Multiplicative Chaos and Integrability

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Log correlated fields and chaos

Log correlated random field $X(x)$ on $x \in \mathbb{R}^n$ has covariance

$$\mathbb{E}X(x)X(y) \sim_{x \to y} \log |x - y|^{-1}$$

Multiplicative chaos measure for $\gamma \in \mathbb{R}$

$$M_\gamma(dx) = \frac{1}{Z_\gamma} e^{\gamma X(x)} dx$$

For **Gaussian** X, M_γ is Gaussian Multiplicative Chaos (GMC)

- Simple models for intermittency in turbulence
- Random landscapes: model of **freezing transition**
- Random matrix theory
- Liouville Quantum Gravity

Review: "Gaussian free field, Liouville quantum gravity and Gaussian multiplicative chaos" by Nathanaël Berestycki and Ellen Powell

Gaussian Free Field (GFF)

GFF can be realised as a random distribution

n=1. GFF on $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$X(\theta) \stackrel{d}{=} \sum_{n \neq 0} |n|^{-\frac{1}{2}} e^{in\theta} X_n, \quad X_n \;\; i.i.d \;\; \mathcal{N}_{\mathbb{C}}(0,1)$$

n=2. (a) $z \in \mathcal{D} \subset \mathbb{C}$ closed domain

$$X(z) \stackrel{d}{=} \sum_{n=1}^{\infty} \lambda_n^{-\frac{1}{2}} e_n(z) X_n, \quad X_n \;\; i.i.d \;\; \mathcal{N}_{\mathbb{R}}(0,1)$$

e_n orthonormal basis of eigenfunctions of Dirichlet Δ,

$$-\Delta e_n = \lambda_n e_n$$

(b) (Σ, g) Riemann surface with metric g, e_n eigenfunctions of Laplace-Beltrami operator Δ_g

In all cases $X \in H^{-s}$, $s > 0$ almost surely.

Gaussian Multiplicative Chaos (GMC)

Regularize $X \to X_\epsilon$ with short distance cutoff ϵ, e.g.

$$X_\epsilon(x) = \epsilon^{-n} \int \rho(\tfrac{x-y)}{\epsilon})X(y)d^n y$$

Then

$$\lim_{\epsilon \to 0} e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}X_\epsilon(x)^2} d^n x = M_\gamma(d^n x)$$

- $e^{-\frac{\gamma^2}{2}\mathbb{E}X_\epsilon(x)^2} \sim e^{\frac{\gamma^2}{2}\log\epsilon} = \epsilon^{\frac{\gamma^2}{2}}$
- Weak limit of measures
- Almost surely or in probability depending on the cutoff
- Limit is **independent** of the regularization scheme.

Scale invariance

Let $r < 1$. From $\log|rx|^{-1} = \log r^{-1} + \log|x|^{-1}$ we infer

$$X(rx) \stackrel{d}{=} \sqrt{|\log r|}N + X(x)$$

where N is unit gaussian. So for $B_r = \{|x| \leq r\}$

$$M_\gamma(B_r) \stackrel{d}{=} r^{n+\frac{\gamma^2}{2}} e^{\gamma\sqrt{|\log r|}N} M_\gamma(B_1)$$

which implies **multifractal** scaling

$$\mathbb{E}(M_\gamma(B_r)^p) = r^{\xi_p}\mathbb{E}(M_\gamma(B_1)^p), \quad \xi_p = (n + \frac{\gamma^2}{2})p - \frac{\gamma^2}{2}p^2$$

Moments

Let $S \subset \mathbb{R}^2$ be unit square and write $S = \cup_{i=1}^{2^d} S_i$, $S_i \sim S/2$. For $p > 1$

$$M_\gamma(S)^p = (\sum_i M_\gamma(S_i))^p \geq \sum_i M_\gamma(S_i)^p$$

By scaling as before

$$M_\gamma(S/2) \overset{law}{=} 2^{-d} e^{\gamma\sqrt{\ln 2}N - \frac{\gamma^2}{2}\ln 2} m_\gamma(S)$$

so that

$$\mathbb{E}M_\gamma(S)^p \geq 2^{d-dp}\mathbb{E}e^{p\gamma\sqrt{\ln 2}N} e^{-p\frac{\gamma^2}{2}\ln 2}\mathbb{E}M_\gamma(S)^p = 2^{d(1-p)+(p^2-p)\frac{\gamma^2}{2}}\mathbb{E}m_{g,\gamma}(S)^p.$$

This is possible only if $d(1-p) + (p^2 - p)\frac{\gamma^2}{2} \leq 0$ i.e. only if $p \leq \frac{2d}{\gamma^2}$

Proposition (a) $\mathbb{E}(M_\gamma(S)^p) < \infty \Leftrightarrow p \leq \frac{2d}{\gamma^2}$

(b) $M_\gamma \neq 0 \Leftrightarrow \gamma^2 < 2d$

Freezing

At $\gamma^2 = 2d$ M_γ has a **freezing transition**:

$$\lim_{\epsilon\to 0} |\log\epsilon|^{a_\gamma} e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}X_\epsilon(x)^2} d^n x = \tilde{M}_\gamma(d^n x)$$

with $a_\gamma = \frac{1}{2}$ for $\gamma = \sqrt{2d}$ and $a_\gamma = \frac{3\gamma}{2\sqrt{2d}}$ for $\gamma > \sqrt{2d}$

- $\tilde{M}_{\sqrt{2d}}$ is a.s. continuous, Hausdorff dimension zero
- $\tilde{M}_\gamma$ is purely atomic for $\gamma > \sqrt{2d}$,:

$$\tilde{M}_\gamma = \sum_i p_i \delta(x - x_i)$$

- Law of the random points x_i determined by the law of $\tilde{M}_{\sqrt{2d}}$, law of p_i depends on γ

Think of $-X$ a random potential, γ^{-1} temperature. For $\gamma > \sqrt{2d}$ particle is localized near the potential minima.

Fyodorov-Bouchaud

In 2008 Fyodorov and Bouchaud derived a remarkable formula for the PDF of the 1d GMC for $\gamma < \sqrt{2}$:

$$\mathbb{P}(M_\gamma(\mathbb{T}) \in dy) = const \times x^{-1-\frac{\gamma^2}{4})} e^{-x^{-\frac{\gamma^2}{4}}}, \quad x = \Gamma(1 - \frac{\gamma^2}{4})y$$

Assuming the freezing hyphothesis they derived for $\beta \to \infty$ the PDF for the renormalised minimum of X:

$$\mathbb{P}(\varphi \in dx) = 2e^{\frac{x}{\sqrt{2}}} K_1(2e^{\frac{x}{\sqrt{2}}})$$

where

$$\varphi = \lim_{\epsilon \to 0} [\min_x X_\epsilon(x) + \sqrt{2} \log \epsilon^{-1} - \frac{3}{2\sqrt{2}} \log \log \epsilon^{-1}]$$

New **universality class** for extreme value statistics.

Fyodorov-Bouchaud

Fyodorov-Bouchaud computed the integer moments of $M_\gamma(\mathbb{T})$ as Selberg integrals and got

$$\mathbb{E}M_\gamma(\mathbb{T})^p = \frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^p} \quad (*)$$

This computation is valid only for $p \in \mathbb{N}$ and $p < \frac{4}{\gamma^2}$ but is consistent with the above PDF.

$(*)$ was proved by Guillaume Remy in 2017 for all $p \in \mathbb{R}$ and $p < \frac{4}{\gamma^2}$ and solving the moment problem to get PDF.

Remy used **Conformal Field Theory** to derive $(*)$.

Conformal invariance

Let Σ be a 2d surface with Riemannian metric g and X_g the GFF.

For a diffeomorphism ψ

$$X_g \circ \psi \stackrel{d}{=} X_{\psi^* g}$$

and for a Weyl transformation $g \to e^{\varphi} g$

$$X_{e^{\varphi} g} \stackrel{d}{=} X_g - c_g(\varphi)$$

where the random variable $c_g(\varphi)$ is given by

$$c_g(\varphi) = \frac{1}{v_{e^{\varphi} g}(S^2)} \int X_g dv_{e^{\varphi} g}$$

These imply for GMC

$$M_{\gamma,g} \circ \psi \stackrel{d}{=} M_{\gamma,\psi^* g}$$
$$M_{\gamma,e^{\varphi} g} \stackrel{d}{=} e^{\frac{\gamma Q}{2}\varphi - \gamma c_g(\varphi)} M_{\gamma,g}.$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

Liouville Conformal Field Theory

We weigh the GMC law using GMC and add the constant mode:

$$\langle F\rangle_{\Sigma,g} := Z_g \int_{\mathbb{R}} \mathbb{E}\big(F(\phi)e^{-\int_\Sigma QR_g\phi dv_g + \mu e^{\gamma c} M_{\gamma,g}(\Sigma)}\big)dc \quad (*)$$

- $\phi = c + X_g$
- v_g Riemannian volume, R_g scalar curvature
- Z_g "partition function of GFF"
- $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\mu > 0$ "cosmological constant"

$\langle\cdot\rangle_{\Sigma,g}$ defines expectation in a positive measure on $H^{-s}(\Sigma)$

$(*)$ makes sense of the formal expression

$$\langle F\rangle_{\Sigma,g} = \int F(\phi)e^{-S(\phi)}D\phi$$

where the action functional is given by

$$S(\phi) = \int_\Sigma (|d\phi|^2 + QR_g\phi + \mu e^{\gamma\phi})dv_g$$

Liouville Theory

Picard, Poincare: **minimizer** ϕ_0 gives rise to constant negative curvature metric $e^{\gamma\phi_0}|dz|^2$.

Probabilistic theory $\langle\cdot\rangle$:

- Noncritical string theory (Polyakov 1981)
- 2d gravity: Knizhnik, Polyakov, Zamolodchikov (1988)
- 4d SuSy Yang-Mills (Alday, Gaiotto, Tachikawa 2010)

Basic observables are

$$V_\alpha(z) = e^{\alpha\phi(z)}$$

and their correlation functions

$$\langle\prod_i V_{\alpha_i}(z_i)\rangle_{\Sigma,g}$$

defined through limits of regularised objects.

Reduction to GMC

c-integral can be done explicitly: by Gauss-Bonnet ($h =$ genus)

$$\int_\Sigma R_g dv_g = 2(1-h)$$

so that

$$\begin{aligned}\langle \prod_i V_{\alpha_i}(z_i)\rangle_{\Sigma,g} &= \mathbb{E}\left(\prod_i e^{\alpha_i X_g(z_i)} \int e^{(\sum_i \alpha_i - 2(1-h))c} e^{-\mu e^{\gamma c} M_{\gamma,g}(\Sigma)} dc\right) \\ &= \frac{\Gamma(s)}{\gamma \mu^s} \mathbb{E}\left[\prod_i e^{\alpha_i X_g(z_i)} M_{\gamma,g}(\Sigma)^{-s}\right]\end{aligned}$$

where we denoted

$$s = \gamma^{-1}(\sum_i \alpha_i - 2(1-h))$$

Existence

Let $G(z,z_i)=\mathbb{E}X_g(z)X_g(z_i)$. Next we shift (Girsanov theorem)

$$X_g \to X_g + \sum_i \alpha_i G(z,z_i)$$

so that **LCFT correlations reduce to GMC moments**:

$$\langle \prod_i V_{\alpha_i}(z_i)\rangle_{\Sigma,g} = \frac{\Gamma(s)}{\gamma\mu^s} \prod_{i<j} e^{\alpha_i\alpha_j G(z_i,z_j)} \mathbb{E}\left[\int \prod_i e^{\gamma\alpha_i G(z,z_i)} dM_{\gamma,g}(z)\right]^{-s}$$

Since $G(z,z_i)=\log|z-z_i|+\mathcal{O}(|z-z_i|)$

$$e^{\gamma\alpha_i G(z,z_i)} \sim |z-z_i|^{-\alpha_i\gamma} \quad as \;\; z\to z_i$$

This singularity is $M_{\gamma,g}$ integrable iff $\alpha_i<Q$ i.e. if $\gamma\alpha_i<2+\frac{\gamma^2}{2}$

Proposition.(David, A.K., Rhodes, Vargas 2016) The LCFT correlation functions are defined and nontrivial if

$$\alpha_i<Q \quad \text{and} \quad \sum_i \alpha_i > 2(1-h) \quad \text{(Seiberg bounds)}$$

Conformal symmetry

LCFT correlations satisfy axioms of CFT:

Diffeomorphism covariance : For $\psi \in Diff(\Sigma)$

$$\langle \prod_i V_{\Delta_i}(\psi(x_i)) \rangle_{\Sigma,g} = \langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,\psi^* g}$$

Weyl covariance : For $\sigma \in C^\infty(\Sigma)$

$$\langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,e^\sigma g} = e^{\frac{c}{96\pi} \int_\Sigma (|d\sigma|^2 + 2R_g\sigma) dv_g} \prod_i e^{-\Delta_i \sigma(x_i)} \langle \prod_i V_{\Delta_i}(x_i) \rangle_{\Sigma,g}$$

c **central charge** of the CFT

Hence correlations defined on **moduli space** of Riemann surfaces

$$g \sim e^\sigma \psi^* g \quad \psi \in Diff(\Sigma), \;\; \sigma \in C^\infty(\Sigma)$$

Structure constants

Take $\Sigma = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For $n = 3$ moduli space is one point so that 3-point function is determined up to a constant which we may take as

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}} := C(\alpha_1, \alpha_2, \alpha_3)$$

It is given by a moment of a chaos integral

$$C_\gamma(\alpha_1, \alpha_2, \alpha_3) = \frac{2\Gamma(s)}{\mu^s \gamma} \mathbb{E} \left(\int_{\mathbb{C}} \frac{|x \vee 1|_+^{\gamma(\alpha_1+\alpha_2+\alpha_3)}}{|x|^{\gamma\alpha_1} |x-1|^{\gamma\alpha_2}} dM_\gamma(x) \right)^{-s}$$

with $s = \frac{\alpha_1+\alpha_2+\alpha_3-2Q}{\gamma}$.

For $s \in \mathbb{N}$ this moment was computed in terms of **divergent** Selberg type integrals by Dorn, Otto (1994) and Zamolodchikov^2 (1996) yielding a conjecture for $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$, the **DOZZ formula**.

DOZZ formula

$$C_{DOZZ}(\alpha_1,\alpha_2,\alpha_3) = \hat{\mu}^{-s} \frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\alpha_1+\alpha_2+\alpha_3-2Q}{2})\Upsilon(\frac{\alpha_2+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_2}{2})}$$

- $\hat{\mu} = \frac{\pi\Gamma(\frac{\gamma^2}{4})(\frac{\gamma}{2})^{\frac{4-\gamma^2}{2}}}{\Gamma(1-\frac{\gamma^2}{4})}\mu$
- Υ is an entire function on $\mathbb{C}$ defined by

$$\Upsilon(\alpha)^{-1} = \Gamma_2(\alpha|\tfrac{\gamma}{2},\tfrac{2}{\gamma})\Gamma_2(2Q-\alpha|\tfrac{\gamma}{2},\tfrac{2}{\gamma})$$

Theorem (K, Rhodes, Vargas, Annals of Mathematics **191**, 81)
Let α_i satisfy the Seiberg bounds. Then $C(\alpha_1,\alpha_2,\alpha_3)$ is given by the DOZZ formula.

Belavin-Polyakov-Zamolodchicov equation

Theorem *4-point function*

$$F(u) := \langle e^{\alpha\phi(u)} e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle$$

with $\alpha = -\frac{\gamma}{2}$ ***or*** $\alpha = -\frac{2}{\gamma}$ *satisfies a hypergeometric equation*

$$\partial_u^2 F + \frac{a}{u(1-u)}\partial_u F - \frac{b}{u(1-u)} F = 0 \quad (*)$$

On the other hand it has a Chaos expression

$$F(u) \propto \mathbb{E}\left(\int \frac{1}{|z-u|^{\gamma\alpha}|z|^{\gamma\alpha_1}|z-1|^{\gamma\alpha_2}|z-\infty|^{\gamma\alpha_3}} M(dz) \right)^{-s} \quad (**)$$

Study asymptotics $u \to 0$:

$$F(u) = C_\gamma(\alpha_1+\alpha, \alpha_2, \alpha_3) + B(\alpha_1) C_\gamma(\alpha_1-\alpha, \alpha_2, \alpha_3)|z|^\eta + \ldots \quad (1)$$

and compare with hypergeometric functions.

Periodicity

Upshot: Let $\alpha = \frac{\gamma}{2}$ or $\alpha = \frac{2}{\gamma}$. Then

$$C(\alpha_1 - \alpha, \alpha_2, \alpha_3) = D(\alpha, \alpha_1, \alpha_2, \alpha_3) C(\alpha_1 + \alpha, \alpha_2, \alpha_3)$$

$$D(\alpha, \alpha_1, \alpha_2, \alpha_3) = -\frac{1}{\pi\mu} \frac{\Gamma(-\alpha^2)\Gamma(-\alpha\alpha_1)\Gamma(-\alpha\alpha_1 - \alpha^2)\Gamma(\frac{\alpha}{2}(2\alpha_1 - \bar{\alpha}))}{\Gamma(\frac{\alpha}{2}(2Q - \bar{\alpha}))\Gamma(\frac{\alpha}{2}(2\alpha_3 - \bar{\alpha}))\Gamma(\frac{\alpha}{2}(2\alpha_2 - \bar{\alpha}))}$$
$$\times \frac{\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2Q))\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2\alpha_3))\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2\alpha_2))}{\Gamma(1 + \alpha^2)\Gamma(1 + \alpha\alpha_1)\Gamma(1 + \alpha\alpha_1 + \alpha^2)\Gamma(1 + \frac{\alpha}{2}(\bar{\alpha} - 2\alpha_1))}$$

Prove that $\alpha_1 \to C(\alpha_1, \alpha_2, \alpha_3)$ has analytic continuation to $\mathbb{R}$. Then the periodicity determines $C = C_{DOZZ}$.

Proof of Fyodorov-Bouchaud formula (G. Remy)

LCFT on the **disk** $\mathbb{D}$: add a boundary term

$$S(\phi) = \int_\Sigma (|d\phi|^2 + QR_g\phi + \mu e^{\gamma\phi}) dv_g + \tilde{\mu} \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}\phi} d\ell_g$$

Then, for $\mu = 0$ we have

$$U_p := \mathbb{E}(\int_\mathbb{T} e^{\frac{\gamma}{2}X(\theta)} d\theta)^p = 2\gamma^{-1}\tilde{\mu}^{-p}\Gamma(-p)\langle V_\alpha(0)\rangle$$

with $p = \frac{2(Q-\alpha)}{\gamma}$.

$\langle V_\alpha(0)\rangle$ is finite for $\alpha > Q$ so that LCFT gives the negative moments of GMC on $\mathbb{T}$.

F-B formula follows like DOZZ by deriving a hypergeometric equation for $\langle V_{-\frac{\gamma}{2}}(z) V_\alpha(0)\rangle$ and a periodicity relation

$$U_p = \frac{2\pi\Gamma(1 - p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})} U_{p-1}$$

Conformal bootstrap

Conformal bootstrap philosophy:

Every correlation function of a CFT is a linear combination of products of 3-point functions

Theorem (GKRV (2022). Let Σ have genus g. Then

$$\langle \prod_{i=1}^{m} V_{\alpha_i}(z_i) \rangle_\Sigma = \int_{\mathbb{R}_+^{3g+m-3}} |\mathcal{F}(\mathbf{q}, \mathbf{P})|^2 \rho(\mathbf{P}) d\mathbf{P}$$

where

- $\mathbf{q} = (q_1, \ldots, q_{3g+m-3})$ are the **moduli** of the surface (Σ, g) with m marked points , $z_1, \ldots, z_n$.
- $\mathbf{P} = (P_1, \ldots, P_{3g+m-3})$ are spectral parameters
- **Conformal block** $\mathcal{F}(\mathbf{q}, \mathbf{P})$ is a representation theoretic function holomorphic in the moduli $\mathbf{q}$.
- $\rho(\mathbf{P})$ is a product of structure constants $C(\alpha, \alpha', \alpha'')$ with $\alpha, \alpha', \alpha'' \in \{\alpha_i, Q \pm iP_j\}$

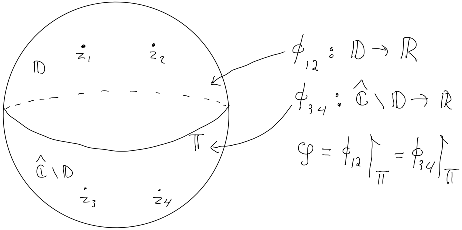
Example: 4-point function

Write 4-point function on the sphere as a **scalar product**

$$\langle \prod_{i=1}^{4} V_{\Delta_i}(z_i) \rangle_{\hat{\mathbb{C}}} = \int_{\varphi:\mathbb{T}\to\mathbb{R}} \mathcal{A}_{12}(\varphi)\mathcal{A}_{34}(\varphi) D\varphi := \langle \mathcal{A}_{12}, \mathcal{A}_{34} \rangle$$

with

$$\mathcal{A}_{12}(\varphi) = \int_{\phi|_{\mathbb{T}}=\varphi} V_{\alpha_i}(z_1) V_{\alpha_2}(z_2) e^{-S_{\mathbb{D}}(\phi)} D\phi$$



Factorisation

Let $\{\psi_n\}$ be an orthonormal basis. By Plancharel

$$\langle \mathcal{A}_{12}, \mathcal{A}_{34} \rangle = \sum_n \langle \mathcal{A}_{12}, \psi_n \rangle \langle \psi_n, \mathcal{A}_{34} \rangle$$

Canonical basis: **eigenfunctions of LCFT Hamiltonian** indexed by (descendants of) scaling fields

$$V_{Q+iP}, \quad P \in \mathbb{R}$$

of scaling dimension $\Delta = \frac{1}{2}(Q^2 + P^2)$.

Then $\langle \mathcal{A}_{12}, \psi_n \rangle \propto C(\alpha_1, \alpha_2, Q + iP)$ and one ends up with the bootstrap formula

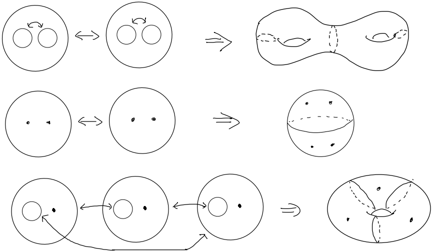
$$\langle \prod_{i=1}^4 V_{\Delta_i}(z_i) \rangle_{\hat{\mathbb{C}}} = \int_{\mathbb{R}} |\mathcal{F}(q, P)|^2 C(\alpha_1, \alpha_2, Q + iP) C(\alpha_3, \alpha_4, Q - iP) dP$$

where q is the modulus (cross ratio of $z_1, \ldots, z_4$).

General case

General case: cut the punctured surface along a homology basis to:

- Pairs of pants $\mathcal{P} \sim \hat{\mathbb{C}} \setminus 3$ disks
- Annuli with one marked point $\hat{\mathbb{C}} \setminus \{2 \text{ disks}, \ 1 \text{ point}\}$
- Disks with two marked points $\hat{\mathbb{C}} \setminus \{1 \text{ disk}, \ 2 \text{ points}\}$



Then use Plancharel at each cutting circle

Spectrum of Liouville theory

Wave functions $\psi(\varphi)$, $\varphi(\theta) = c + \sum_{n\neq 0} \varphi_n e^{in\theta}$. Hamiltonian

$$H = \frac{1}{2}\left(-\frac{d^2}{dc^2} + Q^2 + \sum_{n=1}^{\infty}(a_n^* a_n + \bar{a}_n^* \bar{a}_n)\right) + \mu V$$

$a_n = \partial_{\varphi_n}$ and V multiplication operator with 1d GMC

$$V(\varphi) = e^{\gamma c} \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}\varphi(\theta)^2} d\theta$$

Theorem (Guillarmou, A.K., Rhodes, Vargas (Acta Math. (2022)))
H has a spectral resolution

$$L^2(D\varphi) = \int_{\mathbb{R}_+}^{\oplus} V_P \oplus \tilde{V}_P \, dP$$

where V_P and $\tilde{V}_P$ are highest weight modules of weight Δ_{Q+iP} of two commuting **Virasoro algebras** of central charge $c = 1 + 6Q^2$.

Proof uses **scattering theory** to deform to the $\mu = 0$ GFF case.

Other integrability results

- Using BPZ equations: Structure constants of LCFT on $\mathbb{D}$ (bulk and boundary 1,2,3 point functions): Remy, Zhu, Ang, Sun
- Using conformal welding and SLE: law of modulus of annulus Remy, Ang, Sun
- Representation of conformal blocks as GMS integrals (torus 1-point, sphere 4-point) Remy, Ang, Sun

Challenges

Integrable perturbation of LCFT: Sinh-Gordon model

$$S(\phi) = \int_0^T \int_0^{2\pi R} (\nabla \phi(t,\theta)^2 + \mu \cosh \phi(t,\theta)) d\theta dt$$

- Explicit conjecture for 1-point function (Fateev, Litvinov, Zamolodchikov[2]

$$\lim_{R\to\infty} \lim_{T\to\infty} \langle e^{\alpha\phi(0)} \rangle_\gamma^{R,T}$$

- Explicit conjecture for mass
- By scaling and Feynman-Kac one can prove (GKRV, in preparation)

$$\lim_{T\to\infty} \langle e^{\alpha\phi(0)} \rangle_\gamma^{R,T} = R^{\frac{\alpha^2}{2}} \int_0^\infty e^{\alpha c} \mathbb{E} \left(M_\alpha(\mathbb{T}) \psi_R(c,\phi)^2 \right) dc$$

where ψ_R is the ground state of the Hamiltonian on **unit circle** with coupling constant $R^{\gamma Q} \mu$.

Challenges

- Imaginary Liouville $\gamma \in i\mathbb{R}$ using complex GMC?
- Sine-Gordon model: Again exact conjectures for 1-point function and mass in the symmetry breaking phase

Happy birthday Yan!