \exp\left(\imath\kappa\sqrt{\epsilon_K}\mathbf{d}\cdot\mathbf{x}\right)$$

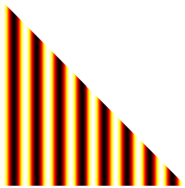
$$\Rightarrow \mathcal{L}_I\varphi = 0$$

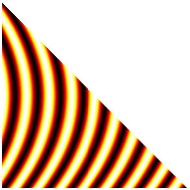
Variable coefficient

$$\mathcal{L}_H = -\Delta - \kappa^2 \epsilon(\mathbf{x})$$

$$\varphi(\mathbf{x}) = \exp\left(\imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}\mathbf{d}\cdot\mathbf{x} + \text{H.O.T.}\right)$$

$$\Rightarrow \mathcal{L}_H\varphi \approx 0$$





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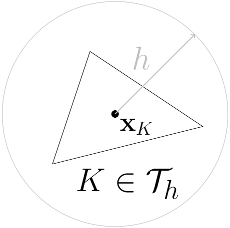
$$\Rightarrow \mathcal{L}_H\varphi \approx 0$$

Definition of a GPW

- For a given point $\mathbf{x}_K$
- For a partial differential operator $\mathcal{L}_H$
- For a parameter q



✓ $\varphi(\mathbf{x}) = \exp P(\mathbf{x})$

✓ $\begin{cases} \text{Taylor expansion at } \mathbf{x}_K \\ \mathcal{L}_H\varphi(\mathbf{x}) = O(h^q) \end{cases}$

Building a GPW : The system

$$-\mathcal{L}_H\varphi(\mathbf{x}) = \left[\partial_x^2 P(\mathbf{x}) + \left(\partial_x P(\mathbf{x})\right)^2 + \partial_y^2 P(\mathbf{x}) + \left(\partial_y P(\mathbf{x})\right)^2 + \kappa^2\epsilon(\mathbf{x})\right] e^{P(\mathbf{x})} = O\left(h^q\right)$$

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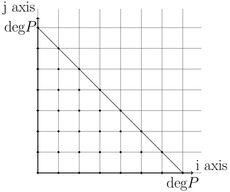
The unknowns $\quad \mathbf{x} = (x, y)$

- $\varphi = \exp P$
- $P(\mathbf{x}) = \sum_{0 \leq i+j \leq \deg P} \lambda_{i,j}(x - x_K)^i (y - y_K)^j$



✓ $\deg P$ and $\{\lambda_{i,j}\}_{0 \leq i+j \leq \deg P}$ $\quad \Rightarrow N_{un} = \frac{(\deg P+1)(\deg P+2)}{2}$

Building a GPW : The system

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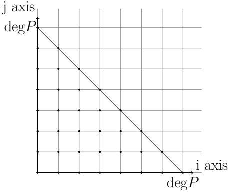


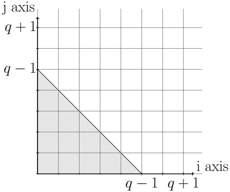
$$\Rightarrow N_{un} = \frac{(\deg P+1)(\deg P+2)}{2}$$

The equations

- $\mathcal{L}_H\varphi(\mathbf{x}) = O\left(h^q\right)$

✓ $\forall(i,j)$ such that $0 \leq i + j < q$

$$\partial_x^i\partial_y^j\left[\mathcal{L}_H(\exp P)/\exp P\right](\mathbf{x}_K) = 0$$



$$\Rightarrow N_{eq} = \frac{q(q+1)}{2}$$

Building a GPW : The system

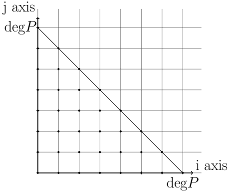
$$-\mathcal{L}_H\varphi(\mathbf{x}) = \left[\partial_x^2 P(\mathbf{x}) + (\partial_x P(\mathbf{x}))^2 + \partial_y^2 P(\mathbf{x}) + (\partial_y P(\mathbf{x}))^2 + \kappa^2\epsilon(\mathbf{x})\right] e^{P(\mathbf{x})} = O\left(h^q\right)$$

The unknowns

$$\mathbf{x} = (x, y)$$

- $\varphi = \exp P$
- $P(\mathbf{x}) = \sum_{0\leq i+j\leq \deg P} \lambda_{i,j}(x-x_K)^i(y-y_K)^j$

✓ $\deg P$ and $\{\lambda_{i,j}\}_{0\leq i+j\leq \deg P}$



$$\Rightarrow N_{un} = \frac{(\deg P+1)(\deg P+2)}{2}$$

The equations

- $\mathcal{L}_H\varphi(\mathbf{x}) = O\left(h^q\right)$

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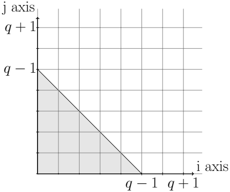


$$\Rightarrow N_{eq} = \frac{q(q+1)}{2}$$

Choice $\deg P = q+1$

Building a GPW : Structure of the non-linearity

$$\left[\partial_x^i \partial_y^j \left(\partial_x^2 P + (\partial_x P)^2 + \partial_y^2 P + (\partial_y P)^2 \right)\right](\mathbf{x}_K) = RHS_{i,j}$$

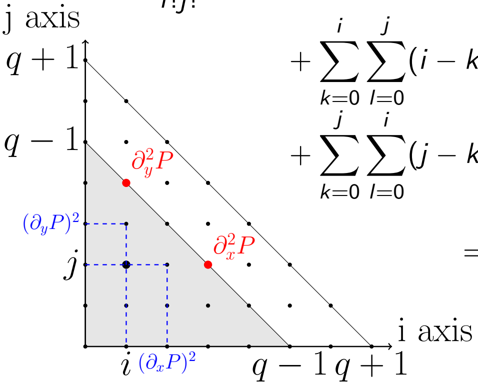
Identify linear and non-linear terms

$\forall (i,j) \in \mathbb{N}^2$ such that $0 \leq i+j \leq q-1$

$$-\kappa^2 \frac{\partial_x^i \partial_y^j \epsilon(\mathbf{x}_K)}{i!j!} = (i+2)(i+1)\lambda_{i+2,j} + (j+2)(j+1)\lambda_{i,j+2}$$

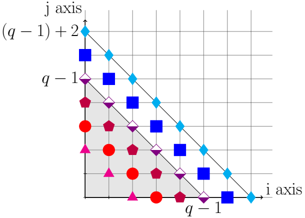
$$+ \sum_{k=0}^{i} \sum_{l=0}^{j} (i-k+1)(k+1)\lambda_{i-k+1,j-l}\lambda_{k+1,l}$$

$$+ \sum_{k=0}^{j} \sum_{l=0}^{i} (j-k+1)(k+1)\lambda_{i-l,j-k+1}\lambda_{l,k+1}$$



$\Rightarrow$ Hierarchy of linear sub-systems of increasing size

Building a GPW : Hierarchy of linear sub-systems



- For all (i,j) such that $i+j=\ell$
  Layer structure
- $X_\ell := [\lambda_{0,\ell+2}, \dots, \lambda_{\ell+2,0}]^T$
- A possible option
  Fix $\lambda_{0,\ell+2}$ and $\lambda_{1,\ell+1}$
- $\Pi_k := (k+2)(k+1)$

$$\underbrace{\left[\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ \hline \Pi_0 & & \Pi_\ell & & \\ & \ddots & & \ddots & \\ & & \Pi_\ell & & \Pi_0 \end{array}\right]}_{L_\ell} \underbrace{\left[\begin{array}{c} \lambda_{0,\ell+2} \\ \lambda_{1,\ell+1} \\ \hline \lambda_{2,\ell} \\ \vdots \\ \lambda_{\ell+2,0} \end{array}\right]}_{X_\ell} = \underbrace{\left[\begin{array}{c} * \\ * \\ \hline \\ RHS \\ \\ \end{array}\right]}_{R_\ell}$$

$\Rightarrow$ Solve by **forward substitution**

$$L_\ell X_\ell = R_\ell$$

Building a GPW : Algorithm *

Algorithm 1 Induction on the global degree $\ell = i + j$

Fix $\lambda_{0,0} = 0$, $(\lambda_{0,1}, \lambda_{1,0}) = \imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}\mathbf{d}$

for $\ell \leftarrow 0, q-1$ **do** $\triangleright\ q$

  Fix $\lambda_{0,\ell+2}$ and $\lambda_{1,\ell+1}$

  $\mathrm{R}_\ell \leftarrow f\left(\{\lambda_{i,j}\}_{i+j\leq\ell+1},\ \kappa, \left\{\partial_x^i\partial_y^j\epsilon(\mathbf{x}_K)\right\}_{i+j\leq\ell}\right)$ $\triangleright\ \mathcal{L}, \mathbf{x}_K$

  for $k \leftarrow 0, \ell$ **do**

    $\lambda_{k+2,\ell-k} := \dfrac{1}{\Pi_{\ell-k}}\Big(\mathrm{R}_\ell[k+2] - \Pi_k\lambda_{k,\ell-k+2}\Big)$ $\triangleright\ \mathcal{L}$

$P(x,y) \leftarrow \sum_{0\leq i+j\leq q+1} \lambda_{i,j}(x-x_K)^i(y-y_K)^j$ $\triangleright\ \mathbf{x}_K, q$

$\varphi(\mathbf{x}) \leftarrow \exp P(\mathbf{x})$

Summary

- **Analytic formula** for $\lambda_{i,j}$ $\oslash\ h$
- $\mathcal{L}_H\varphi = [-\Delta - \kappa^2\epsilon(\mathbf{x})]\varphi = O(h^q)$ $\oslash\ \{\lambda_{i,j}\}_{i\in\{0,1\}}$

Towards approximation properties

Normalization : Choice of $\{\lambda_{i,j}\}_{i\in\{0,1\}}$

- $(\lambda_{0,1}, \lambda_{1,0}) = \imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}\mathbf{d}$ with $\mathbf{d} = (\cos\theta, \sin\theta)$
- $\lambda_{i,j} = 0$ if $i + j \neq 1$ $\forall\ \theta$

Local set of approximated solutions

$\forall \ell$ such that $1 \leq \ell \leq p$, $\theta_\ell = 2\pi\ell/p$

$$\Rightarrow \mathbb{V}_K = Span\{\varphi_\ell\}_{1\leq\ell\leq p}$$

$\triangleright$ $\mathbf{x}_K$, p, q

- $\varphi_\ell(x, y) =$
  $\exp\left(\imath\kappa\sqrt{\epsilon(\mathbf{x}_K)}(\cos\theta_\ell(x - x_K) + \sin\theta_\ell(y - y_K)) + H.O.T\right)$
- $\Delta\varphi_\ell = \underbrace{\left(\partial_x^2 P_\ell + (\partial_x P_\ell)^2 + \partial_y^2 P_\ell + (\partial_y P_\ell)^2\right)}_{=-\kappa^2\epsilon + O(h^q)}\varphi$

Best approximation properties

$$\|u - u_a\| \leq Ch^n$$

Theorem (2D 2nd order) [IG & Sylvand, arxiv]

- $\forall n \in \mathbb{N},\ n > 0$
- $u \in \mathcal{C}^{n+1}$ such that $\mathcal{L}u = 0$, and $\mathcal{C}^n$ coefficients at $\mathbf{x}_K$
- $p = 2n + 1$ basis functions
- $q \geq \max(n-1, 1)$ such that $\mathcal{L}\varphi = O(h^q)\ \forall \varphi \in \mathbb{V}_{\mathbf{x}_K}$
- $\exists\ u_a \in \mathbb{V}_{\mathbf{x}_K}$ such that

$$\begin{cases} |u(\mathbf{x}) - u_a(\mathbf{x})| \leq C(n)\,|\mathbf{x} - \mathbf{x}_K|^{n+1}\,\|u\|_{\mathcal{C}^{n+1}} \\ |\nabla u(\mathbf{x}) - \nabla u_a(\mathbf{x})| \leq C(n)\,|\mathbf{x} - \mathbf{x}_K|^{n}\,\|u\|_{\mathcal{C}^{n+1}} \end{cases}$$

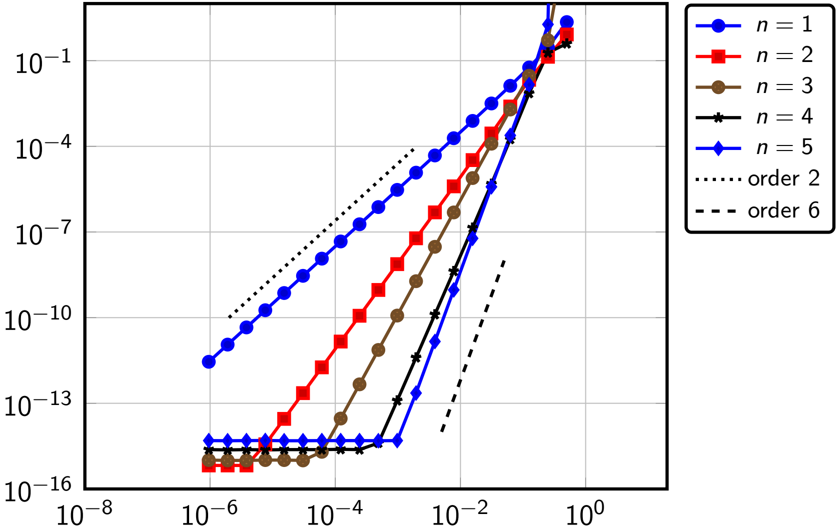
Proof

Numerical Results

$$\|u - u_a\|_{L^\infty\left(\{\mathbf{x}\in\mathbb{R}^2, |\mathbf{x}-\mathbf{x}_K|<h\}\right)} = O\left(h^{n+1}\right)$$

▷ u, $\mathcal{L}$
▷ n, p, q
▷ 50 random points $\mathbf{x}_K$

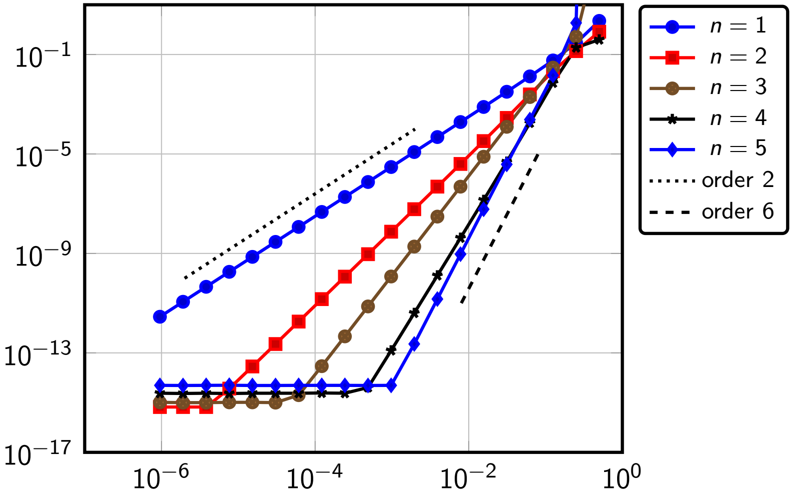
$\mathcal{L} = -\Delta + (x-1)$ and $u(x,y) = Ai(x)\cos y$



$\mathbf{x}_K \in [-2,2] \times [-2,2]$

with $p = 2n+1$ basis functions
at the order $q = \max(n-1, 1)$

$\mathcal{L} = \nabla \cdot (x^2 \nabla) + \cos y \partial_y - x \partial_x + (2x^2 + \sin y - \nu^2)$
$u(x, y) = J_\nu(x) \cos y$

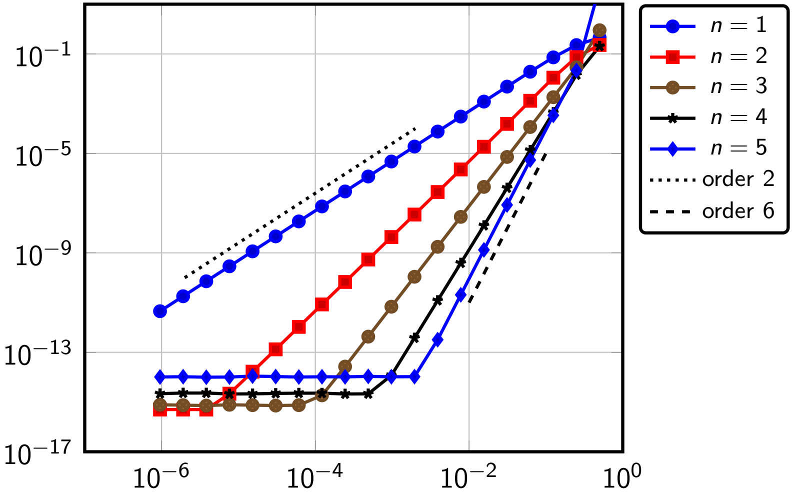


$\mathbf{x}_K \in [1, 4] \times [0, 2\pi]$
$\nu = 1$

with $p = 2n + 1$ basis functions
at the order $q = \max(n - 1, 1)$

$$\mathcal{L} = x^2\partial_x^2 + y^2\partial_y^2 + x\partial_x + y\partial_y + (x^2 + y^2 - 1)$$

$$u(x, y) = J_0(x)J_1(y)$$

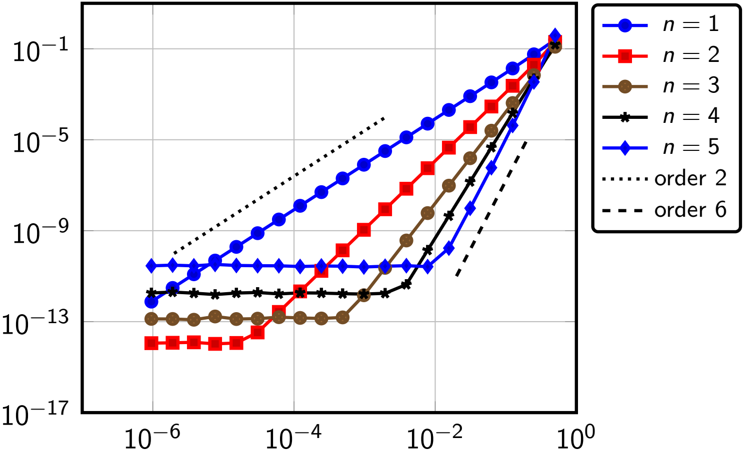


$\mathbf{x}_K \in [1, 3] \times [0, 3]$

with $p = 2n + 1$ basis functions at the order $q = \max(n - 1, 1)$

$$\mathcal{L} = (\partial_x^2 + 2\cos x \sin y \partial_x \partial_y - 2\partial_y^2 + \cos x(y^5 + \cos y)\partial_x + \sin y(x^3 - \sin x)\partial_y + (4\sin x \cos y - 1 - x^3 \cos y + y^5 \sin x$$

$$u(x, y) = \cos(x)\sin(y)$$



$\mathbf{x}_K \in [-1, 1] \times [-1, 1]$

with $p = 2n + 1$ basis functions at the order $q = \max(n - 1, 1)$

Convergence of the numerical method

Convergence of the GPW+T method

GPW space $\mathbb{V}_h$ [IG & Després '14]

$$\mathbb{V}_h \not\subset \mathbb{H} = \{v \in L^2(\Omega) : v \text{ smooth and } -\Delta_h v - \kappa^2 \epsilon v = 0, \forall K \in \mathcal{T}_h\}$$

Stabilized formulation

$$\mathcal{B}_h(u,v) = \mathcal{A}_h(u,v) + \imath\kappa \int_\Omega \gamma(\Delta_h u + \kappa^2 \epsilon u)\overline{(\Delta_h v + \kappa^2 \epsilon v)} dS$$

Coercivity and continuity

$$|||u|||^2 \leq |\mathcal{B}_h(u,u))|$$

$$|\mathcal{B}_h(u,v))| \leq C|||u|||_+ |||v|||$$

Theorem (smooth solution) [IG & Monk '16]

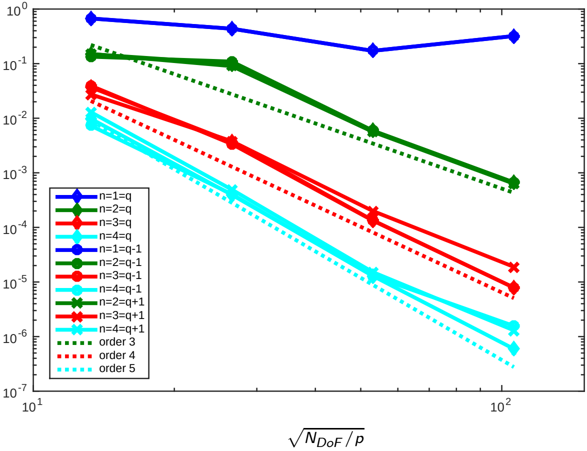
$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^n$$

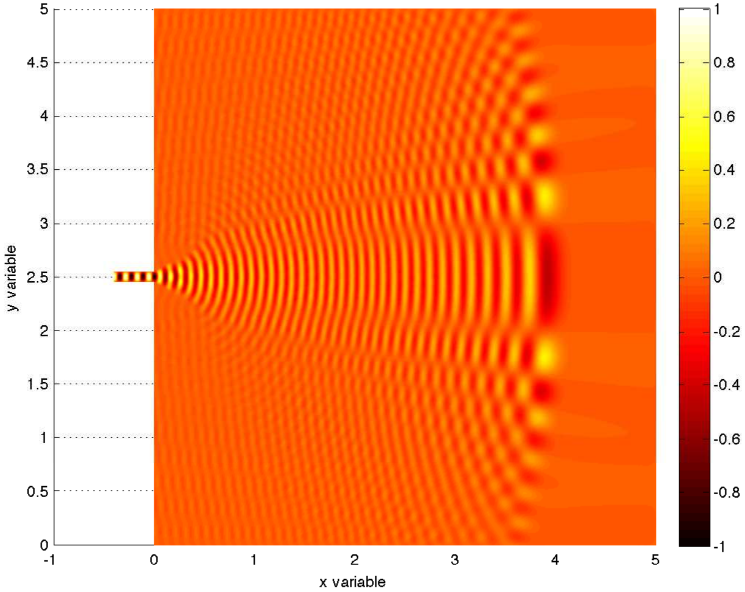
provided that $\alpha = \beta = \delta = 1/2$, $\gamma = h^3$, $p = 2n+1$, $n \geq 2$ and $q = n+1$

GPW + T

2D test case : $L^2(\Omega)$ norm convergence

$u_{ex}(\mathbf{x}) = Ai(\kappa^{2/3}y)$, $\kappa = 15$





y variable
x variable

Construction of GPWs
Maxwell's equation

- $\mathbf{x} \in \mathbb{R}^3$
- Vector valued PDE

The cold plasma model

- Equation for $\mathbf{E}(\mathbf{x})$

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2}\mathbf{E} + \frac{\omega}{c^2}\frac{e^2 n_0}{m_e \epsilon_0}\mathbb{S}\mathbf{E} = 0$$

Define $\omega_p^2 := \dfrac{e^2 n_0}{m_e \epsilon_0}$

- Dielectric tensor $\mathbb{K} := \mathbb{I} - \dfrac{\omega_p^2}{\omega^2}\mathbb{S}$

with $\omega_c := \dfrac{eB_0}{m_e}$

$$S = 1 - \frac{\omega_p^2}{\omega}\left(\frac{\omega}{\omega^2 - \omega_c^2}\right), \; D = -\frac{\omega_p^2}{\omega}\left(\frac{\omega_c}{\omega^2 - \omega_c^2}\right), \; P = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\Rightarrow \mathbb{K} = \begin{pmatrix} S & -\imath D & 0 \\ \imath D & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$\nabla \times \nabla \times \mathbf{E} - \left(\frac{\omega}{c}\right)^2 \mathbb{K}\mathbf{E} = 0$$

Local definition of a **vector valued** GPW

Constant coefficient

$$\mathcal{L} = \nabla \times \nabla \times -\kappa^2 \epsilon_K$$

$$\phi(\mathbf{x}) = \mathbf{A} \exp\left(\imath\kappa\sqrt{\epsilon_K}\mathbf{d}\cdot\mathbf{x}\right)$$

$$\mathbf{d} \times \mathbf{d} \times \mathbf{A} - \kappa^2 \epsilon_K \mathbf{A} = 0$$

$$\Rightarrow \mathcal{L}\phi = 0$$

Variable coefficient

$$\mathcal{L} = \nabla \times \nabla \times -\kappa^2 \epsilon(\mathbf{x})$$

$$\phi_\alpha(\mathbf{x}) = \mathbf{A}_\alpha(\mathbf{x}) \exp\left(\imath\kappa\mathbf{S}_\alpha(\mathbf{x})\right)$$

$$\Rightarrow \mathcal{L}\phi \approx 0$$

Towards Generalized Plane Waves

- 6 scalar polynomials $\{\mathbf{A}_\alpha, \mathbf{S}_\alpha\}_{\alpha\in\{x,y,z\}}$
- 3 scalar constraints $\{(\mathcal{L}\phi)_\alpha\}_{\alpha\in\{x,y,z\}}$
- Compatibility condition **A**, **S**

✓ $\begin{cases} \text{Taylor expansion at } \mathbf{x}_K \\ \mathcal{L}\phi(\mathbf{x}) = O\left(h^q\right) \end{cases}$

Phase function assumption $\mathcal{L}_\epsilon \boldsymbol{\phi} = \nabla \times \nabla \times \boldsymbol{\phi} - \kappa^2 \epsilon \boldsymbol{\phi}$

$$\phi_\alpha = A_\alpha(\mathbf{x}) \exp \imath\kappa S(\mathbf{x}), \quad \nabla S = \text{constant}$$

$$\begin{aligned} \mathcal{L}_\epsilon \boldsymbol{\phi} = \kappa^2 &\Big[-\nabla S \times \nabla S \times \mathbf{A} - \epsilon \mathbf{A} \Big] e^{i\kappa S} \\ &+ i\kappa \Big[\nabla S \times \nabla \times \mathbf{A} + \nabla \times \nabla S \times \mathbf{A} \Big] e^{i\kappa S} \\ &+ \Big[\nabla \times \nabla \times \mathbf{A} \Big] e^{i\kappa S} \end{aligned}$$

I. Local Eikonal equation $\Rightarrow$ Phase

Phase function assumption $\mathcal{L}_\epsilon \phi = \nabla \times \nabla \times \phi - \kappa^2 \epsilon \phi$

$$\phi_\alpha = A_\alpha(\mathbf{x}) \exp \imath\kappa S(\mathbf{x}), \quad \nabla S = \text{constant}$$

$$\begin{aligned}\mathcal{L}_\epsilon \phi = \kappa^2 \Big[-\nabla S \times \nabla S \times \mathbf{A} - \epsilon \mathbf{A} \Big] e^{i\kappa S} \\ + i\kappa \Big[\nabla S \times \nabla \times \mathbf{A} + \nabla \times \nabla S \times \mathbf{A} \Big] e^{i\kappa S} \\ + \Big[\nabla \times \nabla \times \mathbf{A} \Big] e^{i\kappa S}\end{aligned}$$

I. Local Eikonal equation $\Rightarrow$ Phase

II. Layer structure $\Rightarrow$ Amplitude

$$\begin{aligned}&\nabla \times \nabla \times \mathbf{A} \\ &= \kappa^2 \Big[\nabla S \times \nabla S \times \mathbf{A} + \epsilon \mathbf{A} \Big] - i\kappa \Big[\nabla S \times \nabla \times \mathbf{A} + \nabla \times \nabla S \times \mathbf{A} \Big]\end{aligned}$$

Phase function assumption $\mathcal{L}_\epsilon \boldsymbol{\phi} = \nabla \times \nabla \times \boldsymbol{\phi} - \kappa^2 \epsilon \boldsymbol{\phi}$

$$\phi_\alpha = A_\alpha(\mathbf{x}) \exp \imath\kappa S(\mathbf{x}), \quad \nabla S = \text{constant}$$

$$\begin{aligned}
\mathcal{L}_\epsilon \boldsymbol{\phi} = \kappa^2 \Big[- \nabla S \times \nabla S \times \mathbf{A} - \epsilon \mathbf{A} \Big] e^{i\kappa S} \\
+ i\kappa \Big[\nabla S \times \nabla \times \mathbf{A} + \nabla \times \nabla S \times \mathbf{A} \Big] e^{i\kappa S} \\
+ \Big[\nabla \times \nabla \times \mathbf{A} \Big] e^{i\kappa S}
\end{aligned}$$

I. Local Eikonal equation $\Rightarrow$ Phase

II. Layer structure $\Rightarrow$ Amplitude

$$\begin{aligned}
&\nabla \times \nabla \times \mathbf{A} \\
&= \kappa^2 \Big[\nabla S \times \nabla S \times \mathbf{A} + \epsilon \mathbf{A} \Big] - i\kappa \Big[\nabla S \times \nabla \times \mathbf{A} + \nabla \times \nabla S \times \mathbf{A} \Big]
\end{aligned}$$

$\Rightarrow$ Towards a hierarchy of linear sub-systems

Building a GPW : The system $\nabla \times \nabla \times \mathbf{A}$

The unknowns $\mathbf{x} = (x, y, z)$

- $\mathbf{A}_\alpha(\mathbf{x}) = \sum_{0 \le i+j+k \le \mathrm{deg}_\alpha} \lambda^\alpha_{i,j,k} (x - x_K)^i (y - y_K)^j (z - z_K)^k$

✓ deg_α and $\{\lambda^\alpha_{i,j,k}\}_{0 \le i+j+k \le \mathrm{deg}_\alpha}$

$$\Rightarrow N_{un} = \sum_\alpha \frac{1}{6} (\mathrm{deg}_\alpha + 1)(\mathrm{deg}_\alpha + 2)(\mathrm{deg}_\alpha + 3)$$

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$$\Rightarrow N_{un} = \sum_\alpha \frac{1}{6} (\deg_\alpha + 1)(\deg_\alpha + 2)(\deg_\alpha + 3)$$

The equations

- $\mathcal{L}\phi(\mathbf{x}) = O\left(h^q\right)$

✓ $\forall (i, j, k)$ such that $0 \leq i + j + k \leq q - 1$

$$\partial_x^i \partial_y^j \partial_z^k \left[\nabla \times \nabla \times \mathbf{A}\right](\mathbf{x}_K) = \mathbf{F}_{ijk}$$

$$\Rightarrow N_{eq} = 3 \frac{q(q+1)(q+2)}{6}$$

Building a GPW : The system $\nabla \times \nabla \times \mathbf{A}$

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$$\Rightarrow N_{un} = \sum_\alpha \frac{1}{6} (\deg_\alpha + 1)(\deg_\alpha + 2)(\deg_\alpha + 3)$$

The equations

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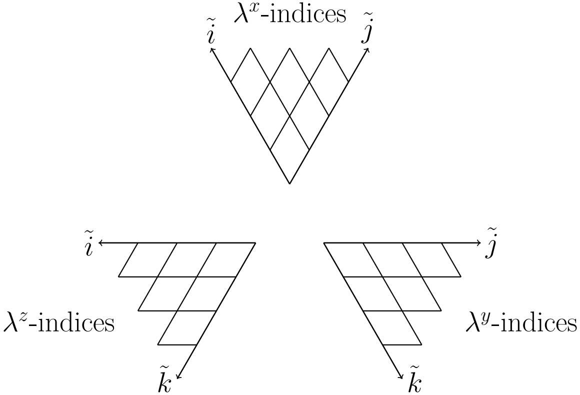
$$\Rightarrow N_{eq} = 3 \frac{q(q+1)(q+2)}{6}$$

Choice $\forall \alpha \ \deg_\alpha = q + 1$

Building a GPW : The unknowns, Hierarchy

- Splitting by level ℓ
  $\{\lambda^x_{i,j,\ell+2-i-j}, 0 \leq i \leq \ell+2, 0 \leq j \leq \ell+2-i\}$,
  $\{\lambda^y_{\ell+2-j-k,j,k}, 0 \leq j \leq \ell+2, 0 \leq k \leq \ell+2-j\}$,
  $\{\lambda^z_{i,\ell+2-k-i,k}, 0 \leq k \leq \ell+2, 0 \leq i \leq \ell+2-k\}$

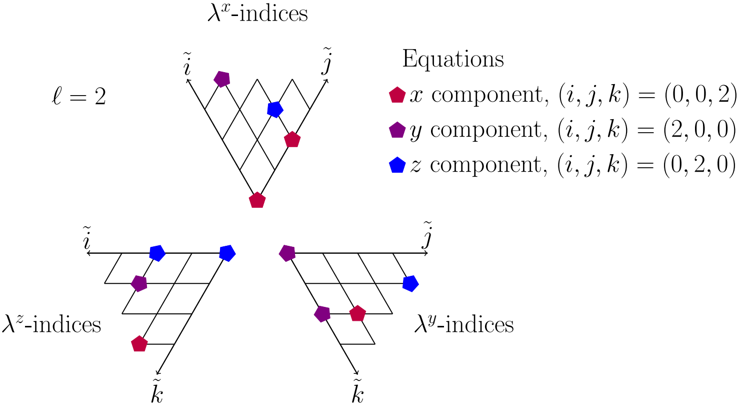


Symmetric representation

$\tilde{i} + \tilde{j} + \tilde{k} = 4$
$\ell = 2$

Building a GPW : The equations

- Coupling $i + j + k = \ell$
  $-(j+2)(j+1)\lambda^x_{i,j+2,k} - (k+2)(k+1)\lambda^x_{i,j,k+2}$
  $+(i+1)(j+1)\lambda^y_{i+1,j+1,k} + (i+1)(k+1)\lambda^z_{i+1,j,k+1} = \mathbf{F}^x_{ijk}$



Building a vector valued GPW : an ill-posed formulation

✓ The Laplacian operator

$$\Delta : \mathbb{C}_{q+1}[X, Y] \to \mathbb{C}_{q-1}[X, Y]$$

✗ The curl-curl operator

$$\nabla \times \nabla \times : (\mathbb{C}_{q+1}[X, Y, Z])^3 \to (\mathbb{C}_{q-1}[X, Y, Z])^3$$

1. $-(j+1)j\lambda^x_{i,j+1,k+1} - (k+3)(k+2)\lambda^x_{i,j-1,k+3}$
   $+(i+1)j\lambda^y_{i+1,j,k+1} + (i+1)(k+2)\lambda^z_{i+1,j-1,k+2}$
2. $-(k+3)(k+2)\lambda^y_{i-1,j,k+3} - (i+1)i\lambda^y_{i+1,j,k+1}$
   $+(j+1)(k+2)\lambda^z_{i-1,j+1,k+2} + (j+1)i\lambda^x_{i,j+1,k+1}$
3. $-(i+1)i\lambda^z_{i+1,j-1,k+2} - (j+1)j\lambda^z_{i,j+2,k}$
   $+(k+3)i\lambda^x_{i,j-1,k+3} + (k+3)j\lambda^y_{i-1,j,k+3}$

$$\frac{1.}{j(k+2)} + \frac{2.}{i(k+2)} + \frac{3.}{ij} = 0$$

What's next then ?

✗ $\nabla \times \nabla \times \phi = -\Delta\phi + \nabla(\nabla \cdot \phi)$

Different ansatz

- $\phi_\alpha = A_\alpha(\mathbf{x}) \exp \imath\kappa S(\mathbf{x})$, with $\nabla S =$ constant
- $\phi_\alpha = \mathbf{A}_\alpha(\mathbf{x}) \exp\left(\imath\kappa \mathbf{S}_\alpha(\mathbf{x})\right)$, with $\mathbf{A} =$ constant
- $\phi_\alpha = \mathbf{A}_\alpha(\mathbf{x}) \exp\left(\imath\kappa \mathbf{S}_\alpha(\mathbf{x})\right)$

Different equation

- $\mathcal{L} = \nabla \times \nabla \times -\kappa^2\epsilon$
- $\mathcal{L} = \nabla \times \mu^{-1}\nabla \times -\kappa^2\epsilon$, with μ^{-1} scalar-valued
- $\mathcal{L} = \nabla \times (\boldsymbol{M}\nabla\times) - \kappa^2\epsilon$, with $\boldsymbol{M}$ matrix-valued

Future work

GPWs

- Approximation properties in 3D
  - With G. Sylvand (Airbus)
- Vector valued equations
  - With J.-F. Fritsch (ENSTA)
- Conditioning & Adaptive discretization
- High frequency regime and numerical integration
- Time domain
  - With A. Moiola (U. Pavia), P. Stocker (U. Vienna)

Trefftz + GPWs

- *h*-convergence and *p*-convergence
  - With P. Monk (U. Delaware), R. Hiptmair (ETH Zurich)
- Parallel implementation, Preconditioning
  - With G. Stadler (NYU)

Thank you.