Large Deviations for the largest eigenvalue of random matrices

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Random Matrices and Random Landscapes
Conference in honor of Yan Fyodorov

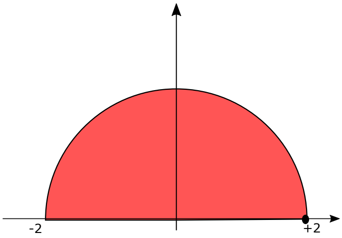
Joint works with F. Augeri, R. Ducatez, J. Husson and N. Cook

Take a $N \times N$ self-adjoint Wigner matrix :

$$X_N = \begin{pmatrix} \frac{x_{11}}{\sqrt{N}} & \frac{x_{1,2}}{\sqrt{N}} & \frac{x_{1,3}}{\sqrt{N}} & \cdots & \cdots \\ \frac{x_{1,2}}{\sqrt{N}} & \frac{x_{2,2}}{\sqrt{N}} & \frac{x_{2,3}}{\sqrt{N}} & \cdots & \cdots \\ \frac{x_{1,3}}{\sqrt{N}} & \frac{x_{2,3}}{\sqrt{N}} & \frac{x_{3,3}}{\sqrt{N}} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

where the random variables $(x_{i,j}, 1 \leq i \leq j \leq N)$ are independent centered variables with covariance 1 outside the diagonal and $2/\beta$ on the diagonal ($\beta = 1$ if the entries are real and $\beta = 2$ if they are complex). We assume $(\sqrt{\frac{2}{\beta}}^{1_{i=j}} x_{ij})_{i \leq j}$ are equidistributed with law μ to simplify.

Almost sure convergence of the spectrum



Let $\lambda_N \leq \lambda_{N-1} \leq \cdots \leq \lambda_1$ be the eigenvalues of X_N.

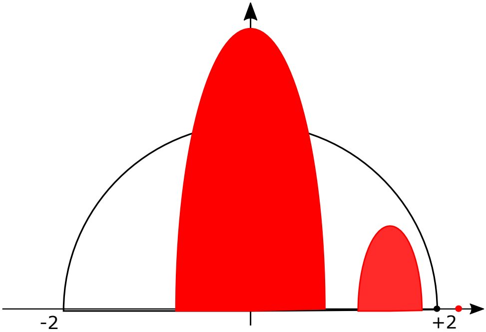
- **Wigner's theorem '56** :

$$\lim_{N\to\infty} \frac{1}{N}\#\{i : \lambda_i \in [a,b]\} = \sigma([a,b]) = \frac{1}{2\pi}\int_a^b \sqrt{4-y^2}dy \qquad a.s.$$

- **Füredi-Komlós' theorem** '81[Bai-Yin '98] If $\int |x|^4 d\mu(x)$ is finite, the largest eigenvalue λ_1 sticks to the bulk :

$$\lim_{N\to\infty} \lambda_1 = 2 \qquad a.s.$$

Large deviations



Goal : Estimate for any $\mu \in \mathcal{P}(\mathbb{R})$ and $x \in \mathbb{R}$

$$\mathbb{P}\Big(\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \simeq \mu\Big) \quad \text{and} \quad \mathbb{P}(\lambda_1 \simeq x)\,.$$

Concentration of measure

Theorem (G-Zeitouni '00)

Assume the entries satisfy log-Sobolev inequality or are compactly supported. Then there exists finite constants $c, C > 0$ such that for all f Lipschitz

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) - \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)]\right| \geq \delta \|f\|_{Lip}\right) \leq Ce^{-cN^2\delta^2}$$

Moreover

$$\mathbb{P}\left(|\lambda_1 - \mathbb{E}[\lambda_1]| \geq \delta\right) \leq Ce^{-cN\delta^2}$$

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Theorem (Bordenave, Caputo,Chafai '11)

For any function f with bounded total variation

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) - \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)]\right| \geq \delta \|f\|_{TV}\right) \leq Ce^{-cN\delta^2}$$

Gaussian ensembles : $\mu = N(0,1)$

$$dP_\beta^N(\lambda) = \frac{1}{Z_\beta^N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp\{-\frac{\beta}{4} N \sum_{i=1}^N \lambda_i^2\} \prod_{1\le i\le N} d\lambda_i$$

Theorem

- *(Ben Arous-G '97) For any probability measure ν,*

$$P_\beta^N(\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i} \simeq \nu) \simeq e^{-\beta N^2 (J(\nu) - \inf J)}$$

where $J(\nu) = \frac{1}{8}\int\int \left(x^2 + y^2 - 4\log|x-y|\right) d\nu(x) d\nu(y)$.

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- *(Ben Arous-Dembo-G '99) For x real,* $P_\beta^N(\lambda_1\simeq x)\simeq e^{-N\beta I_{GOE}(x)}$

where $I_{GOE}(x)=\frac{1}{2}\int_2^x\sqrt{y^2-4}dy$ *if* $x\ge 2$, $+\infty$ *if* $x<2$.

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- *(Majumdar-Schehr '13) If $x < 2$,* $P_\beta^N(\lambda_1 \simeq x) \simeq e^{-N^2 \beta I_-(x)}$

Large deviations for "heavier" tail entries

Assume now that for some $\alpha \in (0,2)$, there exists $a > 0$ so that for all i, j

$$\lim_{t\to\infty} 2^{-1_{i=j}} t^{-\alpha} \log \mathbb{P}(|x_{ij}| \geq t) = -a$$

Theorem

- *(Bordenave-Caputo '12) For any probability measure μ,*

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} \simeq \mu\right) \simeq e^{-N^{1+\frac{\alpha}{2}} J_{\alpha,a}(\mu)}$$

  where $J_{\alpha,a}$ is ∞ unless $\mu = \sigma \boxplus \nu$ and then $= c_a \int |x|^\alpha d\nu(x)$.

- *(Augeri '15) For any $x \geq 2$,*

$$\mathbb{P}\left(\lambda_1 \simeq x\right) \simeq e^{-N^{\frac{\alpha}{2}} I_{\alpha,a}(x)}$$

  where $I_{\alpha,a}(x) = c'_a(\int (x-y)^{-1} d\sigma(y))^{-\alpha}$.

Large deviations for sharp sub-Gaussian entries

μ symmetric with $\mu(x^2) = 1$ has a sub-Gaussian tail if there exists $A \geq 1$ such that for all t

$$\int e^{tx} d\mu(x) \leq e^{A\frac{t^2}{2}} .$$

μ has a sharp subgaussian tail iff $A = 1$.The Gaussian law, Rademacher law $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$ and the uniform measure on $[-\sqrt{3}, \sqrt{3}]$ have sharp sub-gaussian tails.

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Theorem (G-Husson '18)

Assume the entries x_{ij} have a sharp sub-Gaussian tail. Then the law of λ_1 satisfies the same large deviation principle than in the Gaussian case

$$\mathbb{P}(\lambda_1 \simeq x) \simeq e^{-N\beta I_{GOE}(x)}$$

Large deviations in the sub-Gaussian case

Assume μ is symmetric, $\mu(x^2) = 1$ and sub-Gaussian :

$$A = \sup_t \frac{2}{t^2} \log \int e^{tx} d\mu(x) \in [1, +\infty) \, .$$

Theorem (Augeri-G-Husson '19, Cook-Ducatez-G WIP)

Assume $A > 1$.Under some technical hypothesis, the law of λ_1 satisfies large deviation estimates with good rate function I_μ : for x small or large enough

$$\mathbb{P}(\lambda_1 \simeq x) \simeq e^{-\beta N I_\mu(x)} \, ,$$

where $I_\mu(x) \simeq \frac{x^2}{4A}$ for x large and $I_\mu(x) = I_{GOE}(x)$ for x small.

Full LDP for sparse sub-Gaussian entries

μ is symmetric and such that

$$\psi(t) = \frac{2}{t^2} \log \int e^{tx} d\mu(x)$$

is increasing e.g $\mu = p\delta_0 + (1-p)N(0,1)$.

Theorem (Cook-Ducatez-G WIP)

For all $x \in \mathbb{R}$

$$\mathbb{P}(\lambda_1 \simeq x) \simeq e^{-NI_\mu(x)},$$

where $I_\mu(x) \leq I_{GOE}(x)$ for all x and

- *If $I_\mu(x) = I_{GOE}(x)$ for $\eta, \varepsilon > 0$ small*

$$\lim_{N\to\infty} \mathbb{P}\left(\{\|v_1\|_\infty \leq \varepsilon\} | \{|\lambda_1 - x| \leq N^{-\eta}\}\right) = 1$$

- *If $I_\mu(x) < I_{GOE}(x)$, $\exists \gamma(x) > 0$ so that for $\eta > 0$ small*

$$\lim_{N\to\infty} \mathbb{P}\left(\{\|v_1\|_\infty \geq \gamma(x)\} | \{|\lambda_1 - x| \leq N^{-\eta}\}\right) = 1$$

A key tool : Spherical integrals

The spherical integral of X is given for $\theta \geq 0$ by

$$I_N(\theta, X) = \mathbb{E}_e[e^{N\theta\langle e, Xe\rangle}]$$

where e follows the uniform law on the sphere.
By (G-Maida '05), if $\lambda_{max}(X) \to \rho$ and $\frac{1}{N}\sum \delta_{\lambda_i(X)} \to \mu$

$$I_N(\theta, X) \simeq e^{NJ(\rho,\mu,\theta)}$$

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Moreover, for $x \geq 2$ and $\theta \geq 0$, $J(x, \theta) = J(x, \sigma, \theta)$ is given by

$$J(x,\theta) := \begin{cases} \theta^2 & \theta \leq \frac{1}{2}\int \frac{d\sigma(y)}{x-y} \\ \theta x - \frac{1}{2}\int \log(x-\lambda)d\sigma(\lambda) - \frac{1}{2}\log(2e\theta) & \theta \geq \frac{1}{2}\int \frac{d\sigma(y)}{x-y} \end{cases}$$

Proof of LDUB in the sharp sub-Gaussian case

We tilt the measure by spherical integrals : if $X = X^T$ is $N \times N$ with i.i.d variables with law μ and $J(x, \theta) = J(x, \sigma, \theta)$

$$
\begin{aligned}
\mathbb{P}(\lambda_1 \simeq x) &= \mathbb{E}_X[\frac{I_N(X,\theta)}{I_N(X,\theta)} 1_{\lambda_1 \simeq x}] \\
&\leq e^{-N(J(x,\theta)+o(1))} \mathbb{E}_X[I_N(X,\theta)]
\end{aligned}
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&= e^{-N(J(x,\theta)+o(1))} \mathbb{E}_X[\mathbb{E}_e[e^{N\theta\langle e, Xe\rangle}]] \\
&= e^{-N(J(x,\theta)+o(1))} \mathbb{E}_e[\prod_{i\leq j} \int e^{2^{1_{i\neq j}} \theta \sqrt{N} e_i e_j x} d\mu(x)]
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&\leq e^{-N(J(x,\theta)+o(1))}\mathbb{E}_e\left[\prod_{i,j} e^{\theta^2 N e_i^2 e_j^2}\right] \qquad (SSG)
\end{aligned}$$

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&\leq e^{-N(J(x,\theta)+o(1))} \mathbb{E}_e[\prod_{i,j} e^{\theta^2 N e_i^2 e_j^2}] \qquad (SSG) \\
&= e^{-N(J(x,\theta)+o(1))+N\theta^2}
\end{aligned}
$$

Taking $N \to \infty$ and optimizing over θ gives the upper bound.

Large deviation lower bound

$$\mathbb{P}(\lambda_1 \simeq x) = \mathbb{E}_X[\frac{I_N(X,\theta)}{I_N(X,\theta)} 1_{\lambda_1 \simeq x}]$$

$$\simeq e^{-N(J(x,\theta)+o(1))} \frac{\mathbb{E}_X[I_N(X,\theta) 1_{\lambda_1 \simeq x}]}{\mathbb{E}_X[I_N(X,\theta)]} \mathbb{E}_X[I_N(X,\theta)]$$

We need to show that for any $x > 2$ there exists $\theta > 0$ s.t.

$$\frac{\mathbb{E}_X[I_N(X,\theta) 1_{\lambda_1 \simeq x}]}{\mathbb{E}_X[I_N(X,\theta)]} \geq e^{o(N)} \text{ and } \mathbb{E}_X[I_N(X,\theta)] \geq e^{N(\theta^2+o(1))}$$

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$$\bullet \mathbb{E}_X[I_N(X,\theta)] \geq \mathbb{E}_e[1_{\|e\|_\infty \leq N^{-1/3}} \prod_{i \leq j} \int e^{2^{1_{i\neq j}} \theta \sqrt{N} e_i e_j x} d\mu(x)] \simeq e^{N\theta^2}$$

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- $\mathbb{E}_X[I_N(X,\theta)] \ge \mathbb{E}_e[1_{\|e\|_\infty \le N^{-1/3}} \prod_{i \le j} \int e^{2^{1_{i \ne j}} \theta \sqrt{N} e_i e_j x} d\mu(x)] \simeq e^{N\theta^2}$

- $\frac{\mathbb{E}_X[I_N(X,\theta) 1_{\lambda_1 \simeq x}]}{\mathbb{E}_X[I_N(X,\theta)]} \ge \inf_{\|e\|_\infty \le N^{-1/3}} \frac{\mathbb{E}_X[1_{\lambda_1 \simeq x} e^{N\theta \langle e, Xe \rangle}]}{\mathbb{E}_X[e^{N\theta \langle e, Xe \rangle}]}$

where X has approximately the law of $W + \theta e e^T$ under the tilted measure. The BBP transition insures that $\lambda_1 \to \theta + \theta^{-1} = x$.

General result

Theorem (Ducatez-Cook-G. WIP)

Let $\psi(t) = t^{-2}\log\int e^{tx}d\mu(x)$ *so that* $\|\psi\|_\infty < \infty$. *Let* $\eta > 0$ *small. Let* $x \geq 2$. *Then*

$$\frac{1}{N}\log\mathbb{P}(\lambda_1 \simeq x)$$

$$\simeq \sup_{w=\hat{w}Sparse}\inf_{\theta\geq 1}\left\{\frac{1}{N}\log\mathbb{E}_u\mathbb{E}_H(e^{N\theta\langle u,Hu\rangle}1_{\|\hat{u}-\alpha(x,\theta)w\|_2\leq N^{-\eta}}) - J(x,\theta)\right\}$$

where $\alpha(x,\theta) = \sqrt{(1-\frac{G_\sigma(x)}{2\theta})_+}$ *and* $\hat{u} = (1_{|u_i|\geq N^{-\frac{1}{2}+\eta}}u_i)_{1\leq i\leq N}$.

- the first term in the RHS can be estimated.
- When x is small, the supremum over w is taken at $w=0$ yielding the GOE LDP. When ψ is increasing it is taken at $(w,0,\ldots 0)$, yielding a RHS $= I_\mu$. If μ is compactly, for x large enough, it is taken at $(N^{-1/4}m_x,\ldots,N^{-1/4}m_x,0,\ldots,0)$ with dimension $\sqrt{N}/m_x^2$.

Extensions/ Open problems

- LDP for the largest eigenvalue generalizes to band matrices $Y_{i,j} = \sigma_{i,j} X_{i,j}$ (Husson '20), to $X + D$ (Mc Kenna '20), to $A + UBU^*$ (G-Maida '19), to ABA (Mergny-Potters '22), to joint LDP with the eigenvector (Biroli-G '20), to the kth largest eigenvalues (G-Husson '21, Husson-Ko '22)
- LDP for the spectral measure of Wigner matrices is open for sub-Gaussian entries. It should not be universal : if the entries are Rademacher

$$\mathbb{P}(\hat{\mu}_N \simeq \delta_0) \geq (\frac{1}{2})^{N^2}$$

but the rate function is infinite in the Gaussian case.
- LDP for the spectral measure of $A + UBU^*$ and the diagonal entries of UBU^*, see Belinschi-Huang-G '20 and Narayanan-Sheffield '22

$$\begin{pmatrix} H & A & P & P \\ Y & B & I & R \\ T & H & D & A \\ Y & Y & A & N \end{pmatrix}$$