Numerics, a model and statistics for the stochastically forced vorticity equation

Gabriel Lord

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1. Results for PDE/ SPDE
2. A reduced model
3. Comparison SDE/SPDE
4. ... propose different timestep approximation.

Vorticity Equations

Consider

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \qquad \mathbf{u} = \begin{pmatrix} \partial_y(-\Delta^{-1}) \\ -\partial_x(-\Delta^{-1}) \end{pmatrix} \omega. \qquad (1)$$

On the asymmetric torus

$$(x, y) \in D_\delta := [0, 2\pi\delta] \times [0, 2\pi]$$

with $\delta \approx 1$.
Periodic boundary conditions, and viscosity $0 < \nu \ll 1$.
The relation between $\mathbf{u}$ and ω is known as the Biot-Savart law.

Stochastically Forced Vorticity

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega + \frac{\partial \mathcal{W}}{\partial t}, \qquad \mathbf{u} = \begin{pmatrix} \partial_y (-\Delta^{-1}) \\ -\partial_x (-\Delta^{-1}) \end{pmatrix} \omega. \tag{2}$$

The noise is white in time, colored in space, and takes the form, for $\mathbf{k} = (k_1, k_2) \neq (0, 0)$,

$$\mathcal{W}(t, x, y) = \sqrt{2\nu} \sum_{\mathbf{k} \in \mathcal{K} \subset \mathbb{Z}^2 \setminus \{(0,0)\}} \sigma_{\mathbf{k}} e^{\mathrm{i}(k_1 x/\delta + k_2 y)} \beta_{\mathbf{k}}(t). \tag{3}$$

$\beta(t) = \{\beta_{\mathbf{k}}(t)\}$ are i.i.d. Wiener processes.

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Spatial correlation $\sigma_{\mathbf{k}}$
We assume there exist fixed positive constants C_0 and α_0 such that

$$|\sigma_{\mathbf{k}}| \leq C_0 e^{-\alpha_0 |\mathbf{k}|^2}.$$

Solutions then analytic in space [Mattingly 2002].

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Solutions then analytic in space [Mattingly 2002].
To insure the vorticity remains real valued for all times we impose

$$\bar{\sigma}_{\mathbf{k}} = \sigma_{-\mathbf{k}} \quad \bar{\beta}_{\mathbf{k}} = \beta_{-\mathbf{k}}.$$

We choose $\sigma_{(0,0)} = 0$ so property $\int_{D_\delta} \omega = 0$ is preserved.

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The x- and y-bar states

$$\omega_{xbar}(x,t) = e^{-\frac{\nu}{\delta^2}t}\sin(x/\delta), \qquad \omega_{ybar}(y,t) = e^{-\nu t}\sin y,$$

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Dipoles are given by

$$\omega_{dipole}(x,y,t) = e^{-\frac{\nu}{\delta^2}t}\sin(x/\delta) + e^{-\nu t}\sin y,$$

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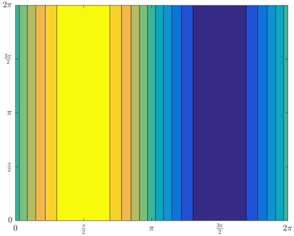
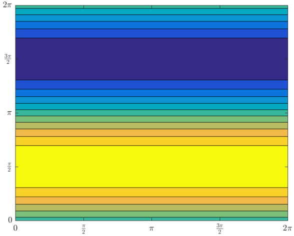
X-bar, Y-bar and Dipoles: with $\delta = 1$





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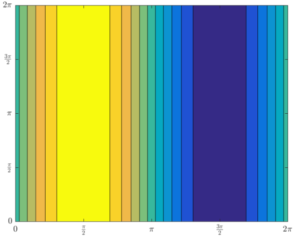

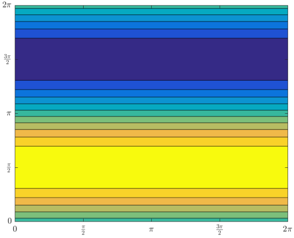

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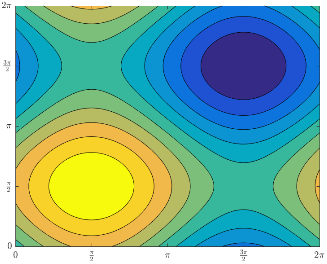

Dipole: $\omega_{dipole} = \sin(x) + \sin(y)$

SPDE in Fourier space :

$$\dot{\hat{\omega}}_{\mathbf{k}} = -\frac{\nu}{\delta^2}|\mathbf{k}|_\delta^2 \hat{\omega}_{\mathbf{k}} - \frac{\delta}{2} \sum_{\mathbf{j}+\mathbf{l}=\mathbf{k}} \langle \mathbf{j}^\perp, \mathbf{l} \rangle \left(\frac{1}{|\mathbf{l}|_\delta^2} - \frac{1}{|\mathbf{j}|_\delta^2} \right) \hat{\omega}_{\mathbf{j}} \hat{\omega}_{\mathbf{l}} + \sqrt{2\nu}\sigma_{\mathbf{k}} \dot{\beta}_{\mathbf{k}},$$

where

$$|\mathbf{k}|_\delta^2 = k_1^2 + \delta^2 k_2^2, \qquad \mathbf{k}^\perp = (k_2, -k_1). \tag{4}$$

- Low Modes :

x-**bar states** : $e^{-\frac{\nu}{\delta^2}t}\cos(x/\delta)$ and $e^{-\frac{\nu}{\delta^2}t}\sin(x/\delta)$ correspond to solutions with energy only in the $\mathbf{k} = (\pm 1, 0)$ modes.

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Dipole states : energy in both the $\mathbf{k} = (\pm 1, 0)$ and $\mathbf{k} = (0, \pm 1)$.
These "low modes" have lowest value of $|\mathbf{k}|_\delta$ defined by (4).
- High Mode : Any mode $\hat{\omega}_{\mathbf{k}}$ with $|\mathbf{k}| > \max\{1, \delta^2\}$

Stochastic Order Parameter

To measure the relative energy in the low modes, define

$$Z_{vort}(t) := \frac{|\hat{\omega}_{(1,0)}(t)|^2}{|\hat{\omega}_{(1,0)}(t)|^2 + |\hat{\omega}_{(0,1)}(t)|^2},$$

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- If $Z_{vort}(t)$ falls toward 0, the system would be observed to be in a y-bar state.
- If $Z_{vort}(t)$ instead stays near $1/2$, the system is in a dipole state with relative energy in the low modes comparable in magnitude.

Numerics

Similar to [Bouchet & Simonnet 09], take $\sum_{\{\mathbf{k}\in\mathcal{K}\}} e^{-\alpha_0|\mathbf{K}|^2} = 1$.
Space Discretization : Fourier
Time Discretization : Finely discretized Tamed Euler-Maruyama method (see later).

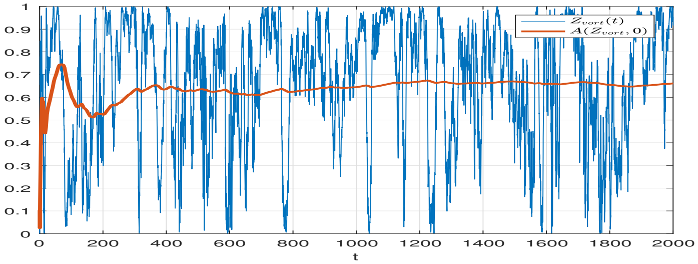
$$\bar{Z}_{vort}(t) = \frac{1}{N}\sum_{i=1}^{N} Z^i_{vort}(t),$$

Time averages of these Monte Carlo averages:
Introduce a "burn-in time", t_{burn}
Define this time average for any function $f(t)$ defined on $t_{burn} \leq t \leq T$ to be

$$A(f, t_{burn}) := \frac{1}{T - t_{burn}} \int_{t_{burn}}^{T} f(t)\,\mathrm{d}t.$$

- $\delta = 1$: dipole solution
- $\delta > 1$: *x*-bar solution
- $\delta < 1$: *y*-bar solution

Vorticity : $\delta > 1$



An individual trajectory transitions among quasi-stationary states.

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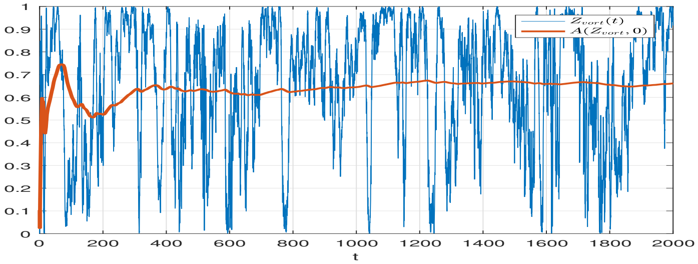

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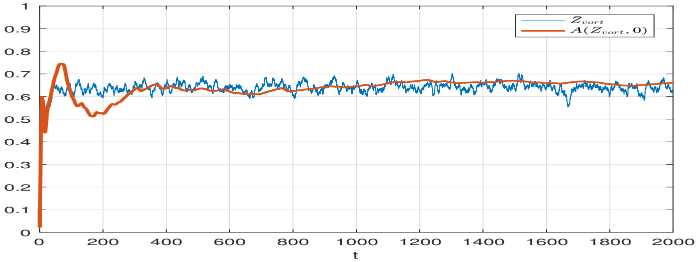
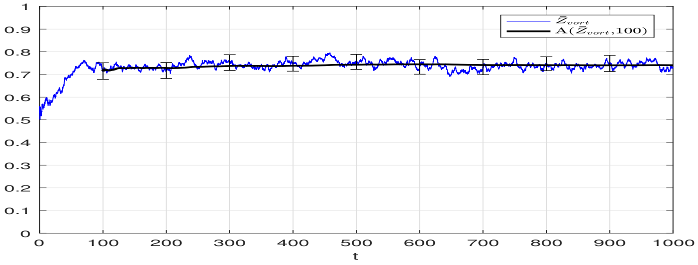
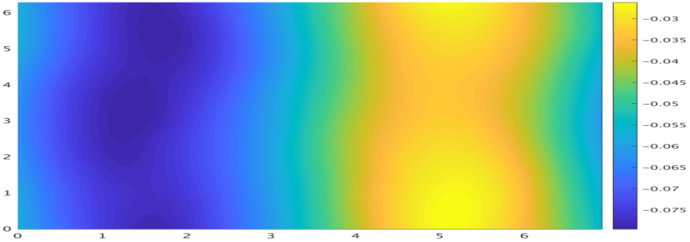


On average, the system is close to an x-bar state.

Vorticity : $\delta = 1.1$

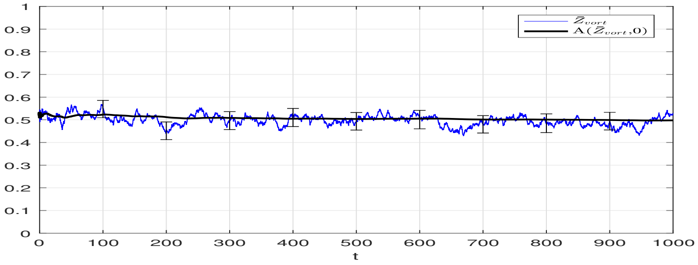


$\bar{Z}_{vort}(t)$ with 95% confidence interval.



Average contour plot of vorticity.

Vorticity: $\delta = 1$



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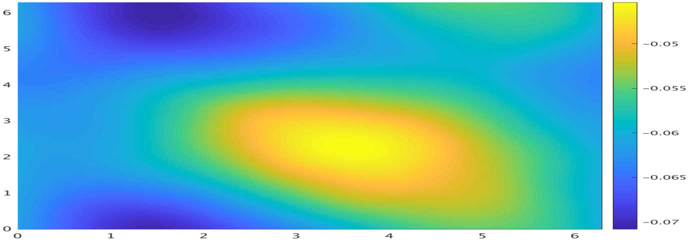

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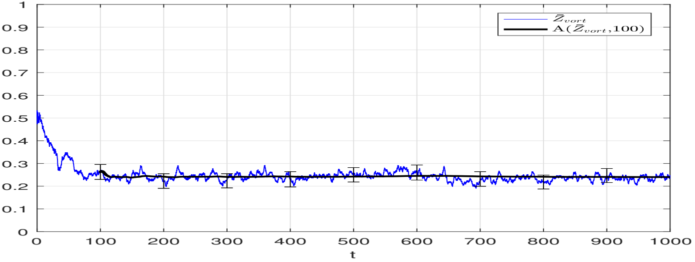
Vorticity: $\delta = 0.9$



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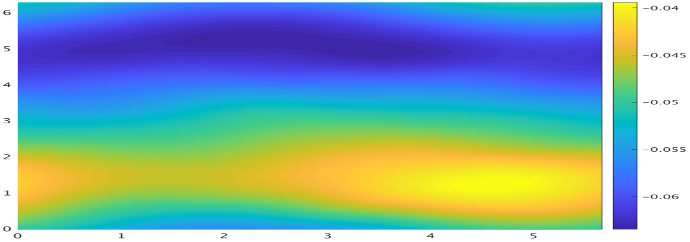

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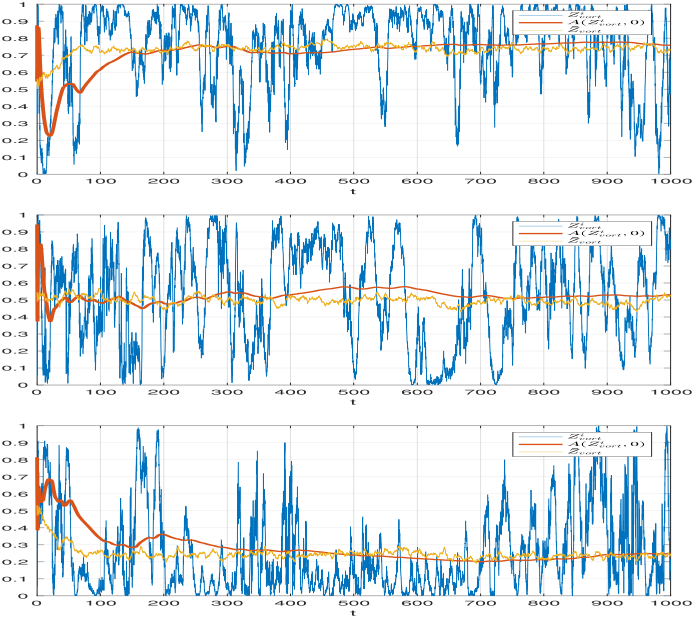
Vorticity : $\delta = 1.1$, $\delta = 1.0$, $\delta = 0.9$



Finite Dimensional system

Use lowest eight Fourier modes:

$$\begin{aligned}\omega_1 &:= \hat{\omega}_{(1,0)}, \quad \omega_2 := \hat{\omega}_{(-1,0)}, \quad \omega_3 := \hat{\omega}_{(0,1)}, \quad \omega_4 := \hat{\omega}_{(0,-1)}, \\ \omega_5 &:= \hat{\omega}_{(1,1)}, \quad \omega_6 := \hat{\omega}_{(-1,1)}, \quad \omega_7 := \hat{\omega}_{(1,-1)}, \quad \omega_8 := \hat{\omega}_{(-1,-1)}.\end{aligned}$$

$\omega_{1,2,3,4}$ correspond to the low modes
$\omega_{5,6,7,8}$ represent the role of all the high modes.

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$\omega_{1,2,3,4}$ correspond to the low modes
$\omega_{5,6,7,8}$ represent the role of all the high modes.
- Since the solution $\omega(x, y)$ is real valued, the following complex conjugacy relationship still hold,

$$\omega_1 = \bar{\omega}_2, \quad \omega_3 = \bar{\omega}_4, \quad \omega_5 = \bar{\omega}_8, \quad \omega_7 = \bar{\omega}_8.$$

Based on centre manifold from deterministic case.

SDE Model

Set $\alpha_1 = (1 + \delta^2)$, $\alpha^4 = (1 + 4\delta^2)$, $\alpha_4 = (4 + \delta^2)$ and $\beta = (\delta^2 - 1)$ then

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Set $\alpha_1 = (1+\delta^2)$, $\alpha^4 = (1+4\delta^2)$, $\alpha_4 = (4+\delta^2)$ and $\beta = (\delta^2 - 1)$ then

$$\dot{\omega}_1 = -\frac{\nu}{\delta^2}\omega_1 + \frac{1}{\delta\alpha_1}[\omega_3\omega_7 - \bar{\omega}_3\omega_5] + \frac{3\delta^6}{2\nu\alpha_4\alpha_1^2}\omega_1(|\omega_5|^2 + |\omega_7|^2) + \sqrt{2\nu}\sigma_1\dot{W}_1$$

$$\dot{\omega}_3 = -\nu\omega_3 + \frac{\delta^3}{\alpha_1}[\bar{\omega}_1\omega_5 - \omega_1\bar{\omega}_7] + \frac{3\delta^2}{2\nu\alpha^4\alpha_1^2}\omega_3(|\omega_5|^2 + |\omega_7|^2) + \sqrt{2\nu}\sigma_3\dot{W}_3$$

$$\dot{\omega}_5 = -\nu\frac{\alpha_1}{\delta^2}\omega_5 - \frac{\beta}{\delta}\omega_1\omega_3 - \frac{\delta^6(3+\delta^2)}{2\nu\alpha_4\alpha_1}\omega_5|\omega_1|^2 - \frac{1+3\delta^2}{2\nu\delta^2\alpha^4\alpha_1}\omega_5|\omega_3|^2 + \sqrt{2\nu}\sigma_5\dot{W}_5$$

$$\dot{\omega}_7 = -\nu\frac{\alpha_1}{\delta^2}\omega_7 + \frac{\beta}{\delta}\omega_1\bar{\omega}_3 - \frac{\delta^6(3+\delta^2)}{2\nu\alpha_4\alpha_1}\omega_7|\omega_1|^2 - \frac{1+3\delta^2}{2\nu\delta^2\alpha^4\alpha_1}\omega_7|\omega_3|^2 + \sqrt{2\nu}\sigma_7\dot{W}_7.$$

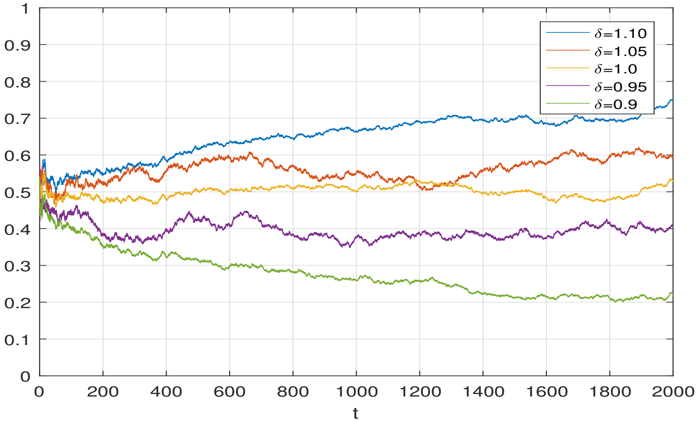
Take

$$\sigma_{1,3} = e^{-\alpha_0} \quad \text{and} \quad \sigma_{5,7} = e^{-2\alpha_0}.$$

Order parameter for SDE model :

$$Z_{red}(t) := \frac{|\omega_1(t)|^2}{|\omega_1(t)|^2 + |\omega_3(t)|^2}.$$

Reduced SDEs

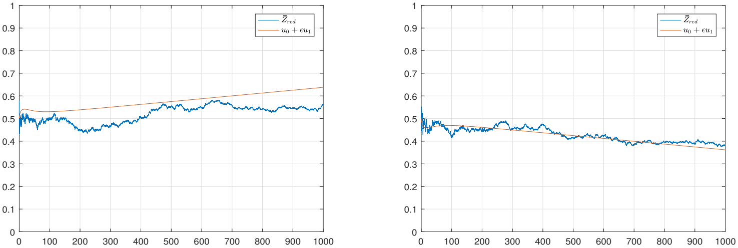


Simulation of $\bar{Z}_{red}(t)$ with noise for $\nu = 0.001$.
For $\delta > 1$, the order parameter increases : x-bar state.
For $\delta < 1$, the order parameter decreases : y-bar state.
When $\delta = 1$, $\bar{Z}_{red}(t)$ remains near $1/2$ indicating a dipole state.

Pertubation Analysis

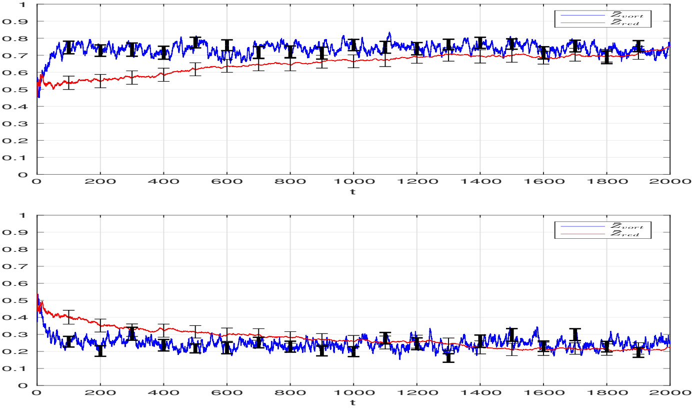
Use backward Kolmogorov equation to derive PDEs for insight on $Z_{red}(t)$ as $\delta \to 1$ ...

0. Set $\delta^2 = 1 \pm \epsilon$
1. Scale – get fast-slow system
2. Obtain backward Kolmogorov equation
3. Average out the fast variables and look at $\mathbb{E}[Z_{red}(t)]$ as $\delta \to 1$



evolves to y-bar state $\hat{\epsilon} = 0.1$ evolves to x-bar state $\hat{\epsilon} = -0.1$.

Comparison SPDE & SDE :$\delta = 1.1$, $\delta = 1.0$, $\delta = 0.9$



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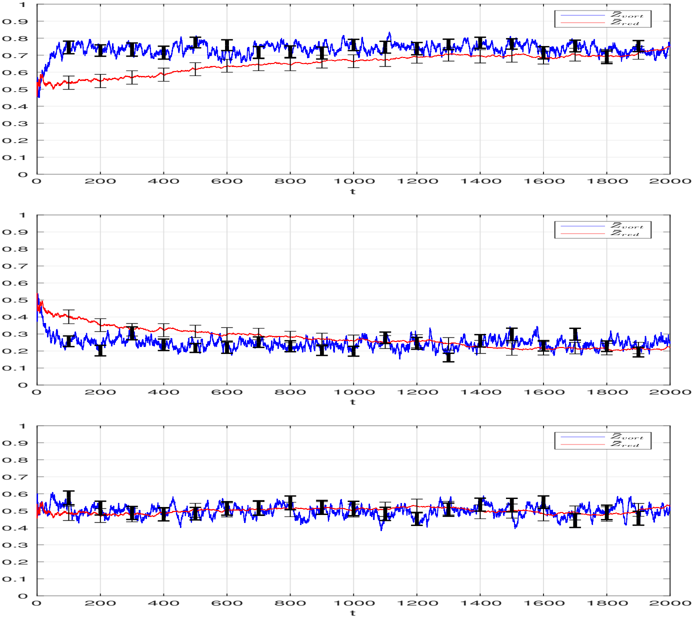

A note on numerics...

We used Tamed Euler methods and fixed steps: the S(P)DEs are nonlinear with non-global Lipschitz drifts

$$du = [Au + f(u)]\,dt + dW.$$

- What goes wrong with EM ? Consider simple 1D SDE

$$dX = -X^3 dt + \sigma dW$$

A note on numerics...

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$$du = [Au + f(u)]\,dt + dW.$$

- What goes wrong with EM ? Consider simple 1D SDE

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- Outside of the basin of attraction : oscillation and growth !

Tamed Euler-Maruyama methods

[Hutzenthaler, Jentzen, Kloeden], [Hutzenthaler,Jentzen], [Gyongy, Sabanis, Siska], etc

- Idea : introduce higher order perturbation of the flow

Simplest : Drift-tamed Euler-Maruyama

$$Y_{n+1} = Y_n + \frac{\Delta t}{1 + \Delta t \| f(Y_n) \|} f(Y_n) + g(Y_n) \Delta W_{n+1}$$

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Prove moment bounds

$$\sup_{n \in \mathbb{N}} \sup_{n \in \{0,1,\ldots,N\}} \mathbb{E}[\|Y_n\|^p] < \infty. \tag{5}$$

Strong convergence

$$\left(\mathbb{E}\left[\sup_{t \in [0,T]} \|X(t) - \bar{Y}_t\|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2}$$

- but use a finite Δt in computations.

Adaptive Alternative

Rather than adapt the flow – adapt the timestep

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$$\Delta t_{\max} = \rho \Delta t_{\min} \qquad 0 < \rho \in \mathbb{R}$$

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If $\Delta t < \Delta t_{\min}$ use backstop method.
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- Admissible step: Take Δt_{n+1} such that

$$\|f(Y_n)\|^2 \leq R_1 + R_2 \|Y_n\|^2.$$

For example $\Delta t_{n+1} \leq \Delta t_{\max} \|f(Y_n)\|^{-1}$.

- Convergence as $\Delta t_{\max} \to 0$. [C. Kelly & G.L. IMAJNA 2017]

SPDE : [S.Campbell & G. L. 2018]

Consider SPDE

$$du = [Au + f(u)]\, dt + G(u)dW.$$

with a mild solution

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(u(s))ds + \int_0^t e^{(t-s)} G(u(s))ds$$

We take trace class noise W and write

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\phi_j(x)\beta_j(t).$$

For f we take : one sided Lipschitz and growth condition
$\|Df(x)\| \leq c(1 + \|x\|^c)$
For G - a global Lipschitz condition.
Under these conditions:-
Existence and bounded moments in [Jentzen and Pusnik].

Space discretization- FE/Spectral

$$dX^h = \Big(A_h X^h + P_h F(X^h)\Big)\, dt + P_h B(X^h) dW.$$

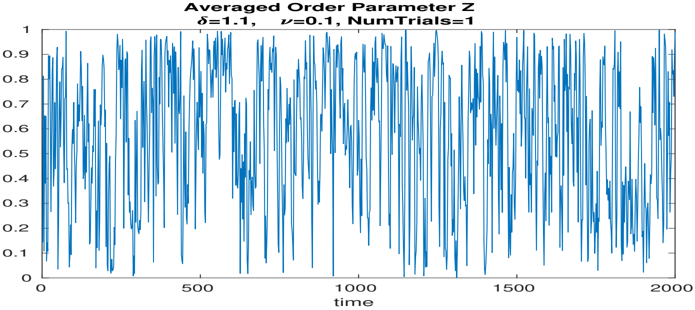
Discretize mild solution in time by Exponential method:

$$X^h_{n+1} = e^{\Delta t_n A_h} X^h_n + A_h^{-1}(e^{\Delta t_n A_h} - I) P_h F(X^h_n) + P_h B(X^h_n) \Delta W_{n+1}.$$

Theorem With admissible timestep Δt_n have strong convergence of exponential method & for initial data $X_0 \in L_2(D(A)^{\gamma/2})$, $0 \leq \gamma \leq 1$

$$\left(\mathbb{E}\left[\|X(T) - Y_N\|^2\right]\right)^{1/2} \leq C(\Delta t_{\max}^{\gamma/2} + h^r).$$

Numerical evidence



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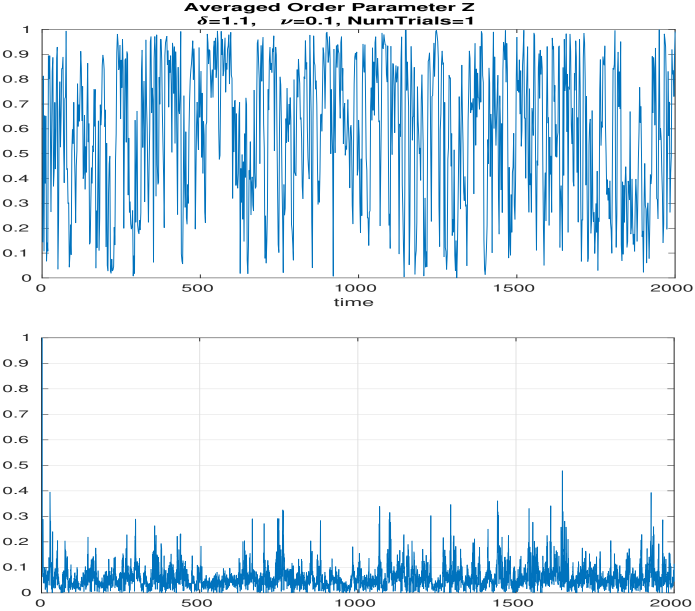

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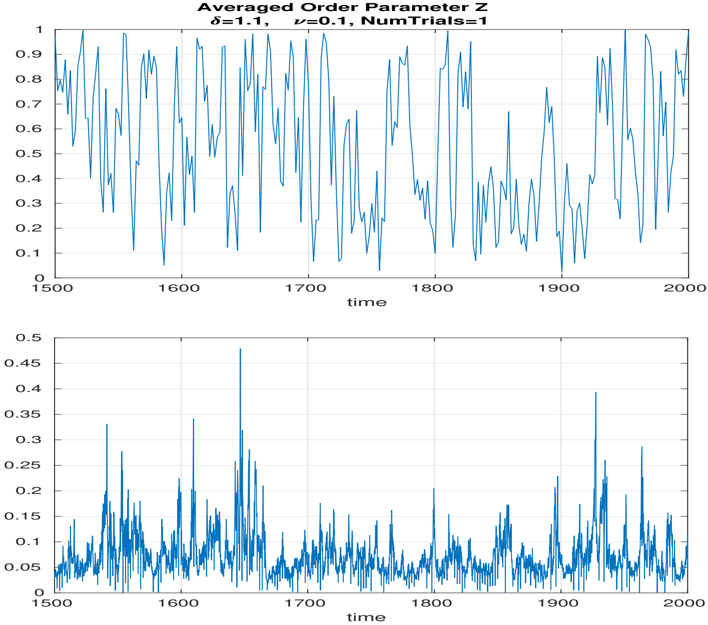
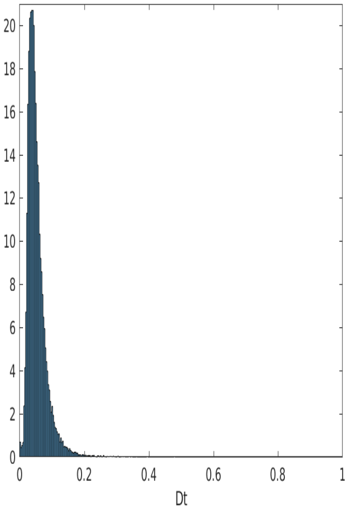


Numerical evidence: timestep distribution



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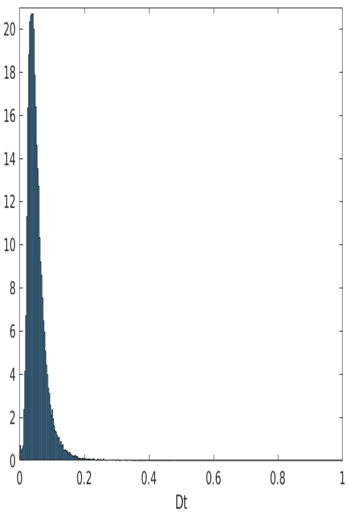

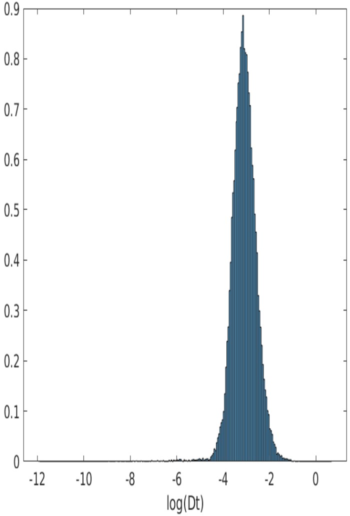

Summary

1. Stochastic vorticity : see different states for $\delta > 1$, $\delta = 1$and $\delta < 1$.
2. Developed a finite dimensional SDE model
3. Numerical studies show a match on transitions between states
4. Would be interesting to take same noise paths : from SPDE to SDE.
5. Would be interesting to look closer as $\delta \approx 1$.
6. Results with adaptive timestepping...