An alternative approach for the mean-field behaviour of spread-out Bernoulli percolation in dimensions $d > 6$

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We dedicate this article to Geoffrey Grimmett on the occasion of his seventieth birthday.

Abstract

This article proposes a new way of deriving mean-field exponents for sufficiently spread-out Bernoulli percolation in dimensions $d > 6$. Among other results, we obtain up-to-constant estimates for the full plane and half-plane two-point functions in the critical and near-critical regimes. In a companion paper, we apply a similar analysis to the study of the weakly self-avoiding walk model in dimensions $d > 4$ [?].

1 Introduction

Grasping the (near) critical behaviour of lattice models is one of the key challenges in statistical mechanics. A possible approach involves determining the models' *critical exponents*. Performing this task is typically very challenging, as it involves the unique characteristics of the models and the geometry of the graphs on which they are constructed.

A significant observation was made for models defined on the hypercubic lattice $\mathbb{Z}^d$: beyond the *upper-critical dimension* d_c, the influence of geometry disappears, and the critical exponents simplify, matching those found on a Cayley tree (or *Bethe lattice*) or on the complete graph. The regime $d > d_c$ forms the *mean-field* regime of a model.

Prominent techniques such as the *lace expansion* [BS85] and the *rigorous renormalisation group* [BBS14, BBS15a, BBS15b, BBS19] have been developed to analyze the mean-field regime. However, a significant drawback of these approaches is their predominantly *perturbative* nature, necessitating the identification of a small parameter within the model. It has been established, using lace expansion, that in several contexts [HS90a, Sak07, Har08, Sak15, FvdH17, Sak22] this small parameter can be taken to be $\frac{1}{d}$, meaning that mean-field behaviour was recovered in these setups in dimensions $d \gg 1$.

In the example of the nearest-neighbour (meaning that bonds are pairs of vertices separated by unit Euclidean distance) Bernoulli percolation, mean-field behaviour was established in dimensions $d > 10$ [HS90a, Har08, FvdH17]. This leaves a gap to fill to reach the expected upper critical dimension of the model $d_c = 6$. It is however possible to provide rigorous arguments to identify d_c by introducing an additional perturbative parameter in the model. In the *spread-out* Bernoulli percolation model, the bonds are pairs of vertices separated by distance between 1 and L, where L is taken to be sufficiently large. According to the deep conjecture of *universality*, the critical exponents of these two models should match. This makes spread-out Bernoulli percolation the natural test ground to develop the analysis of the mean-field regime of Bernoulli percolation.

Lace expansion was successfully applied to study various spread-out models in statistical mechanics, including Bernoulli percolation [HS90a, HHS03], lattice trees and animals [HS90b], the Ising model [Sak07, Sak22], and even some long-range versions of the aforementioned examples [CS15, CS19]. Much more information on the lace expansion approach can be found in [Sla06].

In this paper, we provide an alternative argument to obtain mean field bounds on the two-point function of sufficiently spread-out Bernoulli percolation in dimensions $d > 6$. This technique extends to a number of other (spread-out) models after relevant modifications. In a companion paper [**?**], we provide a treatment of the weakly self-avoiding walk model. However, the strategy developed there does not apply *mutatis mutandis* to the setup of spread-out Bernoulli percolation, and the presence of long finite range interactions requires an additional care.

Notations Consider the hypercubic lattice $\mathbb{Z}^d$ and let $y \sim z$ denote the fact that y and z are neighbors in $\mathbb{Z}^d$. Set $\mathbf{e}_j$ to be the unit vector with j-th coordinate equal to 1. Write x_j for the j-th coordinate of x, and denote its ℓ^∞ norm by $|x| := \max\{|x_j| : 1 \le j \le d\}$. Set $\Lambda_n := \{x \in \mathbb{Z}^d : |x| \le n\}$ and for $x \in \mathbb{Z}^d$, $\Lambda_n(x) := \Lambda_n + x$. Also, set $\mathbb{H}_n := -n\mathbf{e}_1 + \mathbb{H}$, where $\mathbb{H} := \mathbb{Z}_+ \times \mathbb{Z}^{d-1} = \{0, 1, \dots\} \times \mathbb{Z}^{d-1}$. Let ∂S be the boundary of the set S given by the vertices in S with one neighbor outside S. Finally, introduce the set of *generalized blocks* of $\mathbb{Z}^d$,

$$\mathcal{B} := \Big\{ \prod_{i=1}^{d} \{a_i, \dots, b_i\} \subset \mathbb{Z}^d \text{ such that } \forall i \le d, -\infty \le a_i \le 0 \le b_i \le \infty \Big\}. \tag{1.1}$$

If A and B are two percolation events, we write $A \circ B$ the event of *disjoint* occurrence of A and B.

1.1 Definitions and statement of the results

Let $L \ge 1$. Since L will be fixed for the whole article, we omit it from the notations. We will consider the Bernoulli percolation measure $\mathbb{P}_\beta$ such that for every $u, v \in \mathbb{Z}^d$,

$$p_{uv}(\beta) := \mathbb{P}_\beta[uv \text{ is open}] = 1 - \exp(-\beta J_{uv}) = 1 - \mathbb{P}_\beta[uv \text{ is closed}], \tag{1.2}$$

where $J_{uv} = c(L)\mathbb{1}_{1 \le |u-v| \le L}$, and $c(L)$ is a normalization constant which guarantees that $|J| := \sum_{x \in \mathbb{Z}^d} J_{0,x} = 1$ (i.e. $c(L) = (|\Lambda_L| - 1)^{-1}$). Much more general choices can be made for J (see e.g. [HS90a, HS02, HHS03]) but we restrict our attention to the above for simplicity.

We are interested in the model's two-point function which is, for $\Lambda \subset \mathbb{Z}^d$, the probability $\mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y]$. When $\Lambda = \mathbb{Z}^d$, we simply write $\mathbb{P}_\beta[x \leftrightarrow y] = \mathbb{P}_\beta[x \overset{\mathbb{Z}^d}{\longleftrightarrow} y]$. It is well known that this model undergoes a phase transition for the existence of an infinite cluster at some parameter $\beta_c \in (0, \infty)$. Moreover, for $\beta < \beta_c$, $\mathbb{P}_\beta[x \leftrightarrow y]$ decays exponentially fast in $|x - y|$, see [AN84, AB87, DCT16]

Our main result is a near-critical estimate of the two-point function in the full space and in the half space $\mathbb{H}$.

It is convenient to use as a correlation length the *sharp length* L_β defined below (see also [DCT16, Pan23, **?**] for a study of this quantity in the context of the Ising model).

For $\beta \geq 0$ and $S \subset \mathbb{Z}^d$, let

$$\varphi_\beta(S) := \sum_{\substack{y \in S \\ z \notin S \\ y \sim z}} \mathbb{P}_\beta[0 \overset{S}{\longleftrightarrow} y] p_{yz}(\beta). \tag{1.3}$$

The sharp length L_β is defined as follows:

$$L_\beta := \inf\{k \geq 1 : \varphi_\beta(\Lambda_k) \leq 1/e\} \in [1, \infty]. \tag{1.4}$$

Also, let β_0 be such that $\varphi_{\beta_0}(\{0\}) = 1$.

Theorem 1.1. *Let $d > 6$. There exists $L_0 = L_0(d) > 0$ such that for every $L \geq L_0$ the following holds. There exist $c, C > 0$ such that for all $\beta \leq \beta_c$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \delta_0(x) + \frac{C}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} \exp(-c|x|/L_\beta) \qquad \forall x \in \mathbb{Z}^d, \tag{1.5}$$

$$\mathbb{P}_\beta[0 \overset{\mathbb{H}}{\longleftrightarrow} x] \leq \delta_0(x) + \frac{C}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \exp(-c|x_1|/L_\beta) \qquad \forall x \in \mathbb{H}. \tag{1.6}$$

The second main theorem of this article is a set of lower bounds matching the bounds in Theorem 1.1 up to uniform multiplicative constants.

Theorem 1.2. *Let $d > 6$. There exists $L_0 = L_0(d) > 0$ such that for every $L \geq L_0$ the following holds. There exist $c, C > 0$ such that for all $\beta_0 \leq \beta \leq \beta_c$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \geq \frac{c}{L^d} \left(\frac{L}{L \vee |x|} \right)^{d-2} \exp(-C|x|/L_\beta) \qquad \forall x \in \mathbb{Z}^d, \tag{1.7}$$

$$\mathbb{P}_\beta[0 \overset{\mathbb{H}}{\longleftrightarrow} x] \geq \frac{c}{L^d} \left(\frac{L}{|x| \vee L} \right)^{d-1} \exp(-C|x|/L_\beta) \qquad \forall x \in \mathbb{H} \text{ with } x_1 = |x|. \tag{1.8}$$

A direct consequence of Theorem 1.1 is the finiteness at criticality of the so-called *triangle diagram*, which plays a central role in the study of the mean-field regime of Bernoulli percolation, see [AN84, HS90a, BA91].

Corollary 1.3 (Finiteness of the triangle diagram)**.** *Let $d > 6$. There exists $L_0 = L_0(d) > 0$ such that for every $L \geq L_0$,*

$$\nabla(\beta_c) := \sum_{x,y \in \mathbb{Z}^d} \mathbb{P}_{\beta_c}[0 \leftrightarrow x] \mathbb{P}_{\beta_c}[x \leftrightarrow y] \mathbb{P}_{\beta_c}[y \leftrightarrow 0] < \infty. \tag{1.9}$$

We now describe how to recover the mean-field behaviour of the *susceptibility* and the *correlation length* ξ_β defined for $\beta < \beta_c$ by

$$\chi(\beta) := \sum_{x \in \mathbb{Z}^d} \mathbb{P}_\beta[0 \leftrightarrow x], \qquad \xi_\beta^{-1} := \lim_{n \to \infty} -\tfrac{1}{n} \log \mathbb{P}_\beta[0 \longleftrightarrow n\mathbf{e}_1]. \tag{1.10}$$

Corollary 1.4. *Let $d > 6$. There exists $L_0 = L_0(d) > 0$ such that for every $L \geq L_0$ the following holds. There exist $c, C > 0$ such that for all $\beta_0 \leq \beta < \beta_c$,*

$$c(\beta_c - \beta)^{-1} \leq \chi(\beta) \leq C(\beta_c - \beta)^{-1}, \tag{1.11}$$

$$c(\beta_c - \beta)^{-1/2} \leq \xi_\beta \leq C(\beta_c - \beta)^{-1/2}, \tag{1.12}$$

$$c(\beta_c - \beta)^{-1/2} \leq L_\beta \leq C(\beta_c - \beta)^{-1/2}. \tag{1.13}$$

Proof. Let $d > 6$ and L_0 be given by Corollary 1.3. Again using Corollary 1.3, we find that $\nabla(\beta_c) < \infty$ which implies by [AN84] the bounds of (1.11). The bounds (1.12) and (1.13) are obtained using (1.11) and Theorems 1.1 and 1.2 twice: one time to get that $\xi_\beta \asymp L_\beta$ and a second time to get $\chi(\beta) \asymp L_\beta^2$, where $\asymp$ means that the quantities are bounded away from each other by constants that are independent of β. □

1.2 Strategy of the proof

A crucial role will be played by the following two inequalities. We include the proof of this statement in the Appendix.

Lemma 1.5. *For $0 < \beta < \beta_c$, $o \in S \subset \Lambda$, and $x \in \Lambda$,*

$$\mathbb{P}_\beta[o \overset{\Lambda}{\longleftrightarrow} x] \leq \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} x] + \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} y] p_{yz}(\beta) \mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} x], \tag{1.14}$$

$$\mathbb{P}_\beta[o \overset{\Lambda}{\longleftrightarrow} x] \geq \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} x] + \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} y] p_{yz}(\beta) \mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} x] - \sum_{\substack{u \in S \\ v \in \Lambda}} E_\beta^{S,\Lambda}(o; u, v) \mathbb{P}_\beta[v \overset{\Lambda}{\longleftrightarrow} x]. \tag{1.15}$$

where for $u \in S$ and $v \in \Lambda$,use diagram not for next

$$E_\beta^{S,\Lambda}(o; u, v) := \mathbb{1}_{v \in S} \sum_{\substack{y \in S \\ z \in \Lambda \setminus S}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} u] \mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} y] p_{yz}(\beta) \mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} v] \mathbb{P}_\beta[v \overset{\Lambda}{\longleftrightarrow} u] \tag{1.16}$$

$$+ \mathbb{1}_{v \in \Lambda \setminus S} \sum_{u \in S} \sum_{y \neq s \in S} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} u] \mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} y] \mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} s] p_{yv}(\beta) p_{sv}(\beta) \tag{1.17}$$

$$+ \sum_{\substack{y,s \in S \\ y \neq s \\ z,t \in \Lambda \setminus S \\ z \neq t}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} u] \mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} y] \mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} s] p_{yz}(\beta) p_{st}(\beta) \mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} v] \mathbb{P}_\beta[t \overset{\Lambda}{\longleftrightarrow} v] \tag{1.18}$$

$$+ \delta_o(u) \sum_{\substack{y \in S \\ z,t \in \Lambda \setminus S \\ z \neq t}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} y] p_{yz}(\beta) p_{yt}(\beta) \mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} v] \mathbb{P}_\beta[t \overset{\Lambda}{\longleftrightarrow} v]. \tag{1.19}$$

In particular, if $S = \{o\}$, for all $v \in \Lambda$,

$$E_\beta^{\{o\},\Lambda}(o; o, v) = \delta_o(v) \sum_{z \in \Lambda \setminus \{o\}} p_{oz}(\beta) \mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} o] + \sum_{\substack{z,t \in \Lambda \setminus \{o\} \\ z \neq t}} p_{oz}(\beta) p_{ot}(\beta) \mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} v] \mathbb{P}_\beta[t \overset{\Lambda}{\longleftrightarrow} v]. \tag{1.20}$$

When $o = 0$, we simply write $E_\beta^{S,\Lambda}(u, v) := E_\beta^{S,\Lambda}(0; u, v)$.

If $S \ni 0$, the quantity

$$E_\beta^{S,\Lambda} := \sum_{\substack{u \in S \\ v \in \Lambda}} E_\beta^{S,\Lambda}(u, v) \tag{1.21}$$

will be referred to as the *error amplitude*. One of the pivotal steps of our argument is a proof that $E_\beta^{\Lambda_n,\mathbb{Z}^d}$ and $E_\beta^{\mathbb{H}_n,\mathbb{Z}^d}$ are finite and small (in terms of L). This is also where the assumption $d > 6$ becomes crucial.

Remark 1.6. The correction term or "error" term in the lower bound of Lemma 1.5 (illustrated in Figure 1) differs from the corresponding one for the weakly self-avoiding walk model at different levels. We first notice that it makes appear (1.16) that is reminiscent of the *triangle* diagram of percolation. The terms (1.17)-(1.19) come from the possibility of finding multiple candidates for the "first" edge leaving S. Then, (1.18)-(1.19) are "non-local" in the sense that they have a non-zero contribution for $v \notin S + \Lambda_L$. Local errors (like in the case of the weakly self-avoiding walk) are more convenient as they allow for bounds of the type

$$\sum_{\substack{u \in S \\ v \in S+\Lambda_L}} E_\beta^{S,\Lambda}(o; u, v)\mathbb{P}_\beta[v \stackrel{\Lambda}{\longleftrightarrow} x] \leq E_\beta^{S,\Lambda} \cdot \max\left\{\mathbb{P}_\beta[w \stackrel{\Lambda}{\longleftrightarrow} x] : w \in S + \Lambda_L\right\}. \tag{1.22}$$

Such a bound is not immediately available for the remaining terms in the error. However, in the proof, we will argue that in the cases of interest, v will typically be "close" to S. This will allow to treat this additional error term as being "local", at least in average.

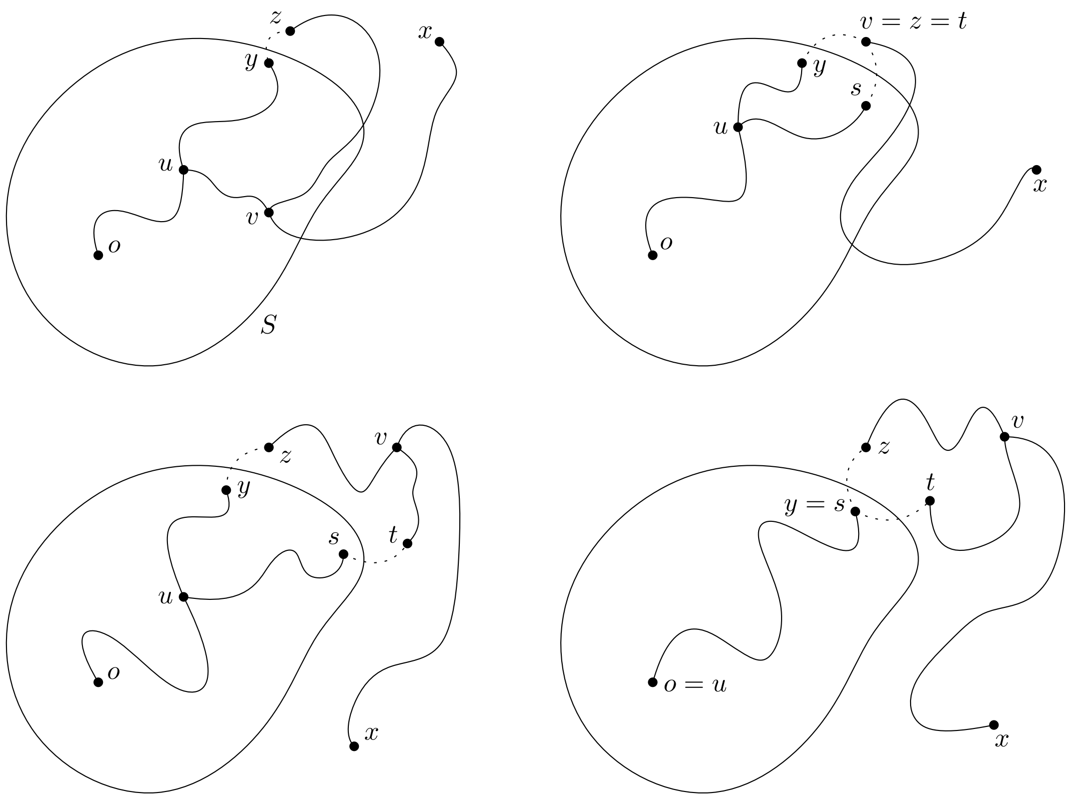

Figure 1: An illustration of the different terms contributing to $E_\beta^{S,\Lambda}(o; u, v)$. The dotted lines represent the open edges leaving S. From left to right and top to bottom, we illustrate (1.16)–(1.19). The two configurations at the top correspond to "local" error terms in the sense that v has to remain in $S + \Lambda_L$. The two configurations at the bottom correspond to "non-local" error terms since v can potentially be far away from S. surprised there cannot be only two terms. For instance 2 and 3 aren't the same with z not different from t

The following definition is motivated by Remark 1.6.

Definition 1.7. Let $0 < \beta < \beta_c$. Let $S \subset \Lambda$, $o \in S$, and $x \in \Lambda$. We introduce $\mathsf{E}_\beta^{S,\Lambda}(o, x)$, the *non-local* error term, defined by

$$\mathsf{E}_\beta^{S,\Lambda}(o, x) := \sum_{\substack{u \in S \\ v \notin S+\Lambda_L}} E_\beta^{S,\Lambda}(o; u, v)\mathbb{P}_\beta[v \stackrel{\Lambda}{\longleftrightarrow} x]. \tag{1.23}$$

When $S = \{o\}$ is a singleton, we just write $\mathsf{E}_\beta^{S,\Lambda}(o,x) = \mathsf{E}_\beta^{\Lambda}(o,x)$.

The random walk distribution naturally associated with $\varphi_\beta(\Lambda_k)$ for $\beta > 0$ and $k \geq 0$ will play a pivotal role in our arguments.

Definition 1.8. Let $\beta > 0$, $k \geq 0$, and $x \in \mathbb{Z}^d$. Define the simple random walk $(X_k^x)_{k \geq 0}$ started at $x \in \mathbb{Z}^d$ and of law $\mathbb{P}^{(k)}_{\mathrm{RW},x,\beta}$ given by the step distribution:

$$\mathbb{P}^{(k)}_{\mathrm{RW},x,\beta}[X_1^x = y] := \frac{\mathbb{1}_{y \notin \Lambda_k(x)}}{\varphi_\beta(\Lambda_k)} \sum_{u \in \Lambda_k(x)} \mathbb{P}_\beta[x \xleftrightarrow{\Lambda_k(x)} u] p_{uv}(\beta). \tag{1.24}$$

When $k = 0$, we just write $\mathbb{P}_{\mathrm{RW},x,\beta} := \mathbb{P}^{(0)}_{\mathrm{RW},x,\beta}$.

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2 Proof of Theorem 1.1

For $\beta > 0$ and $k \geq 0$, define

$$\psi_\beta(\mathbb{H}_k) := \sum_{x \in \partial \mathbb{H}_k} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_k} x]. \tag{2.1}$$

We will use a bootstrap argument (following the original idea from [Sla87]) and prove that an *a priori* estimate on the half-plane two-point function can be improved for sufficiently large L. The idea will be to observe that the two inequalities of Lemma 1.5 provide a good control on $\psi_\beta(\mathbb{H}_k)$, which can be interpreted as an ℓ^1 estimate on the half-plane two-point function at distance k. The point-wise, or ℓ^∞, (half-space) estimate will follow from a *regularity* estimate which allows to compare two-point functions ending at close points. Finally, we will deduce the full space estimate from the half-space one. The improvement of the a priori estimates will be permitted by classical random walk computations involving the random walk introduced in Definition 1.8.

To implement this scheme, we introduce the following parameter β^*.

Definition 2.1. Let $\mathbf{C} > 1$. We define $\beta^* = \beta^*(\mathbf{C})$ to be the largest real number in[1] $[0, 2 \wedge \beta_c]$ such that for every $\beta < \beta^*$,

$$\psi_\beta(\mathbb{H}_n) < \delta_0(n) + \frac{\mathbf{C}}{L} \qquad \forall n \geq 0, \tag{ℓ^1_β}$$

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] < \delta_0(x) + \frac{\mathbf{C}}{L^d}\left(\frac{L}{L \vee |x_1|}\right)^{d-1} \qquad \forall x \in \mathbb{H}. \tag{ℓ^∞_β}$$

The first and second assumptions can be understood as ℓ^1 and ℓ^∞ bounds on the half-space two-point function. Note that when $\mathbf{C}$ is large enough, one has $\beta^* \geq \beta_0$ where β_0

[1]It might be surprising to additionally ask $\beta^* \leq 2$. However, we will see that $\beta_c = 1 + O(L^{-d})$ as L goes to infinity.

is defined[2] by $\varphi_{\beta_0}(\{0\}) = 1$, as a bound by the corresponding random-walk quantities implies that the estimates are true at $\beta = \beta_0$ (this can be seen by iterating infinitely many times the upper bound of Lemma 1.5 with S a singleton and $\beta = \beta_0$).

Our goal is to show that β^* is in fact equal to β_c provided that $\mathbf{C}$ is large enough. The proof goes in three steps. First, we show that we can obtain a bound on $\varphi_\beta(\mathbb{H}_n)$ when $\beta < \beta^*$. This corresponds to the sum of the $\psi_\beta(\mathbb{H}_{n-k})$ for $0 \le k \le L-1$. Second, we control the gradient of the two-point function. Third, we use that the two-point function does not fluctuate too much when moving a little one of the endpoints (thanks to the second point) to turn the bound on $\varphi_\beta(\mathbb{H}_n)$ into an improved bound on $\psi_\beta(\mathbb{H}_n)$. This last quantity is in some sense an improvement of the ℓ^1 bound on the half-space two-point function, which can be use (using the second point once again) to obtain an improved ℓ^∞ bound. From these improvements, we obtain that β^* cannot be strictly smaller than $2 \wedge \beta_c$, since otherwise the improved estimates would remain true (by exponential decay) for β slightly larger than β^*, which would then contradict the definition of β^*. Thus, $\beta^* = 2 \wedge \beta_c$. The last step of the argument then consists of proving that $\beta^* < 2$ which immediately forces $\beta^* = \beta_c$.

2.1 Obtaining bounds on $\varphi_\beta(B)$ with $B \in \mathcal{B}$

The following proposition is the crucial step of our strategy: from the bounds (ℓ^1_β) and (ℓ^∞_β), we obtain a bound on $\varphi_\beta(\mathbb{H}_n)$ that involves the range L. In some sense, for large L this bound will be an improvement on (ℓ^1_β), as we will see in Section 2.3.

Proposition 2.2. *Fix $d > 6$ and $\mathbf{C} > 1$. There exists $K = K(\mathbf{C}, d) > 0$ such that for every $\beta < \beta^*$ and $B \in \mathcal{B}$,*

$$\varphi_\beta(B) < 1 + \frac{K}{L^d}. \tag{2.2}$$

Remark 2.3. It would be interesting to prove that such a bound holds for $\varphi_\beta(S)$ uniformly on every finite set S containing 0 and not only for $B \in \mathcal{B}$.

We start with a number of simple bounds on the two-point function in the bulk and in the half-space obtained thanks to the assumption that $\beta < \beta^*$.

Lemma 2.4. *Fix $d > 6$ and $\mathbf{C} > 1$. For every $\beta < \beta^*$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \le \frac{3\mathbf{C}^2}{L^d} \left(\frac{L}{L \vee |x|}\right)^{d-2} \qquad \forall x \in \mathbb{Z}^d \setminus \{0\}, \tag{2.3}$$

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \frac{\mathbf{C}^2(k+L)}{L^{d+1}} \left(\frac{L}{L \vee (n-k)}\right)^{d-1} \qquad \forall n \ge k \ge 1, \forall x \in \partial\mathbb{H}_{n-k}, \tag{2.4}$$

$$\sum_{x \in \partial\mathbb{H}_{n-k}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \delta_n(k) + \frac{\mathbf{C}^2(k+L)}{L^2} \qquad \forall n, k \ge 0. \tag{2.5}$$

Proof. Let us start with the first inequality. Assume that $x_1 = |x|$. If the connection to x is not included in $\mathbb{H}$, decompose according to the first left-most point z of an open

[2]Note that $\beta_0 \ge 1$ since $1 = \varphi_{\beta_0}(\{0\}) \le \sum_{x \in \mathbb{Z}^d}(1 - e^{-\beta_0 J_{0x}}) \le \beta_0 \sum_{x \in \mathbb{Z}^d} J_{0x} = \beta_0$ since $|J| = 1$.

$\partial\mathbb{H}_n$ $\partial\mathbb{H}_n$ $\partial\mathbb{H}_{n-k}$

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Figure 2: On the left, an illustration of the decomposition of a self-avoiding path connecting 0 to x used in the proof of (2.3). On the right, a similar decomposition used in the proof of (2.4).

self-avoiding path connecting 0 to x; see Figure 2. Using the BK inequality (see [Gri99]), we get when $|x| \geq L$,

$$\begin{aligned}
\mathbb{P}_\beta[0 \leftrightarrow x] &\leq \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] + \sum_{n\geq 1}\sum_{z\in\partial\mathbb{H}_n} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} z]\mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} x] && (2.6)\\
&\overset{(\ell^\infty_\beta)}{\leq} \frac{\mathbf{C}}{L^d}\Big(\frac{L}{|x|\vee L}\Big)^{d-1} + \sum_{n\geq 1}\psi_\beta(\mathbb{H}_n)\frac{\mathbf{C}}{L^d}\Big(\frac{L}{L\vee(|x|+n)}\Big)^{d-1} && (2.7)\\
&\overset{(\ell^1_\beta)}{\leq} \frac{\mathbf{C}}{L^2}\frac{1}{|x|^{d-2}} + \frac{\mathbf{C}^2}{L^2}\frac{1}{(d-2)|x|^{d-2}} \leq \frac{(\mathbf{C}+1)\mathbf{C}}{L^2}\frac{1}{|x|^{d-2}}, && (2.8)
\end{aligned}$$

where on the last inequality, we used that $|x| \geq L$ and that $\sum_{n\geq\alpha}\frac{1}{(n+1)^{d-1}} \leq \frac{1}{(d-2)\alpha^{d-2}}$ for $\alpha \geq 1$. When $1 \leq |x| < L$,

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \frac{\mathbf{C}}{L^d} + \sum_{n=1}^{L-|x|}\psi_\beta(\mathbb{H}_n)\frac{\mathbf{C}}{L^d} + \sum_{n\geq L-|x|+1}\psi_\beta(\mathbb{H}_n)\frac{\mathbf{C}}{L}\frac{1}{(|x|+n)^{d-1}} \leq \frac{\mathbf{C}(2\mathbf{C}+1)}{L^d}. \tag{2.9}$$

For the second inequality, pick $x \in \partial\mathbb{H}_{n-k}$. To bound $\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x]$, decompose an open self-avoiding path connecting 0 to x according to its first left-most point z; see Figure 2. Using the BK inequality one more time, we get

$$\begin{aligned}
\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] &\leq \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-k}} x] + \sum_{j=0}^{k-1}\sum_{z\in\partial\mathbb{H}_{n-j}} \mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} x]\mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} x] && (2.10)\\
&\overset{(\ell^\infty_\beta)}{\leq} \frac{\mathbf{C}}{L^d}\Big(\frac{L}{L\vee(n-k)}\Big)^{d-1} + \sum_{j=0}^{k-1}\sum_{z\in\partial\mathbb{H}_{n-j}}\frac{\mathbf{C}}{L^d}\Big(\frac{L}{L\vee(n-j)}\Big)^{d-1}\mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_{n-j}} x] && (2.11)\\
&\overset{(\ell^1_\beta)}{\leq} \Big(\frac{\mathbf{C}}{L^d} + \frac{\mathbf{C}^2 k}{L^{d+1}}\Big)\Big(\frac{L}{L\vee(n-k)}\Big)^{d-1}. && (2.12)
\end{aligned}$$

For the third inequality, consider the same decomposition as the second one, but use (ℓ^1_β) twice instead of (ℓ^1_β) and (ℓ^∞_β). Consider first the case $0 \leq k \leq n-1$. Summing

(2.10) (which holds for that range of k) over $x \in \partial \mathbb{H}_{n-k}$ gives

$$\sum_{x\in\partial\mathbb{H}_{n-k}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \overset{(\ell^1_\beta)}{\leq} \frac{\mathbf{C}}{L} + \frac{k\mathbf{C}^2}{L^2}. \tag{2.13}$$

We similarly get that for $k = n$,

$$\sum_{x\in\partial\mathbb{H}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \leq 1 + \frac{\mathbf{C}}{L} + \frac{n\mathbf{C}^2}{L^2}. \tag{2.14}$$

The case $k > n$ is handled similarly by replacing $\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-k}} x]$ by $\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x]$ in (2.10). □

We begin with a (rough) bound on $\varphi_\beta(\mathbb{H}_n)$ when $\beta < \beta^*$.

Lemma 2.5. *Fix $d > 6$ and $\mathbf{C} > 1$. For every $\beta < \beta^*$, every $n \geq 0$,*

$$\varphi_\beta(\mathbb{H}_n) \leq 6\mathbf{C}^3. \tag{2.15}$$

Proof. Let $n \geq 0$. Decompose a percolation configuration contributing to one of the summand in $\varphi_\beta(\mathbb{H}_n)$ according to the left-most point u along an open self-avoiding walk connecting 0 to y. This gives

$$\begin{aligned}
\varphi_\beta(\mathbb{H}_n) &\leq \sum_{k=0}^{L-1} \sum_{\substack{y\in\partial\mathbb{H}_{n-k}\\ z\notin\mathbb{H}_n}} p_{yz}(\beta) \sum_{\ell=0}^{k} \sum_{u\in\partial\mathbb{H}_{n-\ell}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u]\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_{n-\ell}} y] && (2.16)\\
&= \sum_{k=0}^{L-1}\sum_{\ell=0}^{k} \sum_{u\in\partial\mathbb{H}_{n-\ell}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u]\Big(\sum_{\substack{y\in\partial\mathbb{H}_{n-k}\\ z\notin\mathbb{H}_n}} p_{yz}(\beta)\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_{n-\ell}} y]\Big) && (2.17)\\
&= \sum_{\ell=0}^{L-1} \sum_{u\in\partial\mathbb{H}_{n-\ell}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u]\Big(\sum_{k=\ell}^{L-1}\sum_{\substack{y\in\partial\mathbb{H}_{n-k}\\ z\notin\mathbb{H}_n}} p_{yz}(\beta)\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_{n-\ell}} y]\Big). && (2.18)
\end{aligned}$$

Now, notice that for $k \leq \ell \leq L-1$ and $y \in \partial\mathbb{H}_{n-k}$, one has,

$$\sum_{z\notin\mathbb{H}_n} p_{yz}(\beta) \leq \frac{L-k}{2L}(|\Lambda_L|-1)(1-e^{-\beta J_{0\mathbf{e}_1}}) \leq \beta\frac{L-k}{2L} \leq 1, \tag{2.19}$$

where we used that $\beta \leq \beta^* \leq 2$. It follows that,

$$\begin{aligned}
\varphi_\beta(\mathbb{H}_n) &\leq \sum_{\ell=0}^{L-1} \sum_{u\in\partial\mathbb{H}_{n-\ell}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u]\Big(\sum_{k=\ell}^{L-1}\sum_{y\in\partial\mathbb{H}_{n-k}} \mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_{n-\ell}} y]\Big) && (2.20)\\
&\overset{(2.5)}{\leq} \sum_{\ell=0}^{L-1} \sum_{u\in\partial\mathbb{H}_{n-\ell}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u]\sum_{k=\ell}^{L-1}\Big(\delta_k(\ell) + \frac{\mathbf{C}^2(k-\ell+L)}{L^2}\Big) && (2.21)\\
&\leq (1+2\mathbf{C}^2)\sum_{\ell=0}^{L-1} \sum_{u\in\mathbb{H}_{n-\ell}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u] && (2.22)\\
&\overset{(\ell^1_\beta)}{\leq} (1+2\mathbf{C}^2)(1+\mathbf{C}) \leq 6\mathbf{C}^3. && (2.23)
\end{aligned}$$

This concludes the proof. □

We now turn to the estimate of the "error" in (1.15) when $\beta < \beta^*$.

Lemma 2.6. *Fix $d > 6$ and $\mathbf{C} > 1$. There exists $K = K(\mathbf{C}, d) > 0$ such that for every $\beta < \beta^*$ and $B \in \mathcal{B}$,*

$$E_\beta^{B,\mathbb{Z}^d} \leq \frac{K}{L^d}. \tag{2.24}$$

Proof. For the first inequality, we use the pointwise bounds of Lemma 2.4. The fact that $d > 7$ implies the existence of K. For completeness, we include the full computation in Appendix 4. The second inequality follows from the first one (by changing K) since

$$E_\beta^{B,\mathbb{Z}^d} \leq \sum_i E_\beta^{\mathbb{H}_{-a_i},\mathbb{Z}^d} + \sum_{i:b_i<\infty} E_\beta^{\mathbb{H}_{b_i},\mathbb{Z}^d}$$

by monotonocity. □

We are now equipped to prove Proposition 2.2.

Proof of Proposition 2.2. Fix $B \in \mathcal{B}$. Summing (1.14) over every $x \in \mathbb{Z}^d$ gives

$$\varphi_\beta(B)\chi(\beta) - \chi(\beta)E_\beta^{B,\mathbb{Z}^d} \leq \chi(\beta), \tag{2.25}$$

which implies the result by dividing by $\chi(\beta)$ and using Lemma 2.2. □

2.2 Control of the gradient

Proposition 2.2 implies a ℓ^1-type bound on the half-space two-point function which involves the range L of the interaction, and which in some sense is better than (ℓ^1_β). The following regularity estimate, which will be the goal of this section, will later allow us on to convert the bound on $\varphi_\beta(\mathbb{H}_n)$ into improved ℓ^1 and ℓ^∞ bounds. We recall that $\mathcal{B}$ is the set of blocks of $\mathbb{Z}^d$.

Proposition 2.7 (Regularity estimate at mesoscopic scales)**.** *Fix $d > 6$ and $\mathbf{C} > 1$. For every $\eta > 0$, there exist $\delta = \delta(\eta, d) \in (0, 1/2)$, $A = A(\eta, d)$, and $L_0 = L_0(\eta, A, \mathbf{C}, d)$ such that for every $L \geq L_0$, every $\beta < \beta^*$, every $n \geq AL$, every $\Lambda \supset \Lambda_{3n}$, every $X \subset \Lambda \setminus \Lambda_{3n}$, and every $u, v \in \Lambda_{\delta n}$,*

$$\Big|\sum_{x\in X} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} x] - \mathbb{P}_\beta[v \overset{\Lambda}{\longleftrightarrow} x]\Big| \leq \eta \max_{w\in\Lambda_{3n}} \sum_{x\in X} \mathbb{P}_\beta[w \overset{\Lambda}{\longleftrightarrow} x] \tag{2.26}$$

$$+ A \max_{w\in\Lambda_{3n}} \max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{3n}\\ S\ni w}} \sum_{x\in X} \mathsf{E}_\beta^{S,\Lambda}(w, x). \tag{2.27}$$

We begin with a regularity estimate at *microscopic* scales of order L.

Lemma 2.8 (Regularity estimate at microscopic scales)**.** *Fix $d > 6$ and $\mathbf{C} > 1$. For every $\eta > 0$, there exist $A_1 = A_1(\eta, d) > 0$ and $L_1 = L_1(\eta, A_1, \mathbf{C}, d) > 0$ large enough such that for every $L \geq L_1$, every $\beta < \beta^*$, every $n \geq A_1 L$, every $\Lambda \supset \Lambda_{2n}$, every $X \subset \Lambda \setminus \Lambda_{2n}$, every $u, v \in \Lambda_n$ with $|u - v| \leq 3L$,*

$$\Big|\sum_{x\in X} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} x] - \mathbb{P}_\beta[v \overset{\Lambda}{\longleftrightarrow} x]\Big| \leq \max_{w,w'\in\Lambda_{2n}} \Big(\eta \sum_{x\in X} \mathbb{P}_\beta[w \overset{\Lambda}{\longleftrightarrow} x] + A_1 \sum_{x\in X} \mathsf{E}_\beta^\Lambda(w', x)\Big). \tag{2.28}$$

Proof. We prove the result for $X = \{x\}$, but the general argument follows similarly. Set $\varphi := \varphi_\beta(\{0\})$. Let $T \geq 1$ to be fixed, $n \geq 2TL$, and assume $u, v \in \Lambda_n$ with $|u - v| \leq 3L$. Iterating (1.15) T times with S a singleton and Λ gives

$$\mathbb{P}_\beta[u \xleftrightarrow{\Lambda} x] \leq \varphi_\beta(\{0\})^T \mathbb{E}_{\mathrm{RW},u,\beta}\Big[\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]\Big] \tag{2.29}$$

$$\mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \geq \varphi_\beta(\{0\})^T \mathbb{E}_{\mathrm{RW},v,\beta}\Big[\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]\Big] - \frac{K}{L^d}\Big(\sum_{t=0}^{T-1}\varphi^t\Big) \max_{w \in \Lambda_{n+(T+1)L}} \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x]$$
$$- \Big(\sum_{t=0}^{T-1}\varphi^t\Big) \max_{w \in \Lambda_{n+TL}} \mathsf{E}_\beta^\Lambda(w, x), \tag{2.30}$$

where K is the constant provided by Lemma 2.6.

A random-walk estimate[3] implies that for every $\eta > 0$, there exists $T = T(\eta, d)$ large enough such that the random walks X_T^u and X_T^v can be coupled to coincide with probability larger than $1 - \eta/2$. This implies

$$\mathbb{E}_{\mathrm{RW},u,\beta}\Big[\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]\Big] - \mathbb{E}_{\mathrm{RW},v,\beta}\Big[\mathbb{P}_\beta[X_T \xleftrightarrow{\Lambda} x]\Big] \leq \frac{\eta}{2} \max\Big\{\mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x] : w \in \Lambda_{n+TL}\Big\}. \tag{2.31}$$

Furthermore, since $\varphi \leq 1 + \frac{K}{L^d}$, we may choose $L_1 = L_1(\eta, T, K, d)$ large enough such that for $L \geq L_1$, $\frac{K}{L^{d-1}} \sum_{t=0}^{T-1} \varphi^t \leq \eta/2$ and $\sum_{t=0}^{T-1} \varphi^t \leq 2T$. The result follows by plugging these estimates and (2.31) in the difference of (2.29) and (2.30), and setting $A_1 := 2T$. □

Let $\Lambda_n^+ := \{x \in \Lambda_n : x_1 > 0\}$ and $H = H(L) := \{v \in \mathbb{Z}^d : |v_1| \leq L\}$. The next result formalizes the fact that when $x \in \Lambda_n$, most of the mass in $\varphi_\beta(\Lambda_n(x))$ comes from the side of Λ_n that is the closest to x.

Lemma 2.9. *Fix $d > 6$ and $\mathbf{C} > 1$. Let K be the constant of Lemma 2.6. There exist $c = c(d), L_2 = L_2(\mathbf{C}, d) > 0$ such that for every $L \geq L_2$, every $\beta < \beta^*$, and every $v \in \Lambda_k^+$ with $k \leq n/2$,*

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \mathbb{P}_\beta[v \xleftrightarrow{\Lambda_n^+} y] p_{yz}(\beta) \leq \Big(1 + \frac{K}{L^d}\Big)\Big(\frac{2k}{n}\Big)^c. \tag{2.32}$$

Proof. Define (n_ℓ) by $n_0 = n$ and then $n_{\ell+1} = \lfloor (n_\ell - 1)/2 \rfloor$. We proceed by induction by proving that for every $\ell \geq 0$ and $v \in \Lambda_{n_\ell}^+$,

$$\sum_{\substack{y \in \Lambda_n^+ \\ z \notin \Lambda_n^+ \cup H}} \mathbb{P}_\beta[v \xleftrightarrow{\Lambda_n^+} y] p_{yz}(\beta) \leq \Big(1 - \frac{1}{2d}\Big)^\ell \Big(1 + \frac{K}{L^d}\Big)^{\ell+1} \tag{2.33}$$

The case $\ell = 0$ follows from Proposition 2.2. Let us transfer the estimate from ℓ to $\ell + 1$. Fix $v \in \Lambda_{n_{\ell+1}}^+$. Let $B := \Lambda_{v_1 - 1}(v)$. By symmetry and Lemma 2.2, we have that

$$\sum_{\substack{r \in B \\ s \notin B \cup H}} \mathbb{P}_\beta[v \xleftrightarrow{B} r] p_{rs}(\beta) \leq \frac{2d-1}{2d}\varphi_\beta(B) \leq \Big(1 - \frac{1}{2d}\Big)\Big(1 + \frac{K}{L^d}\Big). \tag{2.34}$$

[3]For full disclosure we briefly explain how to obtain it. Note that it is sufficient to suppose that u and v differ by only one coordinate, say the first one. Consider a sequence of i.i.d real random variable $(\xi_i)_{i \geq 1}$ of law given by $\mathbb{P}[\xi_i = k] = \mathbb{1}_{-L \leq k \leq L,\, k \neq 0} \frac{(2L+1)}{(2L+1)^d - 1} + \mathbb{1}_{k=0} \frac{2L}{(2L+1)^d - 1}$. Consider an independent copy $(\xi_i')_{i \geq 1}$. Let $S_k := (u_1 - v_1) + \sum_{i=1}^k (\xi_i - \xi_i')$ and write $\mathbb{P}^{u_1 - v_1}$ for the law of the associated random walk (started at $u_1 - v_1$). Let $\eta > 0$. It is sufficient to show that there exists a universal (in particular independent of L) constant $C = C(\eta)$ such that for all $T \geq C$, $\mathbb{P}^{u_1 - v_1}[\tau_0 > T] \leq \eta/2$, where τ_0 is the hitting time of 0. This last fact can be found in [Uch11].

We deduce from Lemma 1.5 and the induction hypothesis that

$$\begin{aligned}\sum_{\substack{y\in\Lambda_n^+\\ z\notin\Lambda_n^+\cup H}} \mathbb{P}_\beta[v \xleftrightarrow{\Lambda_n^+} y]p_{yz}(\beta) &\le \sum_{\substack{y\in\Lambda_n^+\\ z\notin\Lambda_n^+\cup H}} \Big(\sum_{\substack{r\in B\\ s\notin B\cup H}} \mathbb{P}_\beta[v \xleftrightarrow{B} r]p_{rs}(\beta)\mathbb{P}_\beta[s \xleftrightarrow{\Lambda_n^+} y]\Big)p_{yz}(\beta)\\ &= \sum_{\substack{r\in B\\ s\notin B\cup H}} \mathbb{P}_\beta[v \xleftrightarrow{B} r]\Big(\sum_{\substack{y\in\Lambda_n^+\\ z\notin\Lambda_n^+\cup H}} \mathbb{P}_\beta[s \xleftrightarrow{\Lambda_n^+} y]p_{yz}(\beta)\Big)p_{rs}(\beta)\\ &\le \Big(1-\frac{1}{2d}\Big)^{\ell-1}\Big(1+\frac{K}{L^d}\Big)^{\ell} \sum_{\substack{r\in B\\ s\notin B\cup H}} \mathbb{P}_\beta[v \xleftrightarrow{B} r]p_{rs}(\beta)\\ &\overset{(2.34)}{\le} \Big(1-\frac{1}{2d}\Big)^{\ell}\Big(1+\frac{K}{L^d}\Big)^{\ell+1}.\end{aligned}$$

This concludes the proof by choosing L large enough so that $(1-\frac{1}{2d})(1+\frac{K}{L^d}) < 1$ and $c > 0$ small enough. □

We are now in a position to prove the main result of this section.

Proof of Proposition 2.7. We prove the result for $X = \{x\}$, the general case follows similarly. Assume first that $u = k\mathbf{e}_1$ and $v = -k\mathbf{e}_1$ (with $k \le \delta n$). Consider the sets $B^+ := \Lambda_n^+$ and $B^- := -\Lambda_n^+$. Applying Lemma 1.5 twice gives

$$\mathbb{P}_\beta[u \xleftrightarrow{\Lambda} x] \le \sum_{\substack{y\in B^+\\ z\notin B^+}} \mathbb{P}_\beta[u \xleftrightarrow{B^+} y]p_{yz}(\beta)\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x], \tag{2.35}$$

$$\mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \ge \sum_{\substack{y\in B^-\\ z\notin B^-}} \mathbb{P}_\beta[v \xleftrightarrow{B^-} y]p_{yz}(\beta)\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] - \frac{K}{L^d}\max_{w\in B^-+\Lambda_L} \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x] - \mathsf{E}_\beta^{B^-,\Lambda}(v,x). \tag{2.36}$$

We take the difference and use that when $z \in H$, we may associate every pair (y,z) in the sum in (2.35) with the pair (y',z') symmetric with respect to the hyperplane $\{u \in \mathbb{Z}^d : u_1 = 0\}$ in the sum in (2.36), see Figure 3. By doing so, we notice that z and z' are within a distance $2L$ of each other. Hence, if $A_1 = A_1(\eta/2)$ and L_1 are given by Lemma 2.8, providing $L \ge L_1$, we get that for such pairs (y,z) and (y',z'),

$$\Big|\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] - \mathbb{P}_\beta[z' \xleftrightarrow{\Lambda} x]\Big| \le \frac{\eta}{2}\max_{w\in\Lambda_{2n}} \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x] + A_1 \max_{w\in\Lambda_{2n}} \mathsf{E}_\beta^{\Lambda}(w,x). \tag{2.37}$$

Plugging this estimate in the difference of (2.35) and (2.36), and then invoking Lemma 2.9 (to the cost of potentially increasing L again), gives

$$\mathbb{P}_\beta[u \xleftrightarrow{\Lambda} x] - \mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \tag{2.38}$$

$$\le \Big(\frac{\eta}{2}\varphi_\beta(B^+) + \sum_{\substack{y\in B^+\\ z\notin B^+\cup H}} \mathbb{P}_\beta[u \xleftrightarrow{B^+} y]p_{yz}(\beta) + \frac{K}{L^d}\Big)\max_{w\in\Lambda_{2n}} \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x]$$

$$+ \varphi_\beta(B^+)A_1 \max_{w\in\Lambda_{2n}} \mathsf{E}_\beta^{\Lambda}(w,x) + \mathsf{E}_\beta^{B^-,\Lambda}(v,x) \tag{2.39}$$

$$\le \Big[\Big(1+\frac{K}{L^d}\Big)\Big(\frac{\eta}{2}+(2\delta)^c\Big) + \frac{K}{L^d}\Big]\max_{w\in\Lambda_{2n}} \mathbb{P}_\beta[w \xleftrightarrow{\Lambda} x]$$

$$+ \Big(1+\frac{K}{L^d}\Big)K_1 \max_{w\in\Lambda_{2n}} \mathsf{E}_\beta^{\Lambda}(w,x) + \mathsf{E}_\beta^{B,\Lambda}(v,x), \tag{2.40}$$

where we used Proposition 2.2 to obtain that $\varphi_\beta(B^+) \le 1 + \frac{K}{L^d}$. We then write,

$$\Big(1+\frac{K}{L^d}\Big)A_1 \max_{w\in\Lambda_{2n}} \mathsf{E}_\beta^{\Lambda}(w,x) + \mathsf{E}_\beta^{B^-,\Lambda}(v,x) \le 2A_1 \max_{w\in\Lambda_{2n}} \max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{2n}\\ S\ni 0}} \mathsf{E}_\beta^{S,\Lambda}(w,x). \tag{2.41}$$

The proof follows by setting $A = 2A_1$, choosing $\delta = \delta(\eta)$ small enough, and then L large enough.

When $u = k\mathbf{e}_1$ and $v = -(k+1)\mathbf{e}_1$, simply change B^- to $-\mathbf{e}_1 - \Lambda_n^+$. The general case follows by rotating and translating[4] the box. The final result follows by summing over different coordinates and changing η to $d\eta$. □

2.3 Proof of Theorem 1.1

Before moving to the improvement of the ℓ^1 and ℓ^∞ bounds, we begin with two useful estimates on the non-local error term E_β which appears in the regularity estimates of Proposition 2.7 and Lemma 2.8.

Lemma 2.10. *Fix $d > 6$, $\mathbf{C} > 1$, and $A > 0$. Let K be the constant of Lemma 2.6. For every $\beta < \beta^*$, one has,*

$$\max_{w\in\Lambda_{AL}} \sum_{x\in\partial\mathbb{H}_n} \mathsf{E}_\beta^{\mathbb{H}_n}(w,x) \le \frac{K+8\mathbf{C}^2}{L}\frac{\mathbf{C}}{L^d} \qquad \forall n > 2AL. \tag{2.42}$$

Proof. By definition, if $w \in \Lambda_{AL}$,

$$\sum_{x\in\partial\mathbb{H}_n} \mathsf{E}_\beta^{\mathbb{H}_n}(w,x) = \sum_{x\in\partial\mathbb{H}_n}\sum_{v\in\mathbb{H}_n\setminus\Lambda_L}\sum_{\substack{z,t\in\mathbb{H}_n\\ z\neq t}} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z \xleftrightarrow{\mathbb{H}_n} v]\mathbb{P}_\beta[t \xleftrightarrow{\mathbb{H}_n} v]\mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x]. \tag{2.43}$$

The contribution for coming from $v \notin \partial\mathbb{H}_n$ is bounded by

$$\sum_{v\in\mathbb{H}_{n-1}}\sum_{\substack{z,t\in\mathbb{H}_n\\ z\neq t}} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\leftrightarrow v]\mathbb{P}_\beta[t\leftrightarrow v]\Big(\sum_{x\in\partial\mathbb{H}_n}\mathbb{P}_\beta[v \xleftrightarrow{\mathbb{H}_n} x]\Big)$$
$$\overset{(\ell^1_\beta)}{\le} \frac{\mathbf{C}}{L} E_\beta^{\Lambda_0,\mathbb{Z}^d} \le \frac{\mathbf{C}K}{L^{d+1}}, \tag{2.44}$$

[4]This explains the fact that we consider the maximum on Λ_{3n} instead of Λ_{2n}.

where we used Lemma 2.6 in the last inequality. Moreover,

$$\sum_{v\in\partial\mathbb{H}_n}\sum_{\substack{z,t\in\mathbb{H}_n\\ z\neq t}} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\xleftrightarrow{\mathbb{H}_n} v]\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n} v]\Big(\sum_{x\in\partial\mathbb{H}_n}\mathbb{P}_\beta[v\xleftrightarrow{\mathbb{H}_n} x]\Big) \tag{2.45}$$

$$\overset{(\ell^1_\beta)}{\leq}\Big(1+\frac{\mathbf{C}}{L}\Big)\sum_{v\in\partial\mathbb{H}_n}\sum_{\substack{z,t\in\mathbb{H}_n\\ z\neq t}} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\xleftrightarrow{\mathbb{H}_n} v]\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n} v] \tag{2.46}$$

$$\overset{(\ell^\infty_\beta)}{\leq}\Big(1+\frac{\mathbf{C}}{L}\Big)\frac{\mathbf{C}}{L^d}\sum_{z,t\in\mathbb{H}_n} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n} v] \tag{2.47}$$

$$\overset{(\ell^1_\beta)}{\leq}\Big(1+\frac{\mathbf{C}}{L}\Big)\frac{\mathbf{C}}{L^d}\frac{\mathbf{C}}{L}\Big(\sum_{z\in\mathbb{Z}^d} p_{0z}(\beta)\Big)^2 \tag{2.48}$$

$$\leq\frac{4\mathbf{C}^2}{L^{d+1}}\Big(1+\frac{\mathbf{C}}{L}\Big)\leq\frac{8\mathbf{C}^3}{L^{d+1}}, \tag{2.49}$$

where in the third inequality we used the fact that $n>2AL$ (which ensures that $z,t\notin\partial\mathbb{H}_n$). The proof follows readily. □

Lemma 2.11. *Fix $d>6$, $\mathbf{C}>1$. There exists $D_1=D_1(\mathbf{C},d)>0$ such that the following holds. For every $\beta<\beta^*$, for every $n>12L$, and every $x\in\partial\mathbb{H}_n$,*

$$\max_{w\in\Lambda_{n/2}}\max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{n/2}\\ S\ni w}}\mathsf{E}^{S,\mathbb{H}_n}_\beta(w,x)\leq\frac{D_1}{L^4}\frac{\mathbf{C}}{Ln^{d-1}} \tag{2.50}$$

Proof. Fix w and S as above. By definition,

$$\mathsf{E}^{S,\mathbb{H}_n}_\beta(w,x)=(I)+(II)=$$
$$\sum_{\substack{u\in S\\ v\notin S+\Lambda_L}}\sum_{\substack{y,s\in S\\ y\neq s\\ z,t\in\mathbb{H}_n\setminus S\\ z\neq t}}\mathbb{P}_\beta[w\xleftrightarrow{S}u]\mathbb{P}_\beta[u\xleftrightarrow{S}y]\mathbb{P}_\beta[u\xleftrightarrow{S}s]p_{yz}(\beta)p_{st}(\beta)\mathbb{P}_\beta[z\xleftrightarrow{\mathbb{H}_n}v]\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n}v]\mathbb{P}_\beta[v\xleftrightarrow{\mathbb{H}_n}x]$$
$$+\sum_{v\notin S+\Lambda_L}\sum_{\substack{y\in S\\ z,t\in\mathbb{H}_n\setminus S\\ z\neq t}}\mathbb{P}_\beta[w\xleftrightarrow{S}y]p_{yz}(\beta)p_{yt}(\beta)\mathbb{P}_\beta[z\xleftrightarrow{\mathbb{H}_n}v]\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n}v]\mathbb{P}_\beta[v\xleftrightarrow{\mathbb{H}_n}x].$$

Bound on (I) Notice that the contribution coming from $v\in\mathbb{H}_{3n/4}$ is bounded by

$$(2d)^2\max\Big\{\mathbb{P}_\beta[w\xleftrightarrow{\mathbb{H}_n}x]:w\in\mathbb{H}_{3n/4}\Big\}\cdot\max\Big\{E^{\mathbb{H}_k,\mathbb{Z}^d}_\beta:k\geq 0\Big\}\leq(2d)^2\frac{\mathbf{C}(4/3)^{d-1}}{Ln^{d-1}}\frac{K}{L^d}, \tag{2.51}$$

where we used (ℓ^∞_β) and Lemma 2.6. We turn to the contribution for $v\in\mathbb{H}_n\setminus\mathbb{H}_{3n/4}$. Notice that z,t contribute if they are at distance at most L from S, that is $z,t\in\Lambda_{n/2+L}\subset\Lambda_{n/2+n/12}$. If $p\in\{0,\dots,n/4-1\}$, $v\in\partial\mathbb{H}_{n-p}$, and z,t are as above, then $|z-v|,|t-v|\geq n/6$ and

$$\mathbb{P}_\beta[z\xleftrightarrow{\mathbb{H}_n}v]\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n}v]\overset{(2.3)}{\leq}\frac{9\mathbf{C}^4}{L^4}\frac{6^{2d-4}}{n^{2d-4}}. \tag{2.52}$$

Moroever,

$$\sum_{v\in\mathbb{H}_{n-p}}\mathbb{P}_\beta[v\overset{\mathbb{H}_n}{\longleftrightarrow}x]=\psi_\beta(\mathbb{H}_{n-p})\overset{(\ell^1_\beta)}{\leq}\delta_0(p)+\frac{\mathbf{C}}{L}. \tag{2.53}$$

For a fixed $u\in S$, Proposition 2.2 gives

$$\sum_{\substack{y,s\in S\\ y\neq s\\ z,t\in\mathbb{H}_n\setminus S\\ z\neq t}}\mathbb{P}_\beta[u\overset{S}{\longleftrightarrow}y]\mathbb{P}_\beta[u\overset{S}{\longleftrightarrow}s]p_{yz}(\beta)p_{st}(\beta)\leq\varphi_\beta(S)^2\leq\Big(1+\frac{K}{L^d}\Big)^2. \tag{2.54}$$

Finally, we use (2.3) to get $C_1=C_1(\mathbf{C},d)>0$ such that

$$\sum_{u\in S}\mathbb{P}_\beta[w\overset{S}{\longleftrightarrow}u]\leq\sum_{u\in\Lambda_{n/2}}\mathbb{P}_\beta[w\longleftrightarrow u]\leq C_1n^2. \tag{2.55}$$

Putting all the previous displayed equations together, we obtain $C_2=C_2(\mathbf{C},d)>0$ such that

$$(I)\leq\frac{C_2}{L^4n^{d-6}}\frac{\mathbf{C}}{Ln^{d-1}} \tag{2.56}$$

Bound on (II) This is similar and we omit the details. □

Lemma 2.12 (Improving the ℓ^1 bound)**.** *Let $d>6$ and $\varepsilon>0$. For every $\mathbf{C}$ large enough, there exists $L_0=L_0(\varepsilon,\mathbf{C},d)$ such that for $L\geq L_0$ and $\beta<\beta^*$,*

$$\psi_\beta(\mathbb{H}_n)\leq\delta_0(n)+\frac{\varepsilon\mathbf{C}}{L}\qquad\qquad\forall n\geq 0. \tag{2.57}$$

Proof. We divide our proof between large and small values of n. Since $\psi_\beta(\mathbb{H}_n)$ is increasing in β, it is sufficient to prove the result for $\beta_0\leq\beta<\beta^*$ where we recall that $\beta_0\geq 1$ satisfies $\varphi_{\beta_0}(\{0\})=1$. We let A_1,L_1 be given by Lemma 2.8 with $\eta=\varepsilon/2$ and assume that $L\geq L_1$.

Case $n>4A_1L$ Set $\ell:=\lceil L/2\rceil$. Lemma 2.8 applied to $\eta=\varepsilon/2$, $\Lambda=\mathbb{H}_n$, $X=\partial\mathbb{H}_n$, $u=0$ and $v\in\{-\ell\mathbf{e}_1,\dots,-\mathbf{e}_1\}$ implies that for every $0\leq k\leq\ell$,

$$\psi_\beta(\mathbb{H}_n)\leq\psi_\beta(\mathbb{H}_{n-k})+\eta\max_{s\in\{-2A_1L,\dots,2A_1L\}}\psi_\beta(\mathbb{H}_{n+s})+A_1\max_{w\in\Lambda_{2A_1L}}\sum_{x\in\partial\mathbb{H}_n}\mathsf{E}^{\mathbb{H}_n}_\beta(w,x). \tag{2.58}$$

Now, The contribution for coming from $v\notin\partial\mathbb{H}_n$ is bounded by

$$\sum_{v\in\mathbb{H}_{n-1}}\sum_{\substack{z,t\in\mathbb{H}_n\\ z\neq t}}p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\longleftrightarrow v]\mathbb{P}_\beta[t\longleftrightarrow v]\Big(\sum_{x\in\partial\mathbb{H}_n}\mathbb{P}_\beta[v\overset{\mathbb{H}_n}{\longleftrightarrow}x]\Big)$$
$$\overset{(\ell^1_\beta)}{\leq}\frac{\mathbf{C}}{L}E^{\Lambda_0,\mathbb{Z}^d}_\beta\leq\frac{\mathbf{C}K}{L^{d+1}}, \tag{2.59}$$

where we used Lemma 2.6 in the last inequality. Moreover,

$$\sum_{v\in\partial\mathbb{H}_n}\sum_{\substack{z,t\in\mathbb{H}_n\\ z\neq t}} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\xleftrightarrow{\mathbb{H}_n}v]\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n}v]\Big(\sum_{x\in\partial\mathbb{H}_n}\mathbb{P}_\beta[v\xleftrightarrow{\mathbb{H}_n}x]\Big) \tag{2.60}$$

$$\overset{(\ell^1_\beta)}{\leq}\Big(1+\frac{\mathbf{C}}{L}\Big)\sum_{v\in\partial\mathbb{H}_n}\sum_{\substack{z,t\in\mathbb{H}_n\\ z\neq t}} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\xleftrightarrow{\mathbb{H}_n}v]\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n}v] \tag{2.61}$$

$$\overset{(\ell^\infty_\beta)}{\leq}\Big(1+\frac{\mathbf{C}}{L}\Big)\frac{\mathbf{C}}{L^d}\sum_{z,t\in\mathbb{H}_n} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[t\xleftrightarrow{\mathbb{H}_n}v] \tag{2.62}$$

$$\overset{(\ell^1_\beta)}{\leq}\Big(1+\frac{\mathbf{C}}{L}\Big)\frac{\mathbf{C}}{L^d}\frac{\mathbf{C}}{L}\Big(\sum_{z\in\mathbb{Z}^d}p_{0z}(\beta)\Big)^2 \tag{2.63}$$

$$\leq\frac{4\mathbf{C}^2}{L^{d+1}}\Big(1+\frac{\mathbf{C}}{L}\Big)\leq\frac{8\mathbf{C}^3}{L^{d+1}}, \tag{2.64}$$

where in the third inequality we used the fact that $n>2AL$ (which ensures that $z,t\notin\partial\mathbb{H}_n$). The proof follows readily.

Using (ℓ^1_β) and Lemma 2.10,

$$\psi_\beta(\mathbb{H}_n)\leq\psi_\beta(\mathbb{H}_{n-k})+\frac{\mathbf{C}}{L}\Big(\frac{\varepsilon}{2}+A_1\frac{K+8\mathbf{C}^2}{L^d}\Big). \tag{2.65}$$

Now,

$$\sum_{k=0}^{\ell}\psi_\beta(\mathbb{H}_{n-k})=\sum_{k=0}^{\ell}\sum_{y\in\partial\mathbb{H}_{n-k}}\mathbb{P}_\beta[0\xleftrightarrow{\mathbb{H}_{n-k}}y] \tag{2.66}$$

$$\leq\sum_{y\in\mathbb{H}_n\setminus\mathbb{H}_{n-\ell-1}}\mathbb{P}_\beta[0\xleftrightarrow{\mathbb{H}_n}y] \tag{2.67}$$

$$\leq\sum_{y\in\mathbb{H}_n\setminus\mathbb{H}_{n-\ell-1}}\mathbb{P}_\beta[0\xleftrightarrow{\mathbb{H}_n}y]\cdot\frac{4}{\varphi_\beta(\{0\})}\sum_{z\notin\mathbb{H}_n}p_{yz}(\beta) \tag{2.68}$$

$$\leq 4\varphi_\beta(\mathbb{H}_n)\leq 4\Big(1+\frac{K}{L^d}\Big). \tag{2.69}$$

In the third line we used that the sum of the $p_{yz}(\beta)$ over $z\notin\mathbb{H}_n$ is bounded from below by a fourth of the sum over all possible z, which is $\varphi_\beta(\{0\})\geq 1$ for $\beta\geq\beta_0$.

Averaging on $0\leq k\leq\ell$, we deduce that

$$\psi_\beta(\mathbb{H}_n)\leq\frac{1}{\ell+1}4\Big(1+\frac{K}{L^d}\Big)+\frac{\mathbf{C}}{L}\Big(\frac{\varepsilon}{2}+A_1\frac{K+8\mathbf{C}^2}{L^d}\Big). \tag{2.70}$$

Providing $\mathbf{C}>8/\varepsilon$ and then L large enough, this concludes this case.

Case $n\leq 4A_1L$ As before, set $\varphi:=\varphi_\beta(\{0\})$. Let τ be the exit time of $\mathbb{H}_n$. Summing over $x\in\partial\mathbb{H}_n$ and $t\leq T$ the t-th iteration of (1.15) with S being a singleton and $\Lambda=\mathbb{H}_n$ gives

$$\psi_\beta(\mathbb{H}_n)\leq\delta_0(n)+\max\{\varphi^t:t\leq T\}\mathbb{E}_{\mathrm{RW},0,\beta}[\mathcal{N}]+\varphi(\{0\})^T\mathbb{E}_{\mathrm{RW},0,\beta}[\psi_\beta(\mathbb{H}_{n-(X_T)_1})\mathbb{1}_{\tau>T}], \tag{2.71}$$

where $\mathcal{N} := |\{1 \le t \le T \wedge \tau : X_t \in \partial \mathbb{H}_n\}|$.

Classical random walk estimates give the existence of $A_{\mathrm{RW}} = A_{\mathrm{RW}}(A_1, d) > 0$ and $T = T(\varepsilon, A_1, d)$ large enough,

$$\mathbb{E}_{\mathrm{RW},0,\beta}[\mathcal{N}] \le \frac{A_{\mathrm{RW}}}{L}, \tag{2.72}$$

$$\mathbb{P}_{\mathrm{RW},0,\beta}[\tau > T] \le \frac{\varepsilon}{4}. \tag{2.73}$$

Assume that $L \ge L_0 = L_0(T, K, d)$ be such that $(1 + KL^{-d})^T \le 2$. Corollary 2.2 gives

$$\max\{\varphi^t : t \le T\} \le (1 + KL^{-d})^T \le 2 \qquad \forall t \le T. \tag{2.74}$$

Collecting the above work yields

$$\psi_\beta(\mathbb{H}_n) \le \delta_0(n) + \frac{2A_{\mathrm{RW}}}{L} + \frac{\varepsilon \mathbf{C}}{2L}. \tag{2.75}$$

The result follows by choosing $\mathbf{C} \ge 4A_{\mathrm{RW}}/\varepsilon$. □

Lemma 2.13 (Improving the ℓ^∞ bound)**.** *Let $d > 6$, $\mathbf{C} > 0$. For every $\mathbf{C}$ large enough, there exists $L_0 = L_0(d, \mathbf{C})$ such that for $L \ge L_0$ and $\beta < \beta^*$,*

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \delta_0(x) + \frac{\mathbf{C}}{2L^d}\left(\frac{L}{L \vee n}\right)^{d-1} \qquad \forall n \ge 0,\ \forall x \in \partial \mathbb{H}_n. \tag{2.76}$$

Proof. Let η, ε to be fixed later. Again, we divide our proof between large and small values of n. Let $\delta = \delta(\eta)$ and $A = A(\eta)$ be given by Proposition 2.7.

Case $n > 6AL$ Set $V_n := \{y \in \Lambda_{\delta n/6} : y_1 = 0\}$. Proposition 2.7 (applied to $n/6$ and η) gives that for every $\beta < \beta^*$, every $x \in \partial\mathbb{H}_n$ and $y \in V_n$,

$$\begin{aligned}\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} (x-y)] = \mathbb{P}_\beta[y \xleftrightarrow{\mathbb{H}_n} x] \ge \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] - \eta \max\Big\{\mathbb{P}_\beta[w \xleftrightarrow{\mathbb{H}_n} x] : w \in \Lambda_{n/2}\Big\} \\ - A \max_{w \in \Lambda_{n/2}} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{n/2} \\ S \ni w}} \mathsf{E}_\beta^{S,\mathbb{H}_n}(w, x).\end{aligned} \tag{2.77}$$

Using Lemma 2.11, we may choose L large enough such that

$$A \max_{w \in \Lambda_{n/2}} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{n/2} \\ S \ni w}} \mathsf{E}_\beta^{S,\mathbb{H}_n}(w, x) \le \frac{\eta \mathbf{C}}{Ln^{d-1}}. \tag{2.78}$$

Averaging over y gives and choosing L even larger (in terms of ε) yields

$$\frac{\varepsilon \mathbf{C}}{L}\frac{1}{|V_n|} \overset{(2.57)}{\ge} \frac{1}{|V_n|}\psi_\beta(\mathbb{H}_n) \ge \frac{1}{|V_n|}\sum_{y \in V_n} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} (x-y)] \tag{2.79}$$

$$\overset{(\ell^\infty_\beta)}{\ge} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] - \eta\frac{\mathbf{C}}{L(n/2)^{d-1}} - \eta\frac{\mathbf{C}}{Ln^{d-1}}. \tag{2.80}$$

At this stage, consider $\eta = 2^{-d}$, and then $\varepsilon < \delta^d/2$. Choosing $\mathbf{C} = \mathbf{C}(\varepsilon)$ large enough and then L large enough, we find

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \frac{\mathbf{C}}{2Ln^{d-1}}. \tag{2.81}$$

Case $n \le 6AL$ As before, set $\varphi := \varphi_\beta(\{0\})$. Let τ be the exit time of $\mathbb{H}_n$. Summing over $t \le T$ the t-th iteration of (1.14) with S being a singleton and $\Lambda = \mathbb{H}_n$ gives

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \delta_0(x) + \max\{\varphi^t : 0 < t < T\}\mathbb{E}_{\mathrm{RW},0,\beta}[\mathcal{M}] + \varphi^T \mathbb{E}_{\mathrm{RW},0,\beta}\Big[\mathbb{P}[X_T \xleftrightarrow{\mathbb{H}_n} x]\mathbb{1}_{\tau > T}\Big], \tag{2.82}$$

where $\mathcal{M} = |\{1 \le t \le T \wedge \tau : X_t = x\}|$. Classical random walk estimates give the existence of $C_{\mathrm{RW}} = C_{\mathrm{RW}}(A, d) > 0$ such that

$$\mathbb{E}_{\mathrm{RW},0,\beta}[\mathcal{M}] \le \frac{C_{\mathrm{RW}}}{L^d}\left(\frac{L}{L \vee n}\right)^{d-1}, \tag{2.83}$$

$$\mathbb{P}_{\mathrm{RW},0,\beta}[\tau > T] \le \frac{C_{\mathrm{RW}}}{T^{(d-1)/2}}. \tag{2.84}$$

Assuming again that L is chosen so large that $(1 + KL^{-d})^T \le 2$, we finally obtain

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \delta_0(x) + \frac{2A_{\mathrm{RW}}}{L^d}\left(\frac{L}{L \vee n}\right)^{d-1} + \frac{2\mathbf{C}}{L^d}T^{-(d-1)/2}. \tag{2.85}$$

Choosing T large enough that $(6A)^{d-1}T^{-(1-2)/2} \le \frac{1}{8}$ and providing $\mathbf{C} > 8A_{\mathrm{RW}}$, we find again

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \delta_0(x) + \frac{\mathbf{C}}{2L}\Big(\frac{L}{L \vee n}\Big)^{d-1}. \tag{2.86}$$

□

We are now in a position to prove the following proposition.

Proposition 2.14. *Fix $d > 6$. There exist K and L_0 such that for every $L \ge L_0$,*

$$\beta_c \le 1 + \frac{K}{L^d}, \tag{2.87}$$

$$\varphi_{\beta_c}(B) \le 1 + \frac{K}{L^d} \qquad \forall B \in \mathcal{B}, \tag{2.88}$$

$$E^{B,\mathbb{Z}^d}_{\beta_c} \le \frac{K}{L^d} \qquad \forall B \in \mathcal{B}, \tag{2.89}$$

$$\psi_{\beta_c}(\mathbb{H}_n) \le \delta_0(n) + \frac{K}{L} \qquad \forall n \ge 0, \tag{2.90}$$

$$\mathbb{P}_{\beta_c}[0 \leftrightarrow x] \le \frac{K}{L^d}\left(\frac{L}{L \vee |x|}\right)^{d-2} \qquad \forall x \in \mathbb{Z}^d \setminus \{0\}, \tag{2.91}$$

$$\mathbb{P}_{\beta_c}[0 \xleftrightarrow{\mathbb{H}} x] \le \frac{K}{L^d}\left(\frac{L}{L \vee |x_1|}\right)^{d-1} \qquad \forall x \in \mathbb{H} \setminus \{0\}. \tag{2.92}$$

Proof. By Lemmata 2.12 and 2.13, we find that if $\mathbf{C}$ and L are large enough, for every $\beta < \beta^*$,

$$\psi_\beta(\mathbb{H}_n) \le \delta_0(n) + \frac{\mathbf{C}}{2L} \qquad \forall n \ge 0, \tag{2.93}$$

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x] \le \delta_0(x) + \frac{\mathbf{C}}{2L^d}\left(\frac{L}{L \vee n}\right)^{d-1} \qquad \forall n \ge 0,\ \forall x \in \partial\mathbb{H}_n. \tag{2.94}$$

By taking the supremum we find that those bounds still hold true at $\beta = \beta^*$.

Let us now assume by contradiction that $\beta^* < 2 \wedge \beta_c$. Consider $\beta^{**} \in (\beta^*, 2 \wedge \beta_c)$. Exponential decay of correlations imply the existence of $N = N(\beta^{**}, \mathbf{C})$ such that

$$\psi_{\beta^{**}}(\mathbb{H}_n) < \frac{\mathbf{C}}{L} \qquad \forall n \geq N, \tag{2.95}$$

$$\mathbb{P}_{\beta^{**}}[0 \xleftrightarrow{\mathbb{H}_n} x] < \frac{\mathbf{C}}{L^d} \left(\frac{L}{L \vee |x_1|} \right)^{d-1} \qquad \forall x \notin \Lambda_N. \tag{2.96}$$

Using continuity for $n \leq N$ and $|x| \in \Lambda_N$, we deduce that some $\beta \in (\beta^*, \beta^{**})$ satisfies (ℓ^1_β) and (ℓ^∞_β), thus contradicting the definition of β^*.

From all of this, we obtain that $\beta^* = 2 \wedge \beta_c$ and that in addition the properties hold until $2 \wedge \beta_c$. Also, note that Proposition 2.2 implies the right bound on the $\varphi_\beta(\mathbb{H}_n)$ and $\varphi_\beta(\Lambda_n)$ for every $\beta < \beta^*$. Taking the supremum over $\beta < \beta^*$ implies the bounds at β^*.

It remains to show that $\beta^* = \beta_c$. For that, it suffices to notice that for L large enough $\beta^* < 2$. Indeed, the bound on $\varphi_{\beta^*}(\{0\})$ implies that for L large enough,

$$\beta^* \leq 1 + \frac{2K}{L^d}. \tag{2.97}$$

This concludes the proof by choosing L so large that $\frac{2K}{L^d} < 1$. □

We conclude this section by proving Theorem 1.1.

Proof of Theorem 1.1. The previous proposition implies the estimates for $|x| \leq L_\beta$ (changing the constant C to eC). We now turn to the case of $|x| > L_\beta$. Below, Λ denotes either $\mathbb{Z}^d$ or the half-space $\mathbb{H}$. Iterating (1.14) $k := \lfloor |x|/L_\beta \rfloor - 1$ times (or $\lfloor |x_1|/L_\beta \rfloor - 1$ times in the half-space case), we get that

$$\mathbb{P}_\beta[0 \xleftrightarrow{\Lambda} x] \leq \varphi_\beta(\Lambda_{L_\beta})^k \max \Big\{ \mathbb{P}_\beta[x \xleftrightarrow{\Lambda} y] : y \notin \Lambda_{L_\beta}(x) \Big\}. \tag{2.98}$$

We then invoke the definition of L_β and the bounds (2.91) or (2.92) to conclude. □

3 Proof of Theorem 1.2

In this section, we assume that $\mathbf{C}$ and L are large enough such that Proposition 2.14 holds. Let also $K = K(\mathbf{C}, d)$ be given by Proposition 2.14.

3.1 Lower bound on $\psi_\beta(\mathbb{H}_n)$

We start with our basic estimate for this section. It is a strengthening of the lower bound corresponding to the upper bound on $\psi_\beta(\mathbb{H}_n)$ obtained in the previous section. Recall that β_0 is such that $\varphi_{\beta_0}(\{0\}) = 1$. Introduce for $n, k \geq 1$,

$$\psi^{[k]}_\beta(\mathbb{H}_n) := \sum_{\substack{x \in \partial \mathbb{H}_n \\ |x| \leq k}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_n} x]. \tag{3.1}$$

Lemma 3.1. *There exists $c > 0$ such that for every L large enough, every $\beta_0 \leq \beta \leq \beta_c$ and every $1 \leq n \leq L_\beta$,*

$$\psi_\beta(\mathbb{H}_n) \geq \psi^{[n]}_\beta(\mathbb{H}_n) \geq \frac{c}{L}. \tag{3.2}$$

Proof. The first inequality is clear, we therefore focus on the second one. Let $\eta > 0$ to be fixed. We divide the proof between the case $n > 4A_1L$ and $n \leq 4A_1L$, where $A_1 = A_1(\eta)$ is provided by Lemma 2.8. We begin with the former as it is the most interesting one.

Case $n > 4A_1L$ Reproducing the argument of (2.22),

$$\varphi_\beta(\Lambda_n) \le 2d \sum_{\substack{y\in\Lambda_n\\ y_1\in\{n-L+1,\dots,n\}\\ z\notin\mathbb{H}_n}} \mathbb{P}_\beta[0 \xleftrightarrow{\Lambda_n} y] p_{yz}(\beta) \tag{3.3}$$

$$\le 2d \sum_{k=0}^{L-1} \sum_{\substack{y\in\partial\mathbb{H}_{n-k}\\ |y|\le n\\ z\notin\mathbb{H}_n}} p_{yz}(\beta) \sum_{\ell=0}^{k} \sum_{\substack{u\in\partial\mathbb{H}_{n-\ell}\\ |u|\le n}} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}_{n-\ell}} u]\mathbb{P}_\beta[u \xleftrightarrow{\mathbb{H}_{n-\ell}} y] \tag{3.4}$$

$$\le 2d(1+2\mathbf{C}^2) \sum_{\ell=0}^{L-1} \psi_\beta^{[n]}(\mathbb{H}_{n-\ell}). \tag{3.5}$$

Moreover, since $n > 4A_1L$, using the same reasoning as in (2.65) gives (by Lemma 2.8 and (2.90)) for L large enough,

$$\sum_{\ell=0}^{L-1} \psi_\beta^{[n]}(\mathbb{H}_{n-\ell}) \le L\psi_\beta^{[n]}(\mathbb{H}_n) + 2\eta\mathbf{C}. \tag{3.6}$$

Observe that $n \le L_\beta$ gives $\varphi_\beta(\Lambda_n) \ge 1/e$. Considering η small enough (which only influences how big A_1 is, and therefore how big L should be taken) concludes the proof.

Case $n \le 4A_1L$ Set $\varphi := \varphi_\beta(\{0\})$. Summing over $x \in \partial\mathbb{H}_n$ with $|x| \le n$ and $t \le T$ the t-th iteration of (1.15) with S being a singleton and $\Lambda = \mathbb{H}_n$ gives

$$\psi_\beta^{[n]}(\mathbb{H}_n) \ge \mathbb{E}_{\mathrm{RW},0,\beta}[\varphi^{\tau_\partial}\mathbb{1}_{\tau_\partial<T\wedge\tau}] - \Big(\sum_{t=0}^{T-1}\varphi^t\Big)\frac{K}{L^d}\max_{k\ge0}\psi_\beta(\mathbb{H}_k)$$
$$- \Big(\sum_{t=0}^{T-1}\varphi^t\Big)\max_{w\in\mathbb{H}_n}\sum_{x\in\partial\mathbb{H}_n}\mathsf{E}_\beta^{\mathbb{H}_n}(w,x), \tag{3.7}$$

where τ_∂ and τ are respectively the hitting times of $\{x \in \partial\mathbb{H}_n,\ |x| \le n\}$ and the complement of $\mathbb{H}_n$, and where we used (2.89). Using a simple random-walk estimate and the fact that $\beta \ge \beta_0$, we get that for T large enough (in terms of A_1), there exists $c_{\mathrm{RW}} = c_{\mathrm{RW}}(A_1, d) > 0$ such that

$$\mathbb{E}_{\mathrm{RW},0,\beta}[\varphi^{\tau_\partial}\mathbb{1}_{\tau_\partial<T\wedge\tau}] \ge \mathbb{P}_{\mathrm{RW},0,\beta}[\tau_\partial < T\wedge\tau] \ge \frac{c_{\mathrm{RW}}}{L}. \tag{3.8}$$

Reproducing[5] the argument of Lemma 2.10, we get $C_1 = C_1(\mathbf{C}, d) > 0$ such that

$$\max_{w\in\mathbb{H}_n}\sum_{x\in\partial\mathbb{H}_n}\mathsf{E}_\beta^{\mathbb{H}_n}(w,x) \le \frac{C_1}{L^d} \tag{3.9}$$

By (**??**), $\varphi \le 1 + \frac{K}{L^d}$. We take L large enough so that $\Big(\sum_{t=0}^{T-1}\varphi^t\Big) \le 2$. Gathering the previous displayed equations and using (2.90) gives

$$\psi_\beta^{[n]}(\mathbb{H}_n) \ge \frac{c_{\mathrm{RW}}}{L} - 2\frac{K}{L^d}\Big(1+\frac{\mathbf{C}}{L}\Big) - \frac{2C_1}{L^d}. \tag{3.10}$$

It remains to take L large enough as a function of T and $\mathbf{C}$ to conclude. □

[5]We obtain a bound with a diminished power of L because z and t might simultaneously be in $\partial\mathbb{H}_n$ in (2.60).

3.2 A Harnack-type estimate

We will need to turn average estimates into pointwise ones. We therefore show another regularity estimate. For $\varepsilon > 0$, introduce the quantity

$$L_\beta(\varepsilon) := \inf\{n \geq 0 : \varphi_\beta(\Lambda_n) \leq 1 - \varepsilon\}. \tag{3.11}$$

Proposition 3.2 (Regularity estimate at macroscopic scales)**.** *Fix $d > 6$. For every $\alpha > 0$, there exists $C_{\mathrm{RW}} = C_{\mathrm{RW}}(\alpha, d) > 0$ such that for every $\eta > 0$, there exist A and L_0 large enough, and $\varepsilon_0 > 0$ small enough such that the following holds. For every $L \geq L_0$, every $\varepsilon < \varepsilon_0$, every $n \leq L_\beta(\varepsilon)$ satisfying $n \geq AL$, every $\beta \leq \beta_c$, and every $y \notin \Lambda_{(1+\alpha)n} \subset \Lambda$,*

$$\max_{x\in\Lambda_n} \mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] \leq C_{\mathrm{RW}} \min_{x\in\Lambda_n} \mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] + \eta \max_{x\in\Lambda_{(1+\alpha)n}} \mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] + A \max_{u\in\Lambda_{(1+\alpha)n}} \max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{(1+\alpha)n}\\ S\ni u}} \mathsf{E}_\beta^{S,\Lambda}(u, y). \tag{3.12}$$

The idea of the proof is to introduce a well-chosen rescaled random-walk and to observe that its exit probabilities do not drastically depend on the start of the walk. Combined with Proposition 2.7, this will enable us to conclude.

Proof. Fix $m = \lfloor \alpha n/7 \rfloor$. Let $\eta > 0$ to be fixed later. Let $\delta = \delta(\eta)$, $A = A(\eta)$, and $L_0 = L_0(\eta)$ be given by Proposition 2.7. Set $k = \lfloor \delta m \rfloor$, where $\delta = \delta(\eta, d)$. Additionally assume that $n \geq (7\alpha^{-1}AL) \vee (7\alpha^{-1}\delta^{-1}L)$ so that $m \geq AL$ and $k \geq L$. Consider the random walk (X_n^u) defined by

$$\mathbb{P}^{(k)}_{\mathrm{RW},u,\beta}[X_1 = v] := \frac{\mathbb{1}_{v\notin\Lambda_k(u)}}{\varphi_\beta(\Lambda_k)} \sum_{\substack{w\in\Lambda_k(u),\\ w\sim v}} \mathbb{P}_\beta[u \overset{\Lambda_k(u)}{\longleftrightarrow} w] p_{wv}(\beta). \tag{3.13}$$

Note that this random walk does jumps at distance at most $k + L$. Let τ be the hitting time of $\mathbb{Z}^d \setminus \Lambda_{n+m}$. Let $B_1, \ldots, B_s$ be the two layers of boxes of size k, centred at b_i, that are disjoint and covering $\Lambda_{n+m+4k} \setminus \Lambda_{n+m}$, see Figure 3.

We will use two a priori estimates on the random walk and the stopping time, that can be easily obtained from classical random walk analysis[6]: there exist $C_{\mathrm{RW}}(\alpha, d)$ and $\varepsilon_0 = \varepsilon_0(\alpha, \eta, d) > 0$ such that for every $\varphi \in [1 - \varepsilon_0, 1 + \varepsilon_0]$,

$$\mathbb{E}^{(k)}_{\mathrm{RW},x,\beta}\Big[\sum_{s=0}^{\tau} \varphi^s\Big] \leq C_{\mathrm{RW}} \qquad \forall x \in \Lambda_n, \tag{3.14}$$

$$\mathbb{E}^{(k)}_{\mathrm{RW},x,\beta}[\varphi^\tau \mathbb{1}_{X_\tau\in B_i}] \leq C_{\mathrm{RW}} \mathbb{E}^{(k)}_{\mathrm{RW},x',\beta}[\varphi^\tau \mathbb{1}_{X_\tau\in B_i}] \qquad \forall x, x' \in \Lambda_n, \forall i \leq s. \tag{3.15}$$

From now on, we assume that $\varepsilon < \varepsilon_0$. By (**??**), $\varphi_\beta(\Lambda_k) \leq 1 + \frac{K}{L^d}$. We thus fix $L_0 = L_0(\varepsilon_0)$ large enough that for $L \geq L_0$, $\frac{K}{L^d} \leq \varepsilon_0$. By the assumption $n \leq L_\beta(\varepsilon)$, we find

$$1 - \varepsilon \leq \varphi_\beta(\Lambda_k) \leq 1 + \varepsilon_0. \tag{3.16}$$

[6]For the first inequality, simply observe that every $(\alpha\delta)^{-2}$ steps there is a probability c of exiting the box. Hence, as soon as $\varepsilon_0 \ll (\alpha\delta)^2$, the estimate follows easily from a Laplace transform estimate of τ. The second estimate follows from Harnack's inequality for the (coarse grained) exit probabilities.

Below, introduce the short-hand notation $\varphi := \varphi_\beta(\Lambda_k)$. Iterating the two bounds of Lemma 1.5 until the hitting time τ gives

$$\mathbb{P}_\beta[x' \overset{\Lambda}{\longleftrightarrow} y] \quad \leq \quad \mathbb{E}^{(k)}_{\mathrm{RW},x',\beta}\Big[\varphi^\tau \mathbb{P}_\beta[X_\tau \overset{\Lambda}{\longleftrightarrow} y]\Big], \tag{3.17}$$

$$\begin{aligned}\mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] \quad &\overset{(2.89)}{\geq} \quad \mathbb{E}^{(k)}_{\mathrm{RW},x,\beta}\Big[\varphi^\tau \mathbb{P}_\beta[X_\tau \overset{\Lambda}{\longleftrightarrow} y]\Big] - \frac{K}{L^d}\,\mathbb{E}^{(k)}_{\mathrm{RW},x,\beta}\Big[\sum_{s=0}^{\tau}\varphi^s\Big] \max_{u\in\Lambda_{n+m+L}} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} y]\\
&\quad - \mathbb{E}^{(k)}_{\mathrm{RW},x,\beta}\Big[\sum_{s=0}^{\tau}\varphi^s\Big] \max_{w\in\Lambda_{n+m}} \mathsf{E}^{\Lambda_k(w),\Lambda}_\beta(w,x)\\
&\overset{(3.14)}{\geq} \mathbb{E}^{(k)}_{\mathrm{RW},x,\beta}\Big[\varphi^\tau \mathbb{P}_\beta[X_\tau \overset{\Lambda}{\longleftrightarrow} y]\Big] - \frac{K}{L^d} C_{\mathrm{RW}} \max_{u\in\Lambda_{n+m+L}} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} y]\\
&\quad - C_{\mathrm{RW}} \max_{w\in\Lambda_{n+m}} \mathsf{E}^{\Lambda_k(w),\Lambda}_\beta(w,x).\end{aligned} \tag{3.18}$$

Proposition 2.7 gives that for every i,

$$\begin{aligned}\max_{u\in B_i} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} y] \leq \min_{u\in B_i} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} y] + \eta \max_{u\in\Lambda_{3m}(b_i)} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} y]\\
+ A \max_{u\in\Lambda_{3m}(b_i)} \max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{3m}(b_i)\\ S\ni u}} \mathsf{E}^{S,\Lambda}_\beta(u,y).\end{aligned} \tag{3.19}$$

Combining this estimate with (3.15) implies that for every i,

$$\begin{aligned}\mathbb{E}^{(k)}_{\mathrm{RW},x',\beta}\Big[\varphi^\tau \mathbb{1}_{X_\tau\in B_i} \mathbb{P}_\beta[X_\tau \overset{\Lambda}{\longleftrightarrow} y]\Big] \leq C_{\mathrm{RW}} \mathbb{E}^{(k)}_{\mathrm{RW},x,\beta}\Big[\varphi^\tau \mathbb{1}_{X_\tau\in B_i} \mathbb{P}_\beta[X_\tau \overset{\Lambda}{\longleftrightarrow} y]\Big]\\
+ \eta \mathbb{E}^{(k)}_{\mathrm{RW},x',\beta}[\varphi^\tau \mathbb{1}_{X_\tau\in B_i}] \max_{u\in\Lambda_{3m}(b_i)} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} y]\\
+ A \max_{u\in\Lambda_{3m}(b_i)} \max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{3m}(b_i)\\ S\ni u}} \mathsf{E}^{S,\Lambda}_\beta(u,y).\end{aligned} \tag{3.20}$$

Since X_τ belongs to some B_i, the previous estimate together with (3.17) and (3.18) gives[7]

$$\begin{aligned}\mathbb{P}_\beta[x' \overset{\Lambda}{\longleftrightarrow} y] \leq C_{\mathrm{RW}} \mathbb{P}_\beta[x \overset{\Lambda}{\longleftrightarrow} y] + (\eta + \tfrac{K}{L^d} C_{\mathrm{RW}}^2) \max_{u\in\Lambda_{n+7m}} \mathbb{P}_\beta[u \overset{\Lambda}{\longleftrightarrow} y]\\
+ (A + C_{\mathrm{RW}}^2) \max_{u\in\Lambda_{n+7m}} \max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{n+7m}\\ S\ni u}} \mathsf{E}^{S,\Lambda}_\beta(u,y).\end{aligned} \tag{3.21}$$

(We also used one more time (3.14) to get that $\mathbb{E}_{x'}[\varphi^\tau] \leq C_{\mathrm{RW}}$.) It remains to notice that $n + 7m \leq n + \alpha n$, and to pick[8] $\eta = \eta(d)$ small enough, and then L_0 large enough. □

3.3 Proof of the lower bounds

To shorten the notation, we write $L'_\beta := L_\beta(\varepsilon)$. We start by lower bounding the half-space two-point function at scale below $6L'_\beta$ (for some technical reason we will need this multiplicative factor later). Let

$$A_n := \{x \in \mathbb{Z}^d : x_1 = |x| = n\}. \tag{3.22}$$

[7] We additionally used that $n + m + 2k + 3m \leq n + 7m$.

[8] Note that it was fundamental that C_{RW} was depending on d only.

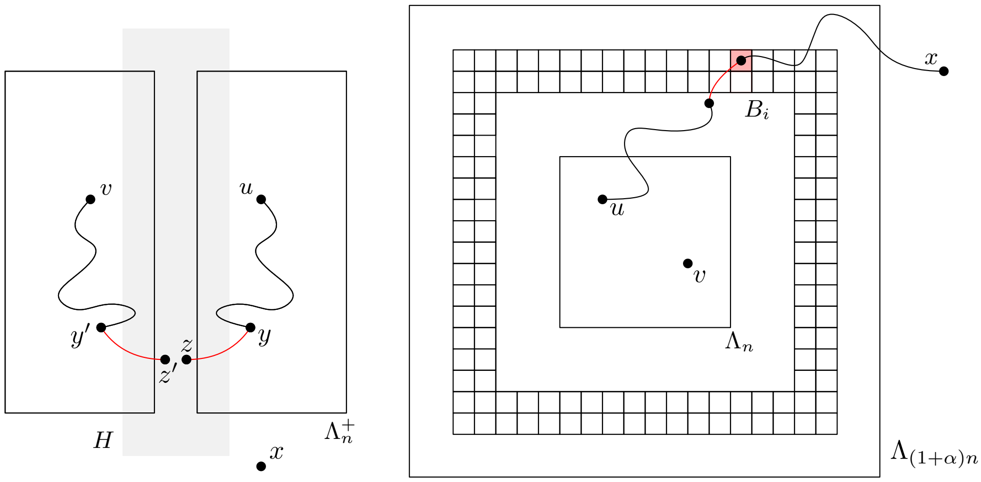

Figure 3: On the left, an illustration of the pairing used in the proof of Proposition 2.7. The grey region corresponds to H. The red path corresponds to a long open edge "jumping" outside Λ_n^+ (resp. $-\Lambda_n^+$). Since u (resp. v) is close to $\{u \in \mathbb{Z}^d : u_1 = 0\}$, a connection from u to x will most likely enter H if it exits Λ_n^+. On the right, an illustration of the proof of Proposition 3.2.

Lemma 3.3. *Fix $d > 6$. There exist $c, \varepsilon_0 > 0$ and $L_0 > 0$ such that for every $L \geq L_0$, every $\varepsilon < \varepsilon_0$, every $\beta_0 \leq \beta \leq \beta_c$, and every $x \in \mathbb{H}$ with $x_1 = |x| \leq 6L'_\beta$,*

$$\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] \geq \frac{c}{L^d}\left(\frac{L}{L \vee |x|}\right)^{d-1}. \tag{3.23}$$

Proof. First, by translation invariance, for every $n \leq L'_\beta$,

$$\frac{1}{|A_n|}\sum_{y \in A_n} \mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} y] = \frac{1}{|A_n|}\psi_\beta^{[n]}(\mathbb{H}_n) \geq \frac{c}{Ln^{d-1}}, \tag{3.24}$$

where $c > 0$ is provided by Lemma 3.1. We want to turn this average estimate into a point wise one. Fix $x \in A_N$ with $N \leq 6L'_\beta$ and set $n := \lfloor N/6 \rfloor \leq L'_\beta$. Let $\eta > 0$ to be fixed.

Let $C_{\mathrm{RW}}, A, \varepsilon_0 > 0$ be given by Proposition 3.2 with $\alpha = \frac{1}{12}$ (see Figure 3.3), η, and $\Lambda = \mathbb{H}$. We consider two cases according to how large n is.

Case $n \geq AL$ In this case, we can apply Proposition 3.2 to get, for all $y \in A_n$,

$$\begin{aligned}\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} y] \leq C_{\mathrm{RW}}\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] + \eta \max\Big\{\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} w] : w_1 \geq n/2\Big\} \\ + A\max\Big\{\mathsf{E}_\beta^{S,\mathbb{H}}(w,0) : w_1 \geq n/2,\ S \in \mathcal{B},\ S \subset \Lambda_{13N/12}(\tfrac{7N}{6}\mathbf{e}_1),\ S \ni w\Big\},\end{aligned} \tag{3.25}$$

Combining the above display with (3.24), the upper bound from (2.92), and (a minor generalisation of) Lemma 2.11 yields

$$C_{\mathrm{RW}}\mathbb{P}_\beta[0 \xleftrightarrow{\mathbb{H}} x] \geq \frac{c}{Ln^{d-1}} - \eta\frac{\mathbf{C}}{Ln^{d-1}} - \frac{KD_1}{L^4}\frac{\mathbf{C}}{Ln^{d-1}}. \tag{3.26}$$

Choosing η small enough and then L large enough concludes the proof in that case.

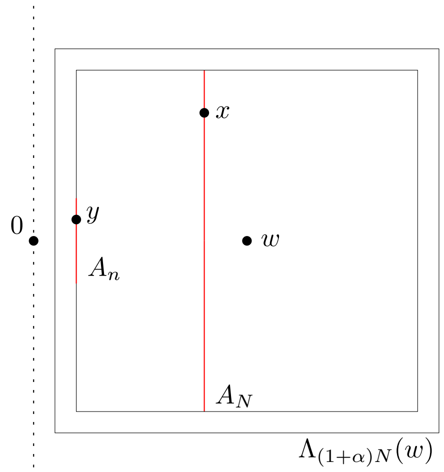

Figure 4: An illustration of how Proposition 3.2 is applied in the proof of Lemma 3.3. The red segments represent the sets A_n and A_N. The boxes are centred at $w = \frac{7N}{6}\mathbf{e}_1$.

Case $n < AL$ To handle the small values of n, we repeat the random walk argument used at several places above. As before, set $\varphi := \varphi_\beta(\{0\})$. Since $\beta \geq \beta_0$, one has $\varphi \geq 1$. Let τ be the exit time of $\mathbb{H}_n$. Summing over $t \leq T$ the t-th iteration of (1.15) with S being a singleton and $\Lambda = \mathbb{H}$ gives

$$\mathbb{P}_\beta[0 \overset{\mathbb{H}}{\longleftrightarrow} x] \geq \mathbb{E}_{\mathrm{RW},0,\beta}[\varphi^{\tau_x}\mathbb{1}_{\tau_x < T\wedge\tau}] - \frac{K}{L^d}\Big(\sum_{t=0}^{T-1}\varphi^t\Big)\max_{w\neq x}\mathbb{P}_\beta[w \overset{\mathbb{H}}{\longleftrightarrow} x] \tag{3.27}$$

$$- \Big(\sum_{t=0}^{T-1}\varphi^t\Big)\max_{\substack{w\neq x\\|w|\leq TL}} E_\beta^{\{w\},\mathbb{H}}(w;w;x) \tag{3.28}$$

$$- \Big(\sum_{t=0}^{T-1}\varphi^t\Big)\max_{\substack{w\neq x\\|w|\leq TL}} \mathsf{E}_\beta^{\mathbb{H}}(w,x), \tag{3.29}$$

where τ_x is the hitting time of x and τ is the exit time of $\mathbb{H}_n$. Note that above, it is possible that in the local error term $v = x$. This explains the additional term (3.28). Classical random walk estimates give the existence of $c_{\mathrm{RW}} = c_{\mathrm{RW}}(A,d), T = T(A,d) > 0$ such that

$$\mathbb{P}_{\mathrm{RW},0,\beta}[\tau_x < T\wedge\tau] \geq \frac{c_{\mathrm{RW}}}{L^d}. \tag{3.30}$$

Using (**??**) and (2.92),

$$K\Big(\sum_{t=0}^{T-1}\varphi^t\Big)\max_{w\neq x}\mathbb{P}_\beta[w \overset{\mathbb{H}}{\longleftrightarrow} x] \leq \frac{K\mathbf{C}}{L^d}\sum_{t=0}^{T-1}(1+\tfrac{K}{L^d})^t, \tag{3.31}$$

which can be made smaller than $\frac{c_{\mathrm{RW}}}{4}$ by choosing L large enough. Moreover, if $w \neq x$,

$$E_\beta^{\{w\},\mathbb{H}}(w;w;x) = \sum_{z\neq t} p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z \overset{\mathbb{H}}{\longleftrightarrow} x]\mathbb{P}_\beta[t \overset{\mathbb{H}}{\longleftrightarrow} x]. \tag{3.32}$$

Using (2.92) we obtain the existence of $C_1 = C_1(\mathbf{C},d) > 0$ such that

$$E_\beta^{\{w\},\mathbb{H}}(w;w;x) \leq \frac{C_1\mathbf{C}}{L^{2d}}, \tag{3.33}$$

and by choosing L large enough we get

$$\Big(\sum_{t=0}^{T-1}\varphi^t\Big)\max_{\substack{w\neq x\\|w|\le TL}} E_\beta^{\{w\},\mathbb{H}}(w;w;x)\le \frac{c_{\mathrm{RW}}}{4L^d}. \tag{3.34}$$

Finally, if $w\neq x$ and $|w|\le TL$,

$$\mathsf{E}_\beta^{\mathbb{H}}(w,x)=\sum_{v\notin\Lambda_L(w)}\sum_{z\neq t}p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\overset{\mathbb{H}}{\longleftrightarrow}v]\mathbb{P}_\beta[t\overset{\mathbb{H}}{\longleftrightarrow}v]\mathbb{P}_\beta[v\overset{\mathbb{H}}{\longleftrightarrow}x] \tag{3.35}$$

$$=\sum_{k\ge0}\sum_{\substack{v\in\partial\mathbb{H}_{-k}\\v\notin\Lambda_L(w)}}\mathbb{P}_\beta[v\overset{\mathbb{H}}{\longleftrightarrow}x]\sum_{z\neq t}p_{wz}(\beta)p_{wt}(\beta)\mathbb{P}_\beta[z\overset{\mathbb{H}}{\longleftrightarrow}v]\mathbb{P}_\beta[t\overset{\mathbb{H}}{\longleftrightarrow}v] \tag{3.36}$$

$$\le\sum_{k\ge0}(2T)^{d-1}\frac{\mathbf{C}^2(2TL+L)}{L^{d+1}}\Big(\frac{L}{L\vee|k-L|}\Big)^{d-1}\Big(\delta_{x_1}(k)+\frac{\mathbf{C}^2(k+L)}{L^2}\Big)\Big(\sum_z p_{0z}\Big)^2\frac{\mathbf{C}}{L^d}, \tag{3.37}$$

where we used that $v\neq z,t$ (since $v\notin\Lambda_L(w)$) and the following:

$$\mathbb{P}_\beta[t\overset{\mathbb{H}}{\longleftrightarrow}v]\overset{(2.4)}{\le}\frac{(2T)^{d-1}\mathbf{C}^2(2TL+L)}{L^{d+1}}\Big(\frac{L}{L\vee|k-L|}\Big)^{d-1},\qquad \mathbb{P}_\beta[z\overset{\mathbb{H}}{\longleftrightarrow}v]\overset{(2.92)}{\le}\frac{\mathbf{C}}{L^d} \tag{3.38}$$

$$\sum_{v\in\partial\mathbb{H}_{-k}}\mathbb{P}_\beta[v\overset{\mathbb{H}}{\longleftrightarrow}x]\overset{(2.5)}{\le}\delta_{x_1}(k)+\frac{\mathbf{C}^2(k+L)}{L^2}. \tag{3.39}$$

We can then obtain the existence of $C_2=C_2(T,\mathbf{C},d)>0$ such that

$$\mathsf{E}_\beta^{\mathbb{H}}(w,x)\le\frac{C_2}{L^{2d}}. \tag{3.40}$$

Once again, if L is large enough,

$$\Big(\sum_{t=0}^{T-1}\varphi^t\Big)\max_{\substack{w\neq x\\|w|\le TL}}\mathsf{E}_\beta^{\mathbb{H}}(w,x)\le\frac{c_{\mathrm{RW}}}{4L^d}. \tag{3.41}$$

This concludes the proof in that case. □

We now turn to the full plane lower bound below scale L'_β.

Lemma 3.4. *Let $d>6$. There exist $c=c(d),\varepsilon>0$ and $L_0>0$ such that for every $L\ge L_0$, every $\varepsilon<\varepsilon_0$, every $\beta_0\le\beta\le\beta_c$, and every $x\in\Lambda_{5L'_\beta}$,*

$$\mathbb{P}_\beta[0\longleftrightarrow x]\ge\frac{c}{L^d}\left(\frac{L}{L\vee|x|}\right)^{d-2}. \tag{3.42}$$

Proof. RP: I will include the proof ASAP. Notice that small values of x are bounded using the half-space bound! □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. We already have the corresponding lower bounds for $|x| \le 5L'_\beta$. Let us turn to the general case. We focus on the full space estimate but the half-space holds the same. Introduce, for $k \ge 0$,

$$m_k := \min\Big\{\mathbb{P}_\beta[x \leftrightarrow 0] : x \in S_k\Big\}, \tag{3.43}$$

where $S_k := \{x \in \mathbb{Z}^d : kL'_\beta \le |x| < (k+1)L'_\beta\}$. We prove by induction for $k \ge 5$ that for some $c_1 > 0$,

$$m_k \ge \frac{1}{L^2}\frac{c'}{(5L'_\beta)^{d-2}}c_1^{k-5} \tag{3.44}$$

For $k = 5$, it is simply (3.42) with $c' = c$. We now assume that $k \ge 6$. For every $x \in S_k$, if $\Lambda = \Lambda_{L'_\beta}(x)$, note that

$$\sum_{\substack{y\in\Lambda\cap A_{k-1}\\ z\notin\Lambda\\ y\sim z}} \mathbb{P}_\beta[y \leftrightarrow x]p_{yz}(\beta) \ge \frac{1}{2d}\frac{1}{2^{d-1}}\varphi_\beta(\Lambda_{L'_\beta}) \ge \frac{1}{2^d d}(1-\varepsilon) =: 3c_0. \tag{3.45}$$

Lemma 1.5 therefore imply that for every $x \in S_k$,

$$\mathbb{P}_\beta[0 \leftrightarrow x] \ge 3c_0 m_{k-1} - \frac{K}{L^d}M_1(x) - \mathsf{E}_\beta^{\Lambda_{L'_\beta}(x),\mathbb{Z}^d}(x,0), \tag{3.46}$$

where

$$M_\ell(x) := \max\{\mathbb{P}_\beta[0 \leftrightarrow y] : y \in \Lambda_{\ell L'_\beta + L}(x)\}. \tag{3.47}$$

Now, define

$$D_\ell(x) := M_\ell(x) + \frac{L^d}{K}\max_{w\in\Lambda_{\ell L'_\beta}(x)}\max_{\substack{S\in\mathcal{B}\\ S\subset\Lambda_{\ell L'_\beta}(x)\\ S\ni w}}\mathsf{E}_\beta^{S,\mathbb{Z}^d}(w,0). \tag{3.48}$$

Notice that

$$D_0(x) = \mathbb{P}_\beta[0 \leftrightarrow x] + \frac{L^d}{K}\mathsf{E}_\beta^{\mathbb{Z}^d}(x,0). \tag{3.49}$$

As a result, we may rewrite (3.46) as

$$\mathbb{P}_\beta[0 \leftrightarrow x] \ge 3c_0 m_{k-1} - \frac{K}{L^d}D_1(x). \tag{3.50}$$

If $\frac{K}{L^d}D_1(x) \le D_0(x)$, then (3.50) gives

$$\mathbb{P}_\beta[0 \leftrightarrow x] \ge \tfrac{3}{2}c_0 m_{k-1} - \frac{L^d}{2K}\mathsf{E}_\beta^{\mathbb{Z}^d}(x,0). \tag{3.51}$$

Lemma 3.5 below allows to bound this non-local error term by $\frac{c_0}{2}m_{k-1}$ provided that L is large enough. As a consequence, we find $m_k \ge c_0 m_{k-1}$ and therefore the induction hypothesis, except if there is $x \in S_k$ such that $\frac{K}{L^d}D_1(x) > D_0(x)$. We show below that this is in fact impossible by proceeding by contradiction.

Let $\eta < 1/(4C_{\rm RW})$ small to be fixed and $K/L^d \le 2\eta$. Also, (potentially) decrease ε so that Proposition 3.2 holds true for this η and $\alpha = 1$ (decreasing ε would not contradict the previous use of Lemmata 3.3 and 3.4 as $L_\beta(\varepsilon)$ is increasing in ε).

For ℓ such that $0 \notin \Lambda_{(\ell+2)L'_\beta}(x)$, Proposition 3.2 applied to all boxes of size L'_β centered on sites in $\Lambda_{\ell L'_\beta}(x)$ gives

$$D_{\ell+1}(x) \le C_{\rm RW} D_\ell(x) + \eta M_{\ell+2}(x) + A \max_{w \in \Lambda_{(\ell+2)L'_\beta}(x)} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{(\ell+2)L'_\beta}(x) \\ S \ni w}} \mathsf{E}_\beta^{S,\mathbb{Z}^d}(w,0) \tag{3.52}$$

$$\le C_{\rm RW} D_\ell(x) + \eta D_{\ell+2}(x), \tag{3.53}$$

provided L is large enough that $AK/L^d \le \eta$. Yet, the choices of L and η, as well as the assumption that

$$D_0(x) < \frac{K}{L^d} D_1(x) \tag{3.54}$$

imply recursively that $D_\ell(x) \le 2\eta D_{\ell+1}(x)$ as long as $0 \notin \Lambda_{(\ell+2)L'_\beta}(x)$. In particular, if $\ell := \lfloor |x|/L'_\beta \rfloor - 3$, we obtain that

$$\frac{c'}{L^2(5L'_\beta)^{d-2}} \overset{(3.42)}{\le} m_4 \le D_\ell(x) \le 2\eta D_{\ell+1}(x) \tag{3.55}$$

$$\overset{(2.91)}{\le} 2\eta \frac{3\mathbf{C}^2}{L^2(L'_\beta)^{d-2}} + 2\eta \frac{L^d}{K} \max_{w \in \Lambda_{(\ell+1)L'_\beta(x)}} \max_{\substack{S \in \mathcal{B} \\ S \subset \Lambda_{(\ell+1)L'_\beta(x)} \\ S \ni w}} \mathsf{E}_\beta^{S,\mathbb{Z}^d}(w,0). \tag{3.56}$$

Need one last Lemma to bound the error term by $O(1)L^{-2}(L'_\beta)^{2-d}$. I am finishing a proof of that. The choice of η leads to a contradiction, therefore concluding the proof. □

Lemma 3.5. *Assume that β' is such that $L'_\beta \ge L$. Assume that $x \in S_k$ with $k \ge 6$. Then, for some $D_2 = D_2(\mathbf{C}, d) > 0$,*

$$\mathsf{E}_\beta^{\mathbb{Z}^d}(x,0) \le \frac{1}{L^{d-6}} \frac{1}{L^d} \frac{D_2}{L^2} \frac{1}{(L'_\beta)^{d-2}} e^{-c(k-5)}, \tag{3.57}$$

and so, under the induction hypothesis, $\mathsf{E}_\beta^{\mathbb{Z}^d}(x,0) = O(L^{6-d}) m_{k-1}$.

Proof. Recall that

$$\mathsf{E}^{\mathbb{Z}^d}(x,0) = \sum_{v \notin \Lambda_L(x)} \sum_{z \ne t} p_{xz}(\beta) p_{xt}(\beta) \mathbb{P}_\beta[z \leftrightarrow v] \mathbb{P}_\beta[t \leftrightarrow v] \mathbb{P}_\beta[v \leftrightarrow 0]. \tag{3.58}$$

We first look at the contribution coming from $v \in \Lambda_{5L'_\beta}(x)$. Using the near-critical full-space bound (we need to state a version with L'_β instead of L_β but this is fine), one has, for such v,

$$\mathbb{P}_\beta[v \leftrightarrow 0] \le \frac{C}{(L'_\beta)^{d-2}} e^{-c(k-5)}. \tag{3.59}$$

Hence,

$$\sum_{v \in \Lambda_{5L'_\beta}(x) \setminus \Lambda_L(x)} \sum_{z \ne t} p_{xz}(\beta) p_{xt}(\beta) \mathbb{P}_\beta[z \leftrightarrow v] \mathbb{P}_\beta[t \leftrightarrow v] \mathbb{P}_\beta[v \leftrightarrow 0]$$

$$\le \frac{A_1}{(L'_\beta)^{d-2}} e^{-c(k-5)} \sum_{z \ne t \in \Lambda_L(x)} \frac{1}{L^{2d}} \frac{1}{L^d} \frac{1}{|z-t|^{d-4}}$$

$$\le \frac{1}{L^{d-4}} \frac{1}{L^d} \frac{A_2}{(L'_\beta)^{d-2}} e^{-c(k-5)},$$

where we used that

$$\sum_{v\in\Lambda_{5L'_\beta}(x)\setminus\Lambda_L(x)}\frac{1}{L^d}\Big(\frac{L}{L\vee|z-v|}\Big)^{d-2}\frac{1}{L^d}\Big(\frac{L}{L\vee|t-v|}\Big)^{d-2}\lesssim\frac{1}{L^d}\frac{1}{|z-t|^{d-4}}. \tag{3.60}$$

The contribution for $v\in\Lambda_{L'_\beta}$ is handled easily too. This reduces the problem to controlling,

$$\frac{1}{L^6}\sum_{\substack{v\in\mathbb{Z}^d\\|v|\geq L'_\beta\\|x-v|\geq L'_\beta}}\frac{e^{-2c|x-v|/L'_\beta}}{|x-v|^{2d-4}}\frac{e^{-c|v|/L'_\beta}}{|v|^{d-2}}. \tag{3.61}$$

□

4 Miscellenaous

The previous analysis also implies that

$$\mathbb{P}_{\beta_c}[0\longleftrightarrow\partial\Lambda_n]\geq\frac{c}{n^2}. \tag{4.1}$$

Indeed, for any finite set $S\subset\Lambda_n$ containing 0, we have

$$\varphi_\beta(\Lambda_n)\leq\varphi_\beta(S)\max\{\varphi_\beta(\Lambda_n(x)):x\in\Lambda_n\}. \tag{4.2}$$

As a consequence, if S is contained in L_β, we deduce that $\varphi_\beta(S)\geq\frac{1}{e}(1+K/L^d)^{-1}=:c_1$. We then deduce from [DCT16] that $\mathbb{P}_{\beta_c}[0\longleftrightarrow\partial\Lambda_{L_\beta}]\geq c_2c_1(\beta_c-\beta)$. When plugging the asymptotic $L_\beta\asymp(\beta_c-\beta)^{-1/2}$, one obtains the result.

Appendix A: proof of Lemma 1.5

Considering a self-avoiding path from o to x we obtain

$$\{o\overset{\Lambda}{\longleftrightarrow}x\}\setminus\{o\overset{S}{\longleftrightarrow}x\}\subset\bigcup_{\substack{y\in S\\z\in\Lambda\setminus S\\y\sim z}}\{o\overset{S}{\longleftrightarrow}y\}\circ\{yz\text{ is open}\}\circ\{z\overset{\Lambda}{\longleftrightarrow}x\}, \tag{A.1}$$

which gives the upper bound by the BK inequality.

For the reverse bound, let

$$\mathcal{N}:=\sum_{\substack{y\in S\\z\in\Lambda\setminus S}}\mathbb{1}[o\overset{S}{\longleftrightarrow}y,\ yz\text{ is open},\ z\xleftrightarrow{(\mathcal{C}^S(o))^c}x], \tag{A.2}$$

where $\mathcal{C}^S(o)$ is the cluster of o in S. Clearly,

$$\{o\overset{\Lambda}{\longleftrightarrow}x\}\setminus\{o\overset{S}{\longleftrightarrow}x\}\supset\{\mathcal{N}\geq1\}. \tag{A.3}$$

Notice that[9]

$$\mathbb{P}_\beta[\mathcal{N}\geq1]\geq2\mathbb{E}_\beta[\mathcal{N}]-\mathbb{E}_\beta[\mathcal{N}^2]. \tag{A.4}$$

[9]This is a consequence of the fact that for $t\in[0,\infty]$, $2t(1-t)\leq\mathbb{1}[t\geq1]$.

Write

$$\mathbb{E}_\beta[\mathcal{N}] = \sum_{\substack{y\in S\\ z\in\Lambda\setminus S}} \sum_{C\ni y} \mathbb{P}_\beta[\mathcal{C}^S(o)=C]p_{yz}(\beta)\mathbb{P}_\beta[z \xleftrightarrow{C^c} x], \tag{A.5}$$

and

$$\mathbb{P}_\beta[z \xleftrightarrow{C^c} x] = \mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] - \Big(\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] - \mathbb{P}_\beta[z \xleftrightarrow{C^c} x]\Big). \tag{A.6}$$

Using [AN84, Proposition 5.2], we find that

$$\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] - \mathbb{P}_\beta[z \xleftrightarrow{C^c} x] \le \sum_{v\in C} \mathbb{P}_\beta[\mathcal{A}(z,v)]\mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x], \tag{A.7}$$

where $\mathcal{A}(z,v)$ is the event that z and v are connected by a path which contains exactly one element of C. Combined with (A.5), and using the fact that $C\subset S$, this yields

$$\begin{aligned} 0 \le &\sum_{\substack{y\in S\\ z\in\Lambda\setminus S}} \mathbb{P}_\beta[0 \xleftrightarrow{S} y]p_{yz}(\beta)\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] - \mathbb{E}_\beta[\mathcal{N}] \\ &\le \sum_{\substack{y\in S\\ z\in\Lambda\setminus S}} \sum_{v\in S} \mathbb{P}_\beta[\{o \xleftrightarrow{S} y,\ o \xleftrightarrow{S} v\}\circ\{z \xleftrightarrow{\Lambda} v\}]p_{yz}(\beta)\mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x]. \end{aligned}$$

Using the BK inequality again yields,

$$\mathbb{P}_\beta[\{o \xleftrightarrow{S} y,\ o \xleftrightarrow{S} v\}\circ\{z \xleftrightarrow{\Lambda} v\}] \le \mathbb{P}_\beta[o \xleftrightarrow{S} y,\ o \xleftrightarrow{S} v]\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} v]. \tag{A.8}$$

Finally, using [AN84, Proposition 4.1], we get

$$\mathbb{P}_\beta[o \xleftrightarrow{S} y,\ o \xleftrightarrow{S} v] \le \sum_{u\in S} \mathbb{P}_\beta[o \xleftrightarrow{S} u]\mathbb{P}_\beta[u \xleftrightarrow{S} v]\mathbb{P}_\beta[u \xleftrightarrow{S} y]. \tag{A.9}$$

We obtained,

$$\mathbb{E}_\beta[\mathcal{N}] \ge \sum_{\substack{y\in S\\ z\in\Lambda\setminus S}} \mathbb{P}_\beta[o \xleftrightarrow{S} u]p_{yz}(\beta)\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} x] \tag{A.10}$$

$$- \sum_{u,v\in S} \sum_{\substack{y\in S\\ z\in\Lambda\setminus S}} \mathbb{P}_\beta[o \xleftrightarrow{S} y]\mathbb{P}_\beta[u \xleftrightarrow{S} y]\mathbb{P}_\beta[u \xleftrightarrow{S} v]p_{yz}(\beta)\mathbb{P}_\beta[z \xleftrightarrow{\Lambda} v]\mathbb{P}_\beta[v \xleftrightarrow{\Lambda} x] \tag{A.11}$$

It remains to analyze $\mathbb{E}_\beta[\mathcal{N}^2]$. Notice that,

$$\mathbb{E}_\beta[\mathcal{N}^2] = \mathbb{E}_\beta[\mathcal{N}] + \sum_{\substack{y,s\in S\\ z,t\in\Lambda\setminus S\\ yz\neq st}} \mathbb{P}_\beta[o \xleftrightarrow{S} y, yz \text{ is open}, z \xleftrightarrow{(\mathcal{C}^S(0))^c} x, o \xleftrightarrow{S} s, st \text{ is open}, t \xleftrightarrow{(\mathcal{C}^S(0))^c} x]. \tag{A.12}$$

Using the same techniques as above, and taking into account that we may have $y=s$ or $z=t$ (but not simultaneously),

$$\mathbb{E}_\beta[\mathcal{N}^2] - \mathbb{E}_\beta[\mathcal{N}] \le (I) + (II) + (III) \tag{A.13}$$

where

$$(I) := \sum_{\substack{u\in S\\ v\in\Lambda}} \sum_{\substack{y,s\in S\\ y\neq s\\ z,t\in\Lambda\setminus S\\ z\neq t}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} u]\mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} y]\mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} s]p_{yz}(\beta)p_{st}(\beta)\mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} v]\mathbb{P}_\beta[t \overset{\Lambda}{\longleftrightarrow} v]\mathbb{P}_\beta[v \overset{\Lambda}{\longleftrightarrow} x], \tag{A.14}$$

$$(II) := \sum_{v\in\Lambda} \sum_{\substack{y\in S\\ z,t\in\Lambda\setminus S\\ z\neq t}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} y]p_{yz}(\beta)p_{yt}(\beta)\mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} v]\mathbb{P}_\beta[t \overset{\Lambda}{\longleftrightarrow} v]\mathbb{P}_\beta[v \overset{\Lambda}{\longleftrightarrow} x], \tag{A.15}$$

$$(III) := \sum_{u\in S} \sum_{\substack{y\neq s\in S\\ z\in\Lambda\setminus S}} \mathbb{P}_\beta[o \overset{S}{\longleftrightarrow} u]\mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} y]\mathbb{P}_\beta[u \overset{S}{\longleftrightarrow} s]p_{yz}(\beta)p_{sz}(\beta)\mathbb{P}_\beta[z \overset{\Lambda}{\longleftrightarrow} x] \tag{A.16}$$

The proof follows readily.

Appendix B: Computation in Lemma 2.6

Let $n \geq 0$. We fix $\beta < \beta^*$ and drop it from the notations. By the definition given in (1.16)–(1.19), we write

$$E_\beta^{\mathbb{H}_n,\mathbb{Z}^d} = \mathcal{E}(1) + \mathcal{E}(2) + \mathcal{E}(3) + \mathcal{E}(4), \tag{A.17}$$

where

$$\mathcal{E}(1) := \sum_{u,v\in\mathbb{H}_n} \sum_{\substack{y\in\mathbb{H}_n\\ z\notin\mathbb{H}_n}} \mathbb{P}[0 \overset{\mathbb{H}_n}{\longleftrightarrow} u]\mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} y]\mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} v]p_{yz}\mathbb{P}[z \leftrightarrow v] \tag{A.18}$$

$$\mathcal{E}(2) := \sum_{\substack{u\in\mathbb{H}_n\\ v\notin\mathbb{H}_n}} \sum_{\substack{y,s\in\mathbb{H}_n\\ y\neq s}} \mathbb{P}[0 \overset{\mathbb{H}_n}{\longleftrightarrow} u]\mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} y]\mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} s]p_{yv}p_{sv} \tag{A.19}$$

$$\mathcal{E}(3) := \sum_{\substack{u\in\mathbb{H}_n\\ v\in\mathbb{Z}^d}} \sum_{\substack{y,s\in\mathbb{H}_n\\ y\neq s\\ z,t\notin\mathbb{H}_n\\ z\neq t}} \mathbb{P}[0 \overset{\mathbb{H}_n}{\longleftrightarrow} u]\mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} y]\mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} s]p_{yz}p_{st}\mathbb{P}_\beta[z \leftrightarrow v]\mathbb{P}[t \leftrightarrow v], \tag{A.20}$$

$$\mathcal{E}(4) := \sum_{v\in\mathbb{Z}^d} \sum_{\substack{y\in\mathbb{H}_n\\ z,t\notin\mathbb{H}_n\\ z\neq t}} \mathbb{P}[0 \overset{\mathbb{H}_n}{\longleftrightarrow} y]p_{yz}p_{yt}\mathbb{P}[z \leftrightarrow v]\mathbb{P}[t \leftrightarrow v]. \tag{A.21}$$

Bound on $\mathcal{E}(1)$ We write

$$\mathcal{E}(1) = \sum_{\ell\geq 0} \sum_{u\in\partial\mathbb{H}_{n-\ell}} \mathbb{P}[0 \overset{\mathbb{H}_n}{\longleftrightarrow} u] \sum_{\substack{y\in\mathbb{H}_n\\ z\notin\mathbb{H}_n}} \mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} y]p_{yz} \sum_{v\in\mathbb{H}_n} \mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} v]\mathbb{P}[z \leftrightarrow v]. \tag{A.22}$$

Using (2.3) and the fact that $|z-v| > 0$, there exists $C_1 = C_1(\mathbf{C}, d) > 0$ such that for all $\ell \geq 0$, and all relevant u and z,

$$\sum_{v\in\mathbb{H}_n} \mathbb{P}[u \overset{\mathbb{H}_n}{\longleftrightarrow} v]\mathbb{P}[z \leftrightarrow v] \leq \sum_{v\in\mathbb{H}_n} \mathbb{P}[u \leftrightarrow v]\mathbb{P}[z \leftrightarrow v] \leq \frac{1}{L^d}\frac{C_1}{(\ell+1)^{d-4}}. \tag{A.23}$$

Using Lemma 2.5, for every $u \in \partial\mathbb{H}_{n-\ell}$,

$$\sum_{\substack{y \in \mathbb{H}_n \\ z \notin \mathbb{H}_n}} \mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} y] p_{yz} = \varphi_\beta(\mathbb{H}_\ell) \leq 6\mathbf{C}^3. \tag{A.24}$$

Finally, using (2.5) of Lemma 2.4,

$$\sum_{u \in \partial\mathbb{H}_{n-\ell}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u] \leq \delta_n(\ell) + \frac{\mathbf{C}^2(\ell + L)}{L^2}. \tag{A.25}$$

Putting all the pieces together we found $C_2 = C_2(\mathbf{C}, d) > 0$ such that

$$\mathcal{E}(1) \leq \frac{C_2}{L^d} \sum_{\ell \geq 0} \frac{1}{(\ell+1)^{d-5}}, \tag{A.26}$$

which converges when $d > 6$.

Bound on $\mathcal{E}(2)$ Write

$$\mathcal{E}(2) = \sum_{\ell \geq 0} \sum_{u \in \mathbb{H}_{n-\ell}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u] \sum_{\substack{y,s \in \mathbb{H}_n \\ y \neq s \\ v \notin \mathbb{H}_n}} \mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} y]\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} s] p_{yv} p_{sv}. \tag{A.27}$$

Since y and s are distinct, one of them is distinct from u. Hence, by symmetry, and using the fact that for a fixed v one has $\sum_{y \in \mathbb{H}_n} p_{yv} \leq \beta^*|J| \leq 2$,

$$\sum_{\substack{y,s \in \mathbb{H}_n \\ y \neq s \\ v \notin \mathbb{H}_n}} \mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} y]\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} s] p_{yv} p_{sv} \overset{(2.4)}{\leq} 4\frac{2\mathbf{C}^2}{L^d} \left(\frac{L}{L \wedge (\ell - L)} \right)^{d-1} \varphi_\beta(\mathbb{H}_\ell) \tag{A.28}$$

$$\overset{(A.24)}{\leq} 48\frac{\mathbf{C}^5}{L^d} \left(\frac{L}{L \wedge (\ell - L)} \right)^{d-1}. \tag{A.29}$$

We obtained,

$$\mathcal{E}(2) \leq \frac{48\mathbf{C}^5}{L^d} \sum_{\ell \geq 0} \left(\frac{L}{L \wedge (\ell - L)} \right)^{d-1} \sum_{u \in \mathbb{H}_{n-\ell}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u]. \tag{A.30}$$

Using (2.5) once again in (A.30), we obtain $C_3 = C_3(\mathbf{C}, d) > 0$ such that

$$\mathcal{E}(2) \leq \frac{C_3}{L^d}. \tag{A.31}$$

Bound on $\mathcal{E}(3)$ By (2.3), there exists $C_4 = C_4(\mathbf{C}, d) > 0$ such that, for all $z \neq t$ as above,

$$\sum_{v \in \mathbb{Z}^d} \mathbb{P}_\beta[z \leftrightarrow v]\mathbb{P}[t \leftrightarrow v] \leq \frac{C_4}{L^d} \frac{1}{|z - t|^{d-4}}. \tag{A.32}$$

Now, write

$$\sum_{\substack{u\in\mathbb{H}_n\\ y,s\in\mathbb{H}_n\\ z,t\notin\mathbb{H}_n\\ z\neq t}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u]\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} y]\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} s]\frac{p_{yz}p_{st}}{|z-t|^{d-4}} \tag{A.33}$$

$$= \sum_{k\geq 0}\sum_{u\in\partial\mathbb{H}_{n-k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u] \sum_{\substack{y,s\in\mathbb{H}_n\\ z,t\notin\mathbb{H}_n\\ z\neq t}} \mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} y]\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} s]\frac{p_{yz}p_{st}}{|z-t|^{d-4}}. \tag{A.34}$$

Looking first at the contribution coming from $|z-t|\geq k+1$, we find, for some $C_5 = C_5(\mathbf{C},d)>0$,

$$\sum_{k\geq 0}\sum_{u\in\partial\mathbb{H}_{n-k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u] \sum_{\substack{y,s\in\mathbb{H}_n\\ z,t\notin\mathbb{H}_n\\ |z-t|\geq k+1}} \mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} y]\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} s]\frac{p_{yz}p_{st}}{|z-t|^{d-4}} \tag{A.35}$$

$$\leq \sum_{k\geq 0}\frac{\varphi_\beta(\mathbb{H}_k)^2}{(k+1)^{d-4}}\sum_{u\in\partial\mathbb{H}_{n-k}}\mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u] \tag{A.36}$$

$$\overset{(2.4)}{\leq} C_5\sum_{k\geq 0}\frac{1}{(k+1)^{d-5}}, \tag{A.37}$$

which is finite when $d>6$, and where we additionally used Lemma 2.5 in the last inequality. We turn to the contribution coming from $|z-t|\leq k$. First, by (2.4) we find that

$$\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} s] \leq \frac{2\mathbf{C}^2}{L^d}\left(\frac{L}{(k-L)\vee L}\right)^{d-1}. \tag{A.38}$$

Then, there exists $C_6 = C_6(d)>0$ such that for fixed y,z as above,

$$\sum_{\substack{t\in\Lambda_k(z)\setminus\{z\}\cap\mathbb{H}_n^c\\ s\in\mathbb{H}_n}} p_{st}\frac{1}{|z-t|^{d-4}} \leq \sum_{\substack{t\in\Lambda_k(z)\setminus\{z\}\cap[-L,L]\times[-k,k]^{d-1}\\ s\in\Lambda_L(t)}} p_{st}\frac{1}{|z-t|^{d-4}} \leq C_6(L\cdot k^3). \tag{A.39}$$

Finally, we obtained,

$$\sum_{k\geq 0}\sum_{u\in\partial\mathbb{H}_{n-k}} \mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u] \sum_{\substack{y,s\in\mathbb{H}_n\\ z,t\notin\mathbb{H}_n\\ |z-t|\leq k}} \mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} y]\mathbb{P}[u \xleftrightarrow{\mathbb{H}_n} s]\frac{p_{yz}p_{st}}{|z-t|^{d-4}}$$

$$\leq \frac{2\mathbf{C}^2C_6}{L^d}\sum_{k\geq 0}\varphi_\beta(\mathbb{H}_k)\left(\frac{L}{(k-L)\vee L}\right)^{d-1} L\cdot k^3\sum_{u\in\partial\mathbb{H}_{n-k}}\mathbb{P}[0 \xleftrightarrow{\mathbb{H}_n} u]$$

$$\leq C_7,$$

where $C_7 = C_7(\mathbf{C},d)>0$. Gathering the last display and (A.32), we obtained,

$$\mathcal{E}(3) \leq \frac{C_4C_7}{L^d}. \tag{A.40}$$

Bound on $\mathcal{E}(4)$ This last term is handled by similar arguments as $\mathcal{E}(3)$. We omit the details.

References

[AB87] Michael Aizenman and David J. Barsky. Sharpness of the phase transition in percolation models. *Communications in Mathematical Physics*, **108**(3):489–526, 1987.

[AN84] Michael Aizenman and Charles M. Newman. Tree graph inequalities and critical behavior in percolation models. *Journal of Statistical Physics*, **36**(1-2):107–143, 1984.

[BA91] David J. Barsky and Michael Aizenman. Percolation critical exponents under the triangle condition. *The Annals of Probability*, **19**(4):1520–1536, 1991.

[BBS14] Roland Bauerschmidt, David C. Brydges, and Gordon Slade. Scaling limits and critical behaviour of the 4-dimensional n-component $|\varphi|^4$ spin model. *Journal of Statistical Physics*, **157**:692–742, 2014.

[BBS15a] Roland Bauerschmidt, David C. Brydges, and Gordon Slade. Critical two-point function of the 4-dimensional weakly self-avoiding walk. *Communications in Mathematical Physics*, **338**:169–193, 2015.

[BBS15b] Roland Bauerschmidt, David C. Brydges, and Gordon Slade. Logarithmic correction for the susceptibility of the 4-dimensional weakly self-avoiding walk: a renormalisation group analysis. *Communications in Mathematical Physics*, **337**:817–877, 2015.

[BBS19] Roland Bauerschmidt, David C. Brydges, and Gordon Slade. *Introduction to a Renormalisation Group Method*, volume **2242**. Springer Nature, 2019.

[BS85] David Brydges and Thomas Spencer. Self-avoiding walk in 5 or more dimensions. *Communications in Mathematical Physics*, **97**(1):125–148, 1985.

[CS15] Lung-Chi Chen and Akira Sakai. Critical two-point functions for long-range statistical-mechanical models in high dimensions. *The Annals of Probability*, **43**(2):639–681, 2015.

[CS19] Lung-Chi Chen and Akira Sakai. Critical two-point function for long-range models with power-law couplings: The marginal case for $d \geq d_c$. *Communications in Mathematical Physics*, **372**(2):543–572, 2019.

[DCT16] Hugo Duminil-Copin and Vincent Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. *Communications in Mathematical Physics*, **343**:725–745, 2016.

[FvdH17] Robert Fitzner and Remco W. van der Hofstad. Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electronic Journal of Probability*, **22**:43, 2017.

[Gri99] Geoffrey Grimmett. *Percolation*, volume **321**. Springer, 1999.

[Har08] Takashi Hara. Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *The Annals of Probability*, **36**(2):530–593, 2008.

[HHS03] Takashi Hara, Remco W. van der Hofstad, and Gordon Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *The Annals of Probability*, **31**(1):349–408, 2003.

[HS90a] Takashi Hara and Gordon Slade. Mean-field critical behaviour for percolation in high dimensions. *Communications in Mathematical Physics*, **128**(2):333–391, 1990.

[HS90b] Takashi Hara and Gordon Slade. On the upper critical dimension of lattice trees and lattice animals. *Journal of Statistical Physics*, **59**:1469–1510, 1990.

[HS02] Remco W. van der Hofstad and Gordon Slade. A generalised inductive approach to the lace expansion. *Probability Theory and Related Fields*, **122**(3):389–430, 2002.

[Pan23] Romain Panis. Triviality of the scaling limits of critical Ising and φ^4 models with effective dimension at least four. *arXiv preprint arXiv:2309.05797*, 2023.

[Sak07] Akira Sakai. Lace expansion for the Ising model. *Communications in Mathematical Physics*, **272**(2):283–344, 2007.

[Sak15] Akira Sakai. Application of the lace expansion to the φ^4 model. *Communications in Mathematical Physics*, **336**:619–648, 2015.

[Sak22] Akira Sakai. Correct bounds on the Ising lace-expansion coefficients. *Communications in Mathematical Physics*, **392**(3):783–823, 2022.

[Sla87] Gordon Slade. The diffusion of self-avoiding random walk in high dimensions. *Communications in Mathematical Physics*, **110**:661–683, 1987.

[Sla06] Gordon Slade. *The Lace Expansion and Its Applications: Ecole D'Eté de Probabilités de Saint-Flour XXXIV-2004*. Springer, 2006.

[Uch11] Kohei Uchiyama. The first hitting time of a single point for random walks. *Electronic Journal of Probability*, **16**:1960–2000, 2011.