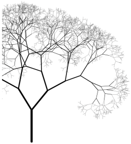
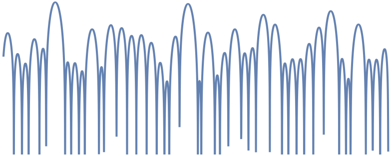
The Fyodorov-Hiary-Keating conjecture(s)





$$\log|\zeta(\tfrac{1}{2}+\mathrm{i}s)|, \qquad 10^6 \leq s \leq 10^6+1$$

Fyodorov-Hiary-Keating conjecture (2012)

Let τ be random, uniform on $[0,T]$. For any real y, as $T \to \infty$ we have

$$\mathbb{P}\left(\max_{|\tau-u|\leq 1}\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}u\right)\right| > \frac{\log T}{(\log\log T)^{3/4}}\cdot e^{y}\right) \to F(y),$$

where $F(y) \sim Cye^{-2y}$, as $y \to \infty$.

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p\in\mathcal{P}}\frac{1}{1-\frac{1}{p^s}},\ \text{ if } \mathrm{Re}(s) > 1.$$

Analytic continuation to $\mathbb{C}$, except at 1. Functional equation :

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

This talk is about : Why ?

Extremes and log-correlation

General setting : Logarithmically-correlated fields. Metric spaces $V_1 \subset V_2 \subset \dots$ with distance d, stochastic process X^N on V_N satisfying

$$\mathbb{E}(X_v^N X_{v'}^N) = -\log\left(d(v,v') + \frac{1}{N}\right) + \text{bounded function}$$

Slow decay of correlations.

Superposition of independent fields on different scales, *with all scales contributing*. Example, random wave model on the circle

$$X_\theta^N = \mathrm{Re}\,\sum_{k=1}^{N} \frac{\mathcal{N}_k(0,1)}{\sqrt{k}} e^{-\mathrm{i}k\theta}.$$

Other log-correlated fields in this talk :

(i) $(\log|\zeta(\frac{1}{2}+\mathrm{i}\tau+\mathrm{i}h))|)_{0\leq h\leq 1}$, $[\mathrm{B}, 2009]$ by a tree structure and Selberg's ideas.

(ii) $(\log|\det(e^{\mathrm{i}\theta}-\mathrm{U}_N)|)_{0\leq\theta\leq 2\pi}$, for U_N uniform on the unitary group [Hughes-Keating-O'Connell, 2001].

Example 1 : the circular logarithmic REM

$$\mathbb{E}\left[V_k V_m\right] = \begin{cases} -2\log|z_k - z_m| & \text{if } k \neq m \\ 2\log M & \text{if } k = m \end{cases}, \quad z_k = e^{\mathrm{i}\frac{2\pi}{M}k}.$$

Fyodorov-Bouchaud (2008)

As $M \to \infty$,

$$\frac{1}{\sqrt{2}}\max V_k = \log M - \frac{3}{4}\log\log M + X_M$$

where X_M has limiting distribution with density $2e^x K_0(2e^{x/2})$.

First crucial observation, based on the Selberg integral : $Z(\beta) = \sum e^{-\beta V_k}$ satisfies

$$\mathbb{E}\left[Z(\beta)^n\right] \sim \begin{cases} M^{1+n^2\beta^2}\cdot \mathrm{O}(1) & \text{if } n > \frac{1}{\beta^2} \\ M^{n(1+\beta^2)}\frac{\Gamma(1-n\beta^2)}{\Gamma(1-\beta^2)} & \text{if } n < \frac{1}{\beta^2} \end{cases}.$$

A density which reproduces these moments is (in the range $Z_e \ll Z \ll M^2$)

$$\frac{1}{\beta^2}\frac{1}{Z}\left(\frac{Z_e}{Z}\right)^{\frac{1}{\beta^2}} e^{-(Z_e/Z)^{1/\beta^2}}, \ \beta < 1, \ Z_e = \frac{M^{1+\beta^2}}{\Gamma(1-\beta^2)}.$$

Second key insight : the $\beta > 1$ regime is governed by a freezing scenario, i.e. some observables keep the same expectation as for $\beta = 1$ (Derrida-Spohn 1988, Carpentier Le-Doussal 2001).

More precisely, define

$$g_\beta(y) = \mathbb{E}\left[e^{-e^{\beta y}\frac{Z}{Z_e}}\right]$$

Then one expects that for any $\beta > 1$ and y

$$g_\beta(y) = g_1(y).$$

Why freezing of the whole generating function (in y) ? A beautiful duality (Fyodorov, Le Doussal, Rosso 2009) shows that $g_\beta(y)$ ($\beta \leq 1$) is a function of $\beta + \beta^{-1}$, in particular for any y

$$\partial_{\beta=1^-}\, g_\beta(y) = 0.$$

Example 2 : the Branching Brownian motion.

Branching rule : After a random time with exponential distribution, a Brownian motion splits into two independent ones. And so on.

$$d(v, v') = e^{t-(v\wedge v')}.$$

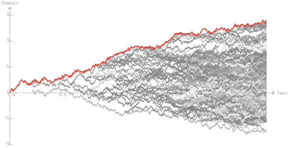


Image : M. Roberts

McKean (1975) connected it to the Fisher-Kolmogorov-Petrovsky-Piskunov reaction-diffusion equation,

$$\partial_t u = \tfrac{1}{2}\partial_{xx} u + u^2 - u, \text{ with step initial condition :}$$
$$u(t,x) = \mathbb{P}(\max_v X_v(t) < x)$$

If e^t ind. Gaussians of variance $t/2$, the maximum $\approx t - \frac{1}{4}\log t$. But :

Theorem (Bramson, 1978)

The maximum $\approx t - \frac{3}{4}\log t$.

Bramson's barrier method.

Let $Z = \#\{v : X_v(t) > t - \frac{3}{4}\log t + b_t\}$, $b_t \to \infty$ slowly.

We have $\mathbb{E}\, Z \to \infty$, first sign that the branching structure matters for the subleading order. The divergence of the expectation comes from atypical events that inflate the expectation.

Look for A_v such that $\widetilde{Z} = \#\{v : X_v(t) > t - \frac{3}{4}\log t + b_t, A_v\}$, satisfies $\mathbb{E}\, \widetilde{Z} \to 0$ but $\mathbb{P}(\cap_v A_v) \to 1$. A pertinent choice :

$$A_v = \{X_v(s) < s + M, s \leq t\}, \;\; M = (\log t)^2 \text{ for example.}$$

Implementation requires the classical Ballot theorem :

Theorem

Conditioned on $B_t = p < M$, the probability that a Brownian motion remains below M up to time t is of order

$$\frac{M(M-p)}{t}.$$

Example 3 : the 2d discrete Gaussian free field.

On a $N \times N$ square of $\mathbb{Z}^2$, density

$$\frac{1}{Z} e^{-\frac{1}{8}\sum_{v\sim v'}(X_v - X_{v'})^2}$$

with zero boundary condition.

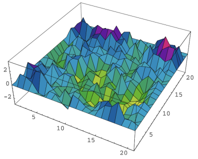


Image : S. Sheffield

$$\mathbb{E}(X_N(v)X_N(v')) = \mathbb{E}_v\left[\sum_{k=1}^{\text{exit time}} \mathbb{1}_{S_k=v'}\right] \sim -C\log\left(d(v,v') + \frac{1}{N}\right).$$

Theorem (Bramson, Ding and Zeitouni, 2013)

$$C_N \max_v X_N(v) = \log N - \frac{3}{4}\log\log N + Z_N$$

with Z_N converging in distribution. The tail is $\asymp \lambda e^{-c\lambda}$ up to $\lambda < \sqrt{\log N}$.

For the proof, help from a branching structure behind X_N obtained by averaging the field on boxes of size $2^{k_0-k} \times 2^{k_0-k}$, $k \leq k_0 = \log N$.

Back to Fyodorov-Hiary-Keating

Fyodorov-Hiary-Keating conjecture(s). As $N, T \to \infty$,

$$\max_{\theta\in[0,2\pi]} \log|\det(e^{\mathrm{i}\theta} - U_N)| = \log N - \frac{3}{4}\log\log N + X_{\mathrm{U}} + \mathrm{o}_{\mathbb{P}}(1),$$

$$\max_{|\tau-u|\leq 1} \log\left|\zeta\left(\tfrac{1}{2} + \mathrm{i}u\right)\right| = \log\log T - \frac{3}{4}\log\log\log T + X_\zeta + \mathrm{o}_{\mathbb{P}}(1).$$

X_{U}, X_ζ have the same right tail

$$\mathbb{P}(X_\zeta > y) \sim C y e^{-2y}, \ y \to \infty.$$

X_{U} has density

$$2e^x K_0(2e^{x/2})$$

(sum of two Gumbel, cf Subag).

Shown for the Gaussian model by Rémy.

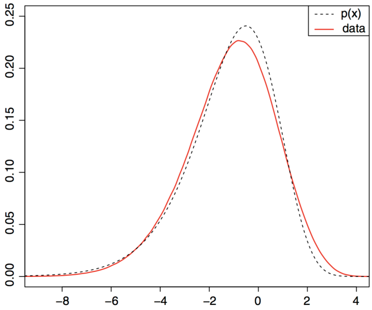
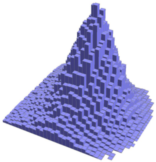


FIG. 1 (color online). Numerical computation (solid red line) compared to theoretical prediction (13) (dashed black line) for $p(x)$.

Numerics : Fyodorov, Hiary, Keating. $T = 10^{28}$!

Why this leading order ?



Approximate
Gaussianity+Log-correlation.

Theorem (Selberg, 1946)

$$\frac{\log|\zeta\left(\frac{1}{2}+\mathrm{i}\tau\right)|}{\sqrt{\frac{1}{2}\log\log T}} \to \mathscr{N}(0,1) \text{ as } T\to\infty.$$

Selberg's proof proceeds in two steps :

1. Key : cut the tail in the Euler product

$$\frac{1}{T}\int_0^T \left|\log\zeta(1/2+\mathrm{i}s) - \sum_{p\leq T}\frac{p^{-\mathrm{i}s}}{\sqrt{p}}\right|^2 \mathrm{d}s < C.$$

2. Then quantify the fact that (U_p's independent on the unit circle)

$$\mathbb{E}\left[\prod_{p\in I}(p^{\mathrm{i}\omega T})^{\alpha_p}\overline{p^{\mathrm{i}\omega T}}^{\beta_p}\right] \approx \mathbb{E}\left[\prod_{p\in I}U_p^{\alpha_p}\overline{U_p}^{\beta_p}\right]$$

What about the full maximum ?

There are two compelling conjectures concerning the maximal size of ζ.

Conjecture 1

$$\max_{0<t<T} \log|\zeta(\tfrac{1}{2}+\mathrm{i}t)| = (C+\mathrm{o}(1))\frac{\log T}{\log\log T}$$

Conjecture 1 would mean that the known upper bounds are close to the truth.

Conjecture 2

$$\max_{0<t<T} \log|\zeta(\tfrac{1}{2}+\mathrm{i}t)| = (\sqrt{2}+\mathrm{o}(1))\sqrt{\log T \log\log T}$$

Conjecture 2 is consistent with $\log|\zeta(\frac{1}{2}+\mathrm{i}\tau)|$ having Gaussian tail with wide uniformity.

This exponential decay seems to suggest that Conjecture 1 is the right order, i.e. the tail in Selberg's CLT is exponential instead of Gaussian.

This extrapolation is not correct. A more precise conjecture is : uniformly in $1 < y < \log\log T$,

$$\mathbb{P}\left(\max_{|\tau-u|\leq 1}\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}u\right)\right| > \frac{\log T}{(\log\log T)^{3/4}}\cdot e^{y}\right) \asymp C y^{-2y} e^{-\frac{y^2}{\log\log T}},$$

a Gaussian decay supporting Conjecture 2.

Remark. Why considering the maximum on intervals of size of order 1 ? This maximum is the building block of maxima on other scales, see the analogy with branching processes in 10min.

Why this correspondence ?

Assume the Riemann hypothesis, and let $\frac{1}{2} \pm \mathrm{i}t_n$ be the ζ zeros.

$$w_n = \frac{t_n}{2\pi} \log \frac{t_n}{2\pi}$$

Theorem (Montgomery, 1972)

If f is a Schwartz function with Fourier transform supported on $(-1, 1)$, then

$$\frac{1}{x} \sum_{1 \leq j,k \leq x, j \neq k} f(w_j - w_k) \underset{x \to \infty}{\longrightarrow} \int_{-\infty}^{\infty} \mathrm{d}y \; f(y) \left(1 - \left(\frac{\sin \pi y}{\pi y}\right)^2\right).$$

All orders correlations coincide : Rudnick, Sarnak (1996), for restricted Fourier support.

Mathematical contributions to the FHK conjecture, for $U(N)$.

$$\max_{\theta\in[0,2\pi]} \log|\det(e^{\mathrm{i}\theta}-U_N)| = \log N - \frac{3}{4}\log\log N + X_{\mathrm{U}} + \mathrm{o}_{\mathbb{P}}(1).$$

Why this conjecture ? Some moments of $\int|\det(e^{\mathrm{i}\theta}-U_N)|^{\beta}\mathrm{d}\theta$ calculated based on the Fisher-Hartwig asymptotics and coincide with those in the circular logarithmic REM.

Theorem (Chhaibi, Najnudel, Madaule, 2016)

The random variable $\max_{\theta\in[0,2\pi]}\log|\det(e^{\mathrm{i}\theta}-U_N)| - (\log N - \frac{3}{4}\log\log N)$ is tight (and extension to circular β-ensemble).

Related results for $\beta = 2$ by

- Arguin Belius B (first order, freezing of the partition function, 2015)

- Paquette and Zeitouni (second order, 2016).

Mathematical contributions to the FHK conjecture, for ζ.

Fyodorov-Keating : *It is not at this stage completely clear to us how, if at all, the arithmetic will modify these (random matrices) expressions, but there are reasons to believe that it will not influence them at leading order.*

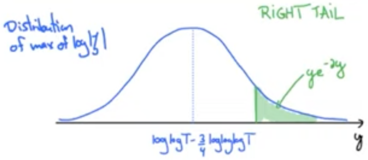
Leading order proved by Joseph Najnudel and ABBRS (2016), upper bound up to $\log_4 T$-error by Adam Harper (2019).

Theorem (Arguin B Radziwill, 2020-22)

Tightness of

$$\frac{(\log\log T)^{3/4}}{\log T}\cdot\max_{|\tau-u|\leq 1}\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}u\right)\right|,$$

and upper tail of order ye^{-2y} up to $y<\sqrt{\log\log T}$.



Proof

Basic idea : Branching. Let

$$Y_\ell(h) = \sum_{e^{\ell-1}<\log p<e^\ell} \frac{\operatorname{Re} p^{-\mathrm{i}(\tau+h)}}{\sqrt{p}}, \quad 1 \leq \ell \leq \log\log T$$

$$S_k(h) = \sum_{\ell=1}^{k} Y_\ell(h)$$

From the prime number theorem, $\mathbb{E}\left[|Y_\ell(h)|^2\right] = \frac{1}{2}$ with good precision. Moreover, log-correlation comes from

$$\mathbb{E}\left[Y_\ell(h_1)Y_\ell(h_2)\right] \approx \frac{1}{2} \text{ if } |h_1 - h_2| \ll e^{-\ell},\ 0 \text{ if } |h_1 - h_2| \gg e^{-\ell}.$$

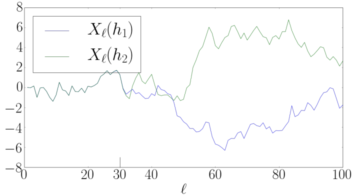


FIGURE – Illustration of the processes $S_k(h_1)$ and $S_k(h_2)$.

Heuristics : Let $n = \log\log T$.Then S_n achieving a high value $\approx n$ requires all $Y_\ell, \ell \leq n$ to be unusually large. These increments need to line up and the partial sums lie in a corridor : $S_k \approx k,\ k \leq n$.

Analytic number theory barrier. To find h such that $S_k(h) \approx k$, we need to identify the moments of the random walk

$$\mathbb{E}\left[(S_k)^{2q}\right] = \mathbb{E}\left[\mathscr{N}(0, k/2)^{2q}\right] + \mathrm{O}\left(\frac{\exp(2qe^k)}{T}\right)$$

up to $q \approx k$.

The leading order can be identified for $k < \log\log T - C \log\log\log T$. This poor control on last increments is a number theory barrier ; one cannot work directly, only, with primes.

This problem is avoided through lower barrier estimates, obtained thanks to twisted moments of ζ : *the proof relies also on the additive nature of ζ, i.e. as a Dirichlet sum.*

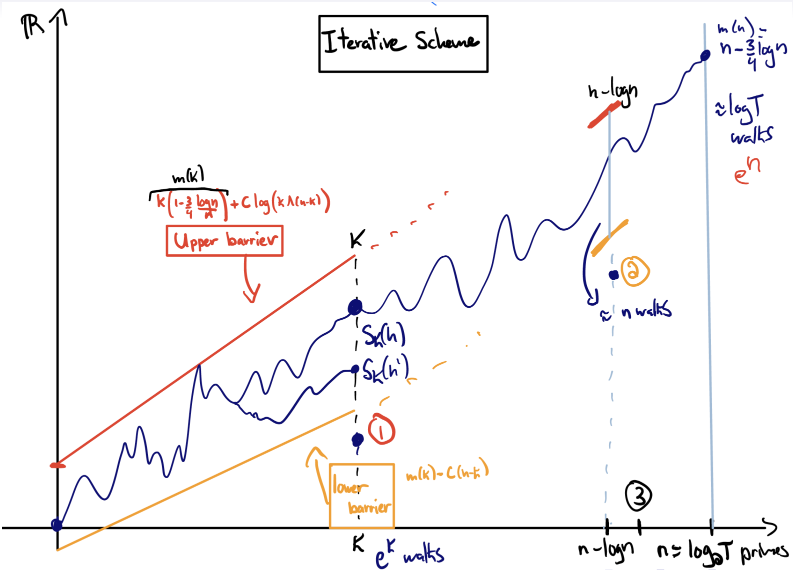


Image : L.-P. Arguin

Multiscale analysis for the upper bound

First, discretize : the maximum over $\log T$ points h is enough (Poisson summation formula).

Let G_k be the set of h such that the walk keeps in the corridor up to time k, and $H = \{h : |\zeta| > \frac{e^n}{n^{3/4}} e^y\}$ The key estimate is (approximately)

$$\mathbb{P}(\exists h \in H \cap G_{n_\ell} \cap G_{n_{\ell+1}}^c) \leq \frac{ye^{-2y}}{(n-n_\ell)^2},\ n_\ell = n - \log_{\ell+1} n.$$

Iterations then show that high points need to be in the corridor. But for $n - n_\ell = \mathrm{O}(1)$ this is unlikely by twisted moments for ζ.

These twisted moments are also used in the proof of the above inequality.

Decoupling between primes : in a primitive version, the **twisted fourth moment** states that

$$\mathbb{E}\Big[|(\zeta\mathcal{M}^{(k)})|^4\cdot|\mathcal{Q}^{(k)}|^2\Big]\ll\mathbb{E}\Big[|(\zeta\mathcal{M}^{(k)})|^4\Big]\cdot\mathbb{E}\Big[|\mathcal{Q}^{(k)}|^2\Big],$$

- $\mathcal{M}^{(k)}$ is a proper approximation of $\zeta^{-1}(s)=\sum\frac{\mu(n)}{n^s}$ which takes into account all primes up to $\exp(e^k)$
- $\mathcal{Q}^{(k)}$ is any Dirichlet series with non-trivial summands supported on multiples of primes smaller than $\exp(e^k)$

In our case, pick $k=n_{\ell+1}$ encode the event "upper barrier up to $n_{\ell+1}$, lower barrier up to n_ℓ" into $\mathcal{Q}^{(n_\ell+1)}$. Then the final gap cannot be too large because of the above decoupling.

It is remarkable that

(i) no higher moments than 4 are needed to obtain tightness ;

(ii) twisted moment are accessible with current number theory technology only up to 4th order.

Additional ideas, with inspiration from a simple, alternate proof of Selberg's CLT by Radziwill and Soundararajan.

Shift from the critical axis for the lower bound to have better approximations for ζ. For us this is harmless because ($\varepsilon > 0$, $V > 1$ and $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log T)^{-1/2-\varepsilon}$)

$$\mathbb{P}\left(\max_{|\tau-u|\leq 1} |\zeta(1/2+\mathrm{i}u)| > V\right) \geq \mathbb{P}\left(\max_{|\tau-u|\leq \frac{1}{4}} |\zeta(\sigma+\mathrm{i}u)| > 2V\right) + \mathrm{o}(1).$$

Avoid large multiplicities for the approximation of ζ^{-1} and $\mathcal{Q}^{(k)}$,as they hurt in large moments. This can be achieved in accordance with the Erdős-Kac theorem, which states that N typically has $\log_2 N$ prime factors (and $\sqrt{\log\log N}$ normal fluctuations) :

$$\mathcal{M}_\ell(h) = \sum_{\substack{p|m \implies p\in(T_{\ell-1},T_\ell] \\ \Omega_\ell(m)\leq (n_\ell-n_{\ell-1})^{10^5}}} \frac{\mu(m)}{m^{\sigma+\mathrm{i}\tau+\mathrm{i}h}},$$

and $\mathcal{M}^{(k)} = \prod_\ell \mathcal{M}_\ell$.

Known. At the level of extremes, the universality class of log-correlated fields includes non-Gaussian models such as $|\det(z - U_N)|$ and $|\zeta|$.

Conjetured based on moments/freezing scenario. The proofs rely on underlying branching structures.

For ζ, the key inputs (in a multiscale analysis involving lower barriers) are twisted moments.

Unknown. Convergence in distribution ?

Maximum for L-functions associated to GL_n, $n \geq 2$?

Averages in families of L-functions ?

Universality of the FHK prediction in non-Gaussian models (Gaussian fields : Ding-Roy-Zeitouni) ?

Thank you, Yan, and happy birthday !