Random Matrices and Random Landscapes

Congratulations Yan and best wishes

Sparse Random Block Matrices : universality

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The ensemble.
Start with the Adjacency matrix $A_{N\times N}$ of a random graph (Erdös-Renyi), insert $d\times d$ random matrices $X_{i,j}$, obtain the ensemble of sparse random block matrices $A_{Nd\times Nd}$

$$A_{N\times N} = \begin{pmatrix} 0 & \alpha_{1,2} & \alpha_{1,3} & \dots & \alpha_{1,N} \\ \alpha_{2,1} & 0 & \alpha_{2,3} & \dots & \alpha_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{N,1} & \alpha_{N,2} & \alpha_{N,3} & \dots & 0 \end{pmatrix} \quad , \quad \alpha_{j,i} = \alpha_{i,j}$$

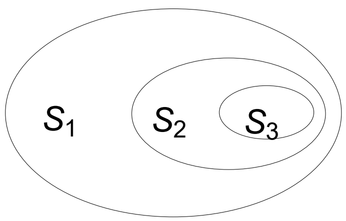
$\{\alpha_{i,j}\}$, $1 \leq i < j \leq N$ is a set of $N(N-1)/2$ i.i.d. random variables , with probability distribution :

$$P(\alpha) = \left(\frac{Z}{N}\right)\delta(\alpha-1) + \left(1 - \frac{Z}{N}\right)\delta(\alpha), \quad Z = <\sum_{j=1}^{N} \alpha_{i,j}> \text{ is the average connectivity.}$$

$$A_{Nd\times Nd} = \begin{pmatrix} 0 & \alpha_{1,2}X_{1,2} & \alpha_{1,3}X_{1,3} & \dots & \alpha_{1,N}X_{1,N} \\ \alpha_{2,1}X_{2,1} & 0 & \alpha_{2,3}X_{2,3} & \dots & \alpha_{2,N}X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{N,1}X_{N,1} & \alpha_{N,2}X_{N,2} & \alpha_{N,3}X_{N,3} & \dots & 0 \end{pmatrix}$$

The goal
To evaluate all the spectral moments of the ensemble $A_{Nd\times Nd}$, in the limit $N\to\infty$ and the limiting spectral distribution.

More precisely. First write the spectral moments of $A_{Nd\times Nd}$ for $N\to\infty$ for d finite or infinite and any probability distribution of the real symmetric random blocks $X_{i,j}$. The contributions form the set S_1 of closed walks on trees.



S_3 is the set of Wigner paths : each travelled edge is travelled exactly twice.

S_2 is the set of paths without the pattern *..a..b..a..b..* that is S_2 corresponds to the non-crossing partitions.

Example
The blocks $X_{i,j}$ are i.i.d. The identification in a product is irrelevant. It is useful a relabeling of the blocks that only records if the blocks are equal or different to other ones in the product. For instance :

$$X_{1,3}X_{3,1}X_{1,3}X_{3,4}X_{4,7}X_{7,4}X_{4,3}X_{3,1} \qquad \texttt{is relabeled}$$
$$(X_1)^3X_2(X_3)^2X_2X_1$$

The lowest order crossing contribution appears at order 8

$$\begin{aligned}\frac{1}{N}\texttt{Tr}\,A^8 \;=\; & Z\,\texttt{tr}\,X_1^8 + Z^2\,\texttt{tr}\left[8\,X_1^6X_2^2 + 4\,X_1^4X_2^4 + 2\,X_1^2X_2^2X_1^2X_2^2\right] + \\ & Z^3\,\texttt{tr}\left[8\,X_1^4X_2^2X_3^2 + 8\,X_1^4X_2X_3^2X_2 + 8\,X_1^3X_2^2X_1X_3^2 + 4\,X_1^2X_2^2X_1^2X_3^2\right] + \\ & Z^4\,\texttt{tr}\left[8\,X_1^2X_2^2X_3X_4^2X_3 + 4\,X_1^2X_2X_3X_4^2X_3X_2 + 2\,X_1^2X_2^2X_3^2X_4^2\right]\end{aligned}$$

The contribution $Z^2\texttt{tr}\,2\,X_1^2X_2^2X_1^2X_2^2$ has the pattern *..a..b..a..b..* that is a crossing partition. All other contributions at this order are non-crossing. The 3 terms $Z^4\,\texttt{tr}\left[8\,X_1^2X_2^2X_3X_4^2X_3 + 4\,X_1^2X_2X_3X_4^2X_3X_2 + 2\,X_1^2X_2^2X_3^2X_4^2\right]$ are Wigner paths.

The results, r finite.

In the limit $d \to \infty$, $Z \to \infty$ with finite ratio $Z/d \geq 2$,
If the rank r of the random blocks X is finite,
the crossing contributions do not contribute.
The non-crossing contributions yield the same limiting moments, *universality=concentration of the measure*, for a large class of probability distributions of the entries of the blocks (isotropy, sub-gaussian,..)
The limiting moments are the spectral moments of the Effective Medium Approximation, with parameter $t = rZ/d$. The generating function of the moments, $g(z) = \sum_{k=0} \frac{\mu_{2k}}{z^{2k+1}}$, is solution of the cubic equation

$$[g(z)]^3 + \frac{t-1}{z}[g(z)]^2 - g(z) + \frac{1}{z} = 0 \qquad , \qquad t = \frac{r\,Z}{d}$$

Analogous results are obtained for the Sparse Laplacian Block Matrices and for the Random Regular Block Ensemble.

The results, maximal rank, $r = d$.

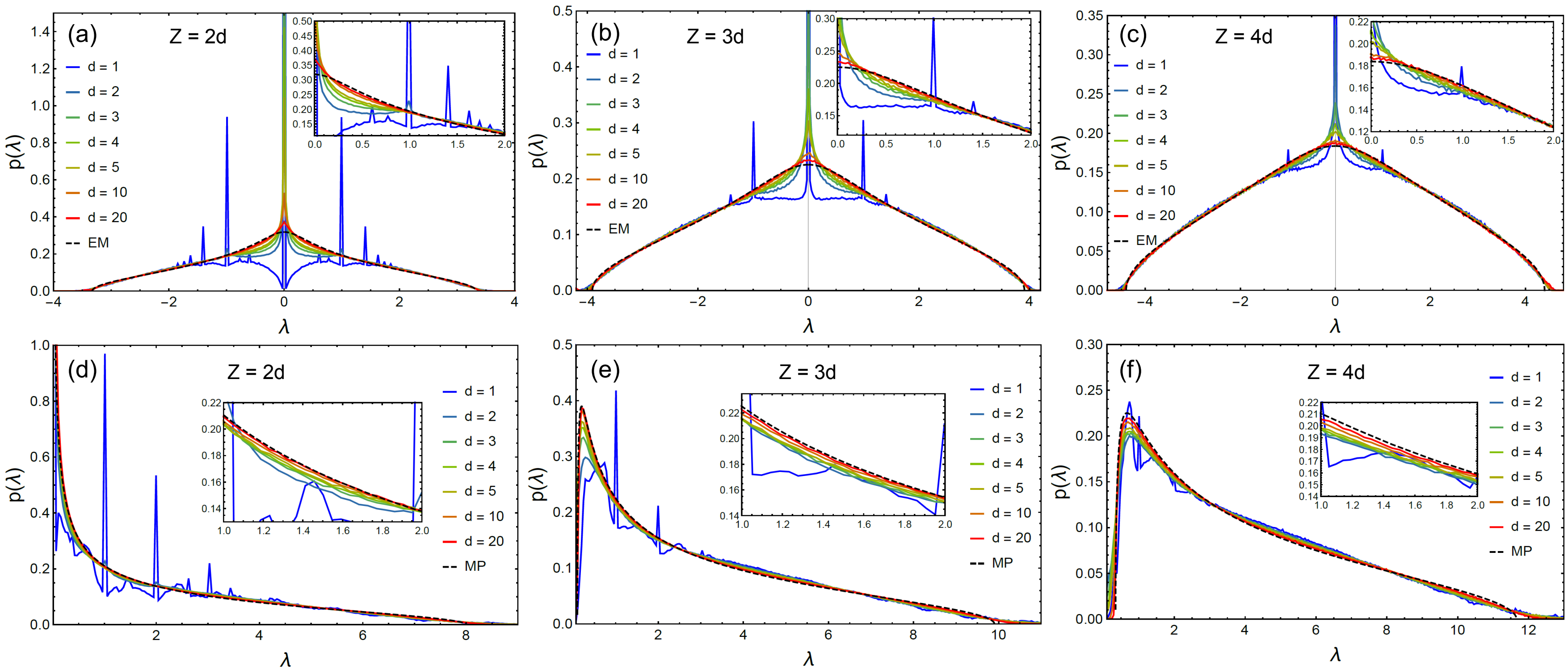
In the limit $d \to \infty$, $Z \to \infty$ with finite ratio $Z/d \geq 2$,
If the rank r of the random blocks X is full, $r = d$, the relevant contributions yield the same limiting moments, *universality=concentration of the measure*, for a large class of probability distributions of the entries of the blocks (isotropy, sub-gaussian,..), gaussian GOE, GUE, restricted trace, etc..

Since the crossing and the non-crossing contributions contribute in the limit, we are unable to identify the limiting spectral density.
The reason :

$$< (\texttt{crossing contribution}) >= \frac{r}{d} < (\texttt{non-crossing contribution}) >$$

Simulations, rank $r = 1$.

Upper row : plots of the eigenvalue spectra of the adjacency ensemble , $d = 1, 2, 3, 4, 5, 10, 20$ and corresponding $N = 15000, 7500, 5000, 4000, 3000, 2000, 1000$. The plots approach the spectrum of Effective Medium Approximation, for increasing d.



Lower row : plots of the eigenvalue spectra of the Laplacian ensemble, same system. The plots approach the Marchenko-Pastur distribution for increasing d.

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M. Pernici and G.M. Cicuta, *Proof of a conjecture on the infinite dimension limit of a unifying model for random matrix theory*, J.of Stat.Phys. **175** (2019) 384-401.

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