COVARIANCE STABILITY & EIGENVECTOR OVERLAPS: AN RMT APPROACH

(A TALK TO HONOR YAN FOR HIS 60TH BIRTHDAY)

J.-Ph. Bouchaud (CFM/X-CFM)

(Joint work with Romain Allez, Joel Bun, Iacopo Mastromatteo, Marc Potters, Pierre-Alain Reigneron and Konstantin Tikhonov, 2014 – 2022)

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Statistical mechanics of a single particle in a multiscale random potential: Parisi landscapes in finite-dimensional Euclidean spaces

Yan V Fyodorov[1] **and Jean-Philippe Bouchaud**[2]

[1] School of Mathematical Sciences, University of Nottingham, Nottingham NG72RD, UK
[2] Science & Finance, Capital Fund Management 6-8 Bd Haussmann, 75009 Paris, France

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Abstract
We construct a N-dimensional Gaussian landscape with multiscale, translation invariant, logarithmic correlations and investigate the statistical mechanics of a single particle in this environment. In the limit of high dimension $N \to \infty$ the free energy of the system and overlap function are calculated exactly using the replica trick and Parisi's hierarchical ansatz. In the thermodynamic limit, we recover the most general version of the Derrida's generalized random energy model (GREM). The low-temperature behaviour depends essentially on the spectrum of length scales involved in the construction of the landscape. If the latter consists of K discrete values, the system is characterized by a K-step replica symmetry breaking solution. We argue that our construction is in fact valid in any finite spatial dimensions $N \geqslant 1$. We discuss implications of our results for the singularity spectrum describing multifractality of the associated Boltzmann–Gibbs measure. Finally we discuss several generalizations and open problems, the dynamics in such a landscape and the construction of a generalized multifractal random walk.

(A hierarchical, translation invariant generalisation of the GFF)

$$\mathbf{M} = \mathbf{C} + \mathbf{O}\mathbf{B}\mathbf{O}^{\dagger}$$

Randomly Perturbed Matrices

Questions in this talk:

- ➢ How similar are
- • the eigenvectors of a « pure » matrix **C** and those of a noisy observation of **C**? (eigenvalues are well known)
- • the eigenvectors of two independent noisy observations of **C**?
- ➢ So what?

Models of Randomly Perturbed Matrices

(Free) Additive noise

$$\mathbf{M} = \mathbf{C} + \mathbf{O}\mathbf{B}\mathbf{O}^{\dagger}$$

« Pure system »
« Signal »

4

« Noise »
B diagonal
O random rotation

(Free) Multiplicative noise

$$\mathbf{M} = \sqrt{\mathbf{C}}\mathbf{O}\mathbf{B}\mathbf{O}^{\dagger}\sqrt{\mathbf{C}}$$

« Pure system »
« Signal »

4

« Noise »
B diagonal
O random rotation

➢ A classic multiplicative example:

- Empirical **M** vs. « True » covariance matrix **C**;
  OBO^t = **XX**^{t} = **W**(ishart), where **X** is a N x T white noise matrix

→ The Marcenko-Pastur distribution

Object of interest: Overlaps

$$\Phi\left(\lambda_i, c_j\right) := N \mathbb{E}\left[\langle \mathbf{u}_i | \mathbf{v}_j \rangle^2\right]$$

« Overlap » Eigenvector of **M** Eigenvector of **C**

Note:

- *N* = size of the matrices, *N* >> 1 in the sequel
- **E**[..]: average over small intervals of λ, of width >> 1/*N*
- The overlaps are quickly of order 1/*N* as a function of the perturbation (« fast » local equilibrium) – but with some remaining structure!

$$d|\psi_i^t\rangle = -\frac{1}{2N}\sum_{j\neq i}\frac{dt}{(\lambda_i(t)-\lambda_j(t))^2}|\psi_i^t\rangle + \frac{1}{\sqrt{N}}\sum_{j\neq i}\frac{dw_{ij}(t)}{\lambda_i(t)-\lambda_j(t)}|\psi_j^t\rangle$$

(Dyson Brownian motion for eigenvectors)

Basic tools

Resolvent:

$$\mathbf{G}_{\mathbf{M}}(z) := (z\mathbf{I}_N - \mathbf{M})^{-1}$$

Stieltjes transform and spectral density (or eigenvalue distribution)

$$\mathrm{Im}\,\mathfrak{g}_{\mathbf{M}}(\lambda - i\eta) \equiv \mathrm{Im}\,\frac{1}{N}\mathrm{Tr}\left[\mathbf{G}_{\mathbf{M}}(\lambda - i\eta)\right] = \pi\,\rho_{\mathbf{M}}(\lambda)$$

Overlaps:

$$\langle \mathbf{v}_i | \mathrm{Im}\,\mathbf{G}_{\mathbf{M}}(\lambda - i\eta) | \mathbf{v}_i \rangle \approx \pi \rho_{\mathbf{M}}(\lambda) \Phi(\lambda, c_i)$$

Note: everywhere the « resolution » $\eta \to 0$ but $\gg 1/N$

Basic tools

R-Transform

$$\mathcal{B}_{\mathbf{M}}(\mathfrak{g}_{\mathbf{M}}(z)) = z. \qquad \mathcal{R}_{\mathbf{M}}(z) := \mathcal{B}_{\mathbf{M}}(z) - \frac{1}{z}$$

e.g. the ***R***-transform of a Wigner matrix is ***R***(z)=σ^2 z

S-Transform

$$\mathcal{T}_{\mathbf{M}}(z) = z\mathfrak{g}_{\mathbf{M}}(z) - 1, \qquad \mathcal{S}_{\mathbf{M}}(z) := \frac{z+1}{z\mathcal{T}_{\mathbf{M}}^{-1}(z)}$$

e.g. the ***S***-transform of a Wishart matrix is ***S***(z)=1/(1+qz) with: **q=N/T**

A Matrix Subordination Law (Allez, Bun, Bouchaud, Potters)

Additive noise

$$\langle \mathbf{G}_{\mathbf{M}}(z) \rangle = \mathbf{G}_{\mathbf{C}}(Z(z))$$

$$Z(z) = z - \mathcal{R}_{\mathbf{B}}(\mathfrak{g}_{\mathbf{M}}(z))$$

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Multiplicative noise

$$z\langle \mathbf{G}_{\mathbf{M}}(z) \rangle = Z(z)\mathbf{G}_{\mathbf{C}}(Z(z))$$

$$Z(z) = z\mathcal{S}_{\mathbf{B}}(z\mathfrak{g}_{\mathbf{M}}(z) - 1)$$

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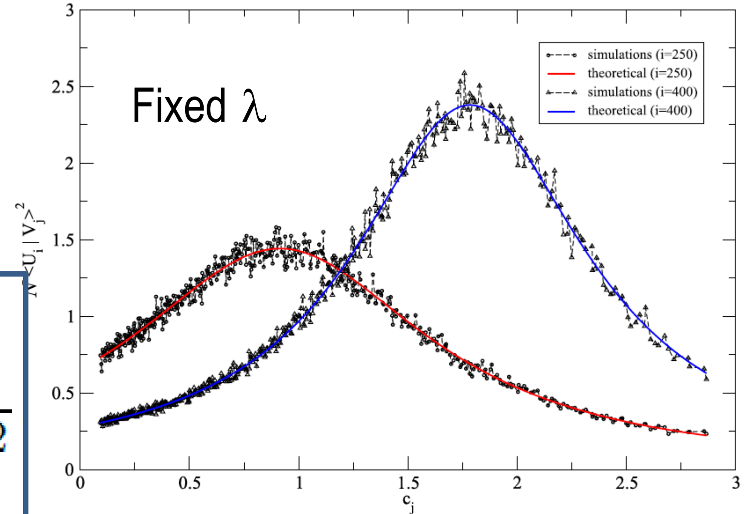
Notes:

- Results obtained using a replica representation of the resolvent + low rank HCIZ
- Taking the trace of these matrix equalities recovers the « free » convolution rules and the corresponding spectra of eigenvalues:

$$\mathcal{R}_{\mathbf{M}}(z) = \mathcal{R}_{\mathbf{C}}(z) + \mathcal{R}_{\mathbf{B}}(z)$$

$$\mathcal{S}_{\mathbf{M}}(u) = \mathcal{S}_{\mathbf{C}}(u)\mathcal{S}_{\mathbf{B}}(u)$$

Overlaps: simplified results (bulk)



Additive noise when **B=Wigner** (cf. Wilkinson)

$$\Phi(\lambda, c) = \frac{\sigma^2}{(c - \lambda + \sigma^2 \mathfrak{h}_{\mathbf{M}}(\lambda))^2 + \sigma^4 \pi^2 \rho_{\mathbf{M}}(\lambda)^2}$$

Notes:

- Tends to a delta function when σ=0 (no noise)
- Cauchy-like formula with a power-law tail for large $|c - \lambda|$ → « Lévy flight »
- Note: True for all « Wigner-like » matrices (not necessarily Gaussian)

Empirical covariance matrices (multiplicative noise)

$$\Phi(\lambda, c) = \frac{qc\lambda}{(c(1-q) - \lambda + qc\lambda \mathfrak{h}_{\mathbf{M}}(\lambda))^2 + q^2 \lambda^2 c^2 \pi^2 \rho_{\mathbf{M}}(\lambda)^2}$$

Notes:

- First obtained by Ledoit & Péché, can be generalized to a broader class of noise
- Tends to a delta function when q=0 (infinite T for a fixed N)

From Overlaps to Rotationally Invariant Estimators

- Assume one has no prior about $\mathbb{C}$
- What is the best L_2 estimator $\boldsymbol{\Xi}(\mathbf{M})$ of $\mathbb{C}$ knowing **M**?
- Without any indication about the directions of the eigenvectors of $\mathbb{C}$, one is stuck with those of **M:**

$$\boldsymbol{\Xi}(\mathbf{M}) = \sum_{i=1}^{N} \xi_i \left|\mathbf{u}_i\right\rangle \left\langle\mathbf{u}_i\right|$$

- The L_2 –optimal ξ are in principle given by:

$$\widehat{\xi}_i = \sum_{j=1}^{N} \langle \mathbf{u}_i | \mathbf{v}_j \rangle^2 c_j$$

- Looks silly: the c's and **v**'s are assumed to be unknown!

From Overlaps to Rotationally Invariant Estimators

$$\widehat{\xi}_i = \sum_{j=1}^{N} \langle \mathbf{u}_i | \mathbf{v}_j \rangle^2 c_j$$

- The high dimensional « miracle »

$$\widehat{\xi}_i \underset{N\to\infty}{=} \int c\, \rho_{\mathbf{C}}(c)\, \Phi(\lambda_i, c)\, \mathrm{d}c.$$
$$= \frac{1}{N\pi\rho_{\mathbf{M}}(\lambda_i)} \lim_{z\to\lambda_i - i0+} \operatorname{Im}\operatorname{Tr}\left[\mathbf{G}_{\mathbf{M}}(z)\mathbf{C}\right]$$

- Note : result only depends on the observable **M** ! (Ledoit-Péché)

- Exemple: Wishart

$$F_2(\lambda) = \frac{\lambda}{(1-q+q\lambda\mathfrak{h}_{\mathbf{M}}(\lambda))^2 + q^2\lambda^2\pi^2\rho_{\mathbf{M}}^2(\lambda)}$$

- Note : F_2 becomes linear if $\mathbb{C}$ is assumed to be an Inverse-Wishart matrix (conjugate prior) → « Linear shrinkage »

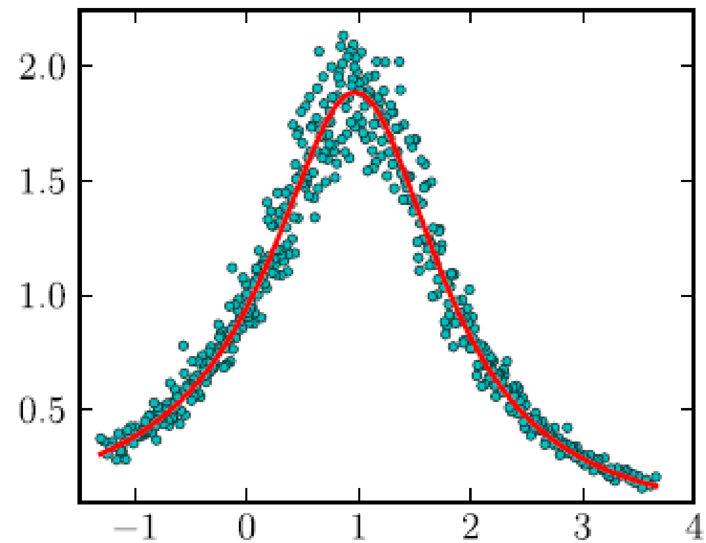
Overlaps between independent realisations

- Extending the above tricks allows us to compute the overlap

$$\Phi(\lambda, \tilde{\lambda}) := N\mathbb{E}\left[\langle \mathbf{u}_\lambda , \tilde{\mathbf{u}}_{\tilde{\lambda}} \rangle^2\right]$$

for *two independent* realisations, e.g. $\mathbf{M} = \mathbb{C} + \mathbf{W}$ and $\tilde{\mathbf{M}} = \mathbb{C} + \tilde{\mathbf{W}}$

- The result is cumbersome but explicit, *both for the multiplicative & additive cases*

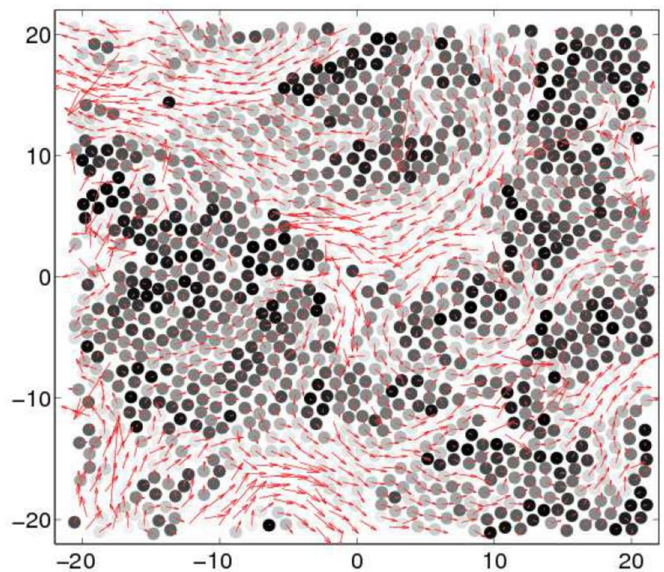


Overlap for a fixed $\tilde{\lambda}$ as a function of λ

- The formula again does not depend explicitly on the (possibly unknown) $\mathbb{C}$
- It can be used to test whether $\mathbf{M}$ and $\tilde{\mathbf{M}}$ originate from the same (unknown) $\mathbb{C}$
- Again, universal within the whole class of Wigner/Wishart like matrices

Overlaps between independent realisations

- The covariance matrix in non-stationary environment
- The Hessian matrix of (slowly) evolving glassy configurations

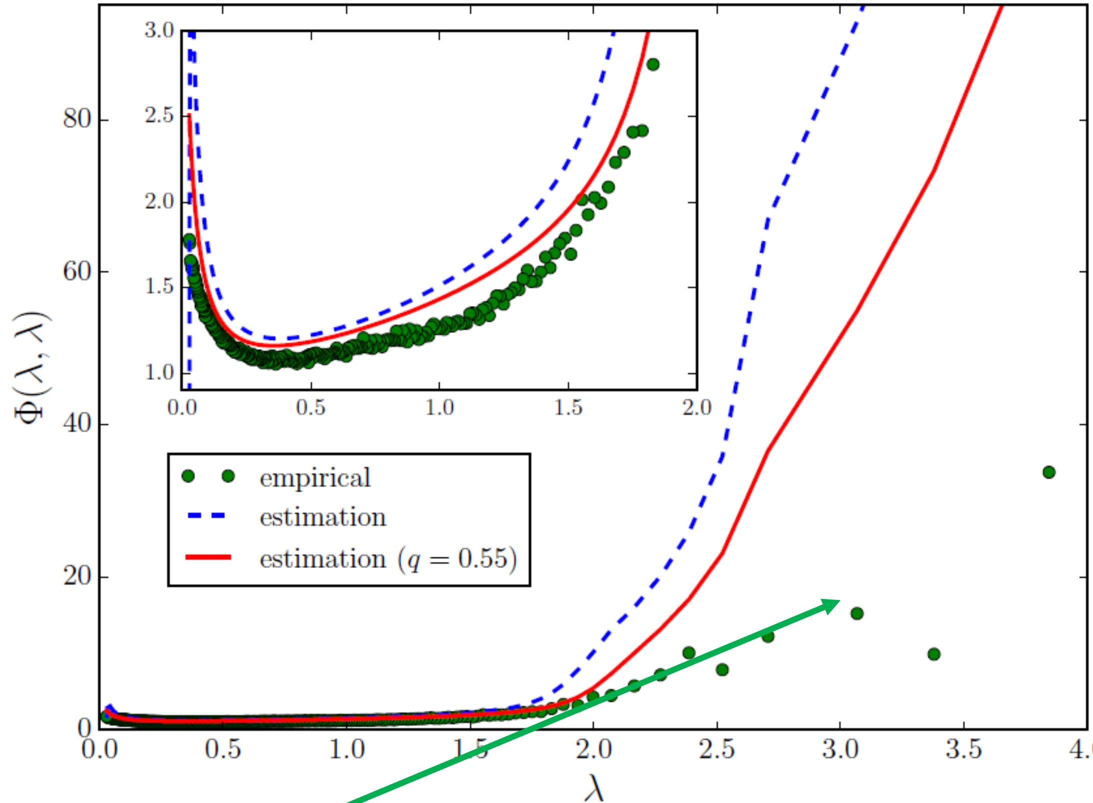


with F. Lechenault, O. Dauchot, G. Biroli

Overlaps between independent realisations

➢ The case of financial covariance matrices: is the « true » underlying correlation structure stable in time?

(Non overlapping time periods)

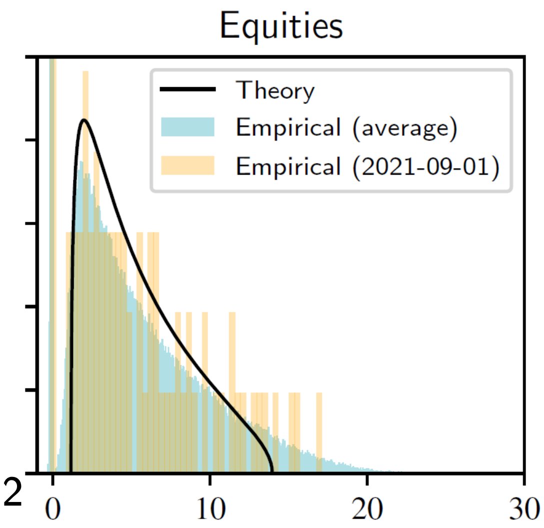


➢ Large eigenvectors are **unstable** (cf. R Allez, JPB and J. Bun, A. Knowles)

➢ Important for portfolio optimisation (uncontrolled risk exposure to large modes)

➢ « Eyeballing » test: should be turned into a true statistical test

A simpler, global test: « fleeting modes »



- Is the « true » underlying correlation structure stable in time?

- Consider the N x N matrix $\mathbb{D} = (\mathbb{E}_{\text{in}})^{-1/2}\, \mathbb{E}_{\text{out}}\, (\mathbb{E}_{\text{in}})^{-1/2}$

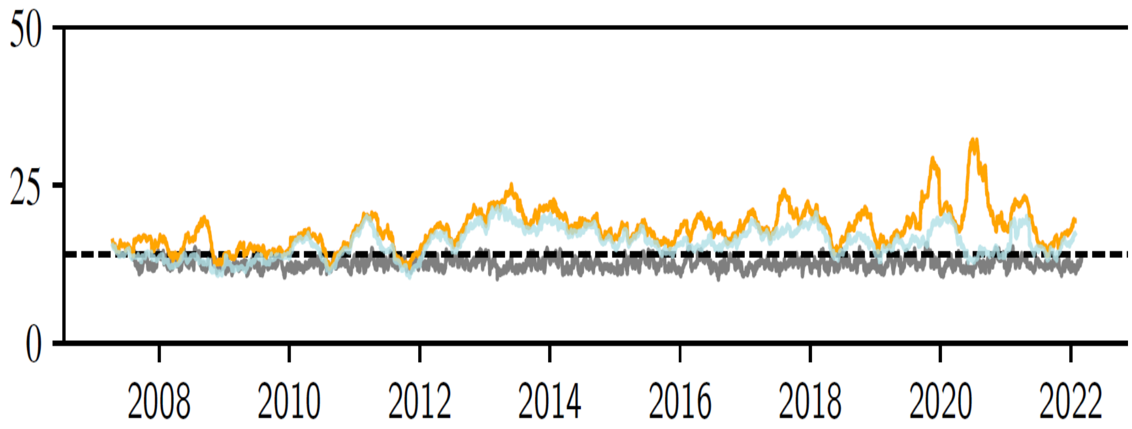
- Where $\mathbb{E}_{\text{in}}$ is the in-sample empirical covariance matrices, defining unit-risk, decorrelated in-sample portfolios

- The eigenvalues/eigenvectors of $\mathbb{D}$ contain relevant information, with max λ's corresponding to maximally over-realizing directions

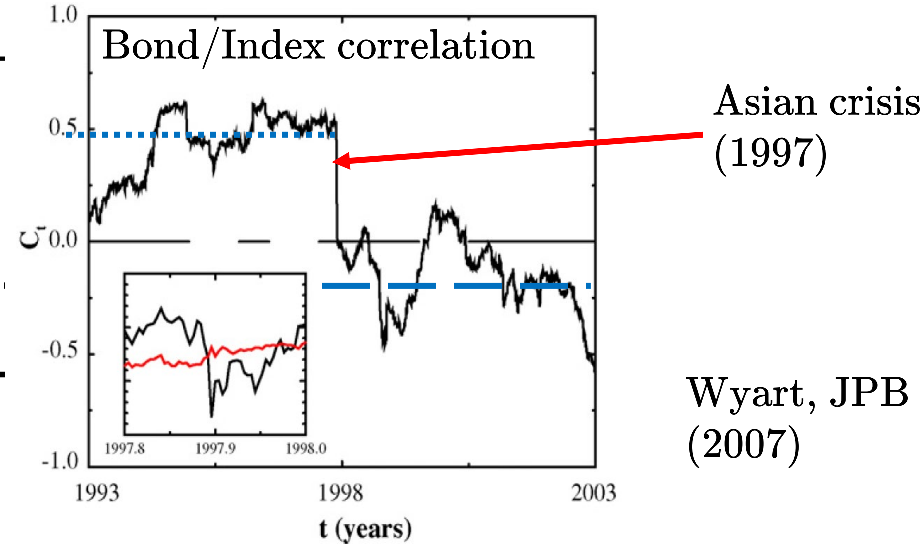
- Null-hypothesis <u>independent</u> of the true covariance matrix $\mathbb{C}$, related to the Jacobi ensemble and only dependent on q_{in} and q_{out}

$$\rho(\lambda) = \frac{1-q_{\text{in}}}{2\pi} \frac{\sqrt{[(\lambda_{\max}-\lambda)(\lambda-\lambda_{\min})]^+}}{\lambda(q_{\text{in}}\lambda+q_{\text{out}})} + \left[1-q_{\text{out}}^{-1}\right]^+ \delta(\lambda)$$

IM, MP, KT, JPB



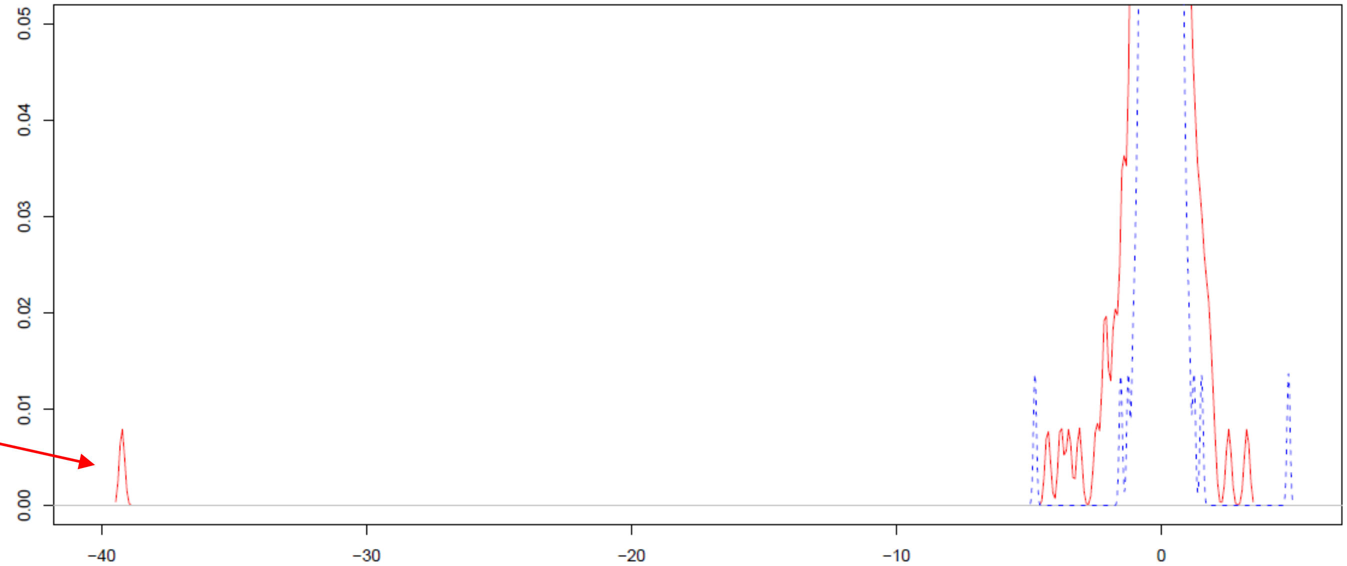
max λ (equities)



Correlations are time dependent

- ➢ What is driving such time dependence?
- Long term evolutions: new firms, evolving business models, macroeconomic effects (e.g. Bond/Index correlation)
- Trading impacts prices → « fleeting modes » reflect traded portfolios (e.g. momentum)
- Behavioural effects, e.g. index I(t) down drives correlations up

Signal: correlations rotate towards (1,1,. . .1) in down markets



Correlations are time dependent

- ➢ Determining the impact of some macro-variables on correlations
- « Principal Regression Analysis »

$$R_i(t)\, R_j(t) = \mathbb{E}_{ij} + \mathrm{I}(t\text{-}1)\, \mathbb{F}_{ij} + \text{noise}$$

- …and RMT again to the rescue: the significant eigenvalues of $\mathbb{F}$ determine which factors influence correlations

RA, PAR, JPB

- ➢ Free Random Matrices results for Stieltjes transforms can be extended to the full resolvant matrix → access to overlaps
- ➢ Large dimension « miracles »:
- The Oracle estimator can be estimated
- The hypothesis that large matrices are generated from the same underlying matrix $\mathbb{C}$ can be tested without knowing $\mathbb{C}$

Conclusions/Extensions

- Overlaps: a true statistical test at large N ?
- RIE for cross-correlation SVDs (*with* F Benaych & M Potters)
- Overlaps for covariances matrices computed on overlapping periods?
- Beyond RIE? Prior on eigenvectors?
- Other uses of RMT in economics/finance: firm networks (and ecology), complex games, cone-wise linear dynamics….

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Letter

Non-self-averaging Lyapunov exponent in random conewise linear systems

Théo Dessertaine [1,2] and Jean-Philippe Bouchaud [2,3]

[1] *LadHyX UMR CNRS 7646, Ecole polytechnique, 91128 Palaiseau Cedex, France*

[2] *Chair of Econophysics & Complex Systems, Ecole polytechnique, 91128 Palaiseau Cedex, France*

[3] *Capital Fund Management, 23 Rue de l'Université, 75007 Paris, France*

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We consider a simple model for multidimensional conewise linear dynamics around cusplike equilibria. We assume that the local linear evolution is either $\mathbf{v}' = \mathbb{A}\mathbf{v}$ or $\mathbb{B}\mathbf{v}$ (with $\mathbb{A}, \mathbb{B}$ independently drawn from a rotationally invariant ensemble of symmetric $N \times N$ matrices) depending on the sign of the first component of $\mathbf{v}$. We establish strong connections with the random diffusion persistence problem. When $N \to \infty$, we find that the Lyapunov exponent is non-self-averaging, i.e., one can observe apparent stability and apparent instability for the same system, depending on time and initial conditions. Finite N effects are also discussed and lead to cone trapping phenomena.

Note: related to the 3d diffusion persistence

Happy 60 yan !