Parareal for hyperbolic problems just does not work, or does it?

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1 Introduction

The Parareal algorithm [16] is a non-intrusive method for the parallel-in-time solution of evolution problems, a topic with a long history [6]. Its convergence is generally superlinear on bounded time intervals and can be linear on unbounded ones, see [11] for parabolic and hyperbolic PDEs, and [7] for nonlinear ODEs. Parareal performs well for parabolic problems and can even converge without a coarse solver in certain settings [10]. For hyperbolic problems, convergence is hampered by the absence of damping [11, 5]. A closely related method is MGRIT [4], whose two-level variant can be interpreted as Parareal with temporal overlap [8]. Considerable effort has been devoted to adapting MGRIT to discretized hyperbolic problems with numerical damping, starting with linear advection [1] (dissipative fine and numerically matched coarse operators), [2, 3] (dissipative semi Lagrangian coarse operators), and [14, 15] using these as componants for more complex problems. The recent review [12] shows that there are other effective time parallel methods for solving hyperbolic problems at the continuous level, many of which are also effective for parabolic problems, whereas multilevel type methods like Parareal use the parabolic nature of the problem or numerical damping in an essential way for convergence. We explain here in detail why this is, and show that there are specific configurations of hyperbolic problems which can effectively be solved by Parareal techniques.

2 Solution behavior of parabolic and hyperbolic problems

In order to well illustrate the idea, we use as our parabolic model problem the heat equation in 1D on the domain $\Omega := (0, 1)$,

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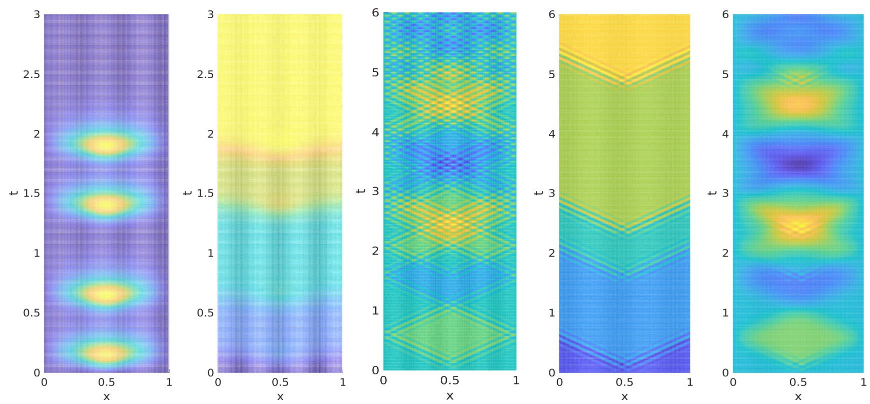

Fig. 1 Two left plots: solution of the heat equation and homogeneous Dirichlet and Neumann boundary conditions using the same source term. Three right plots: solution of the wave equation and homogeneous Dirichlet and transparent boundary conditions, using the same source term, and approximate solution using the SDIRK-1 time integrator (Backward Euler).

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx} u(x,t) + f(x,t) && \text{in } \Omega \times (0,T], \\ u(x,0) &= u_0(x) && \text{in } \Omega, \\ \mathcal{B}_0 u(0,t) &= \mathcal{B}_1 u(0,t) = 0, \end{aligned} \tag{1}$$

where the right hand side function f and the initial value u_0 are given, and the boundary operators $\mathcal{B}_j$ can be for example Dirichlet, $\mathcal{B}_j = I$ the identity, or $\mathcal{B}_j = \partial_x$ for Neumann conditions. For simplicity, we chose zero boundary conditions, but all what we say also holds for non-zero boundary conditions. We show in Figure 1 example solutions for these two cases in the left two plots. We see that the boundary conditions influence the solution of the parabolic problem in an essential way: using the right hand side function f to heat in the center of the spatial interval at four different points in time, the one dimensional structure becomes indeed warm in the center, but cools down rather rapidly with the zero Dirichlet boundary conditions, as we see in the left plot in Figure 1. Such problems can be solved very easily parallel in time, since for example the temperature distribution just before time $t = 2$ does not really depend on if the 1D structure was heated long before or not, and in such cases the Parareal algorithm does not even need a coarse propagator [10]. Using separation of variables, one sees that solutions are of the form $u(x,t) = \sum_{k=1}^{\infty} \hat{u}_k e^{-k\pi t} \sin(k\pi x)$, and thus all Fourier modes $\sin(k\pi x)$ decay rapidly, and even more so if the frequency k is large. In the second plot in Figure 1, we only replaced the zero Dirichlet conditions with zero Neumann conditions, and now the situation is different, since zero Neumann conditions represent a perfectly insulated structure and the heat remains within forming an equilibrium state, and to predict the temperature just before $t = 2$, we must know if the heater was on even long before and for how much. In such situations, the Parareal algorithm needs a coarse propagator to be effective. It can however be very coarse in space, since only the constant mode in space propagates far in time, the heat that is put into the 1D structure by the right hand side f very quickly becomes equidistributed and this equilibrium state is the only information that propagates far in time. Solutions with zero Neumann conditions are of the form $u(x,t) = \sum_{0=1}^{\infty} \hat{u}_k e^{-k\pi t} \cos(k\pi x)$, and we see that the constant mode $k = 0$ indeed does not decay at all, in contrast to all the other modes which still decay rapidly, and even more so if k is large.

As a hyperbolic model problem, we use the second order wave equation in 1D on the same domain $\Omega := (0,1)$,

$$\begin{aligned} \partial_{tt} u(x,t) &= \partial_{xx} u(x,t) + f(x,t) \qquad \text{in } \Omega \times (0,T], \\ u(x,0) &= u_0(x),\ u_t(x,0) = \tilde{u}_0(x) \text{ in } \Omega, \\ \mathcal{B}_0 u(0,t) &= \mathcal{B}_1 u(0,t) = 0. \end{aligned} \tag{2}$$

We show in Figure 1 in panels 3 and 4 two example solutions, first with zero Dirichlet boundary conditions, and then with transparent boundary conditions, i.e. $\mathcal{B}_0 = \partial_x - \partial_t$ and $\mathcal{B}_1 = \partial_x + \partial_t$. We see that with Dirichlet boundary conditions, the early source terms produce complicated high frequency patterns later in time due to reflections on the boundary (the Neumann case would be similar). Using separation of variables, solutions are of the form $u(x,t) = \sum_{k=1}^{\infty} \hat{u}_k e^{-ik\pi t} sin(k\pi x)$, and thus none of the Fourier modes $sin(k\pi x)$ are decaying, energy is conserved, like in many hyperbolic problems which are thus often called hyperbolic conservation laws. With transparent boundary conditions, the solution in the fourth panel looks like the heat equation solution in the second panel of Figure 1: only the constant equilibrium mode remains from earlier source terms f. But modes are not damped like in the heat equation, they leave the domain at the boundaries. In the last panel of Figure 1, we show a numerical solution using an SDIRK-1 time integrator for the Dirichlet boundary condition case. Comparing with the solution in the third panel, we see that the solution is not a good approximation except for very short time, since SDIRK methods are numerically strongly damping, especially high frequency modes, k large, which can be important parts of the solution in hyperbolic problems. The approximate solution then looks more like a heat equation solution.

3 Performance of Parareal on such problems

The Parareal algorithm is using two propagators for the approximate solution of the evolution problem, a fine expensive one called $\mathbf{F}(T_{n+1}, T_n, \mathbf{U}_n)$ which solves the evolution problem (either the heat equation (1) or the wave equation (2)) on the time interval (T_n, T_{n+1}) very accurately using as initial condition $\mathbf{U}_n$ at $t = T_n$, and a coarse, cheap one $\mathbf{G}(T_{n+1}, T_n, \mathbf{U}_n)$ doing the same but much less accurately. The Parareal algorithm starts with an initial guess at the coarse time points $T_0 < T_1 < T_2 < \ldots T_n < \ldots$ for example obtained using the coarse propagator sequentially, $\mathbf{U}_{n+1}^0 = G(T_{n+1}, T_n, \mathbf{U}_n^0)$, with $\mathbf{U}_0^0 = u(x,0)$ the initial condition, and then performs for iteration index $k = 0, 1, \ldots$ the correction iteration

$$\mathbf{U}_{n+1}^{k+1} := \mathbf{F}(T_{n+1}, T_n, \mathbf{U}_n^k) + \mathbf{G}(T_{n+1}, T_n, \mathbf{U}_n^{k+1}) - \mathbf{G}(T_{n+1}, T_n, \mathbf{U}_n^k). \tag{3}$$

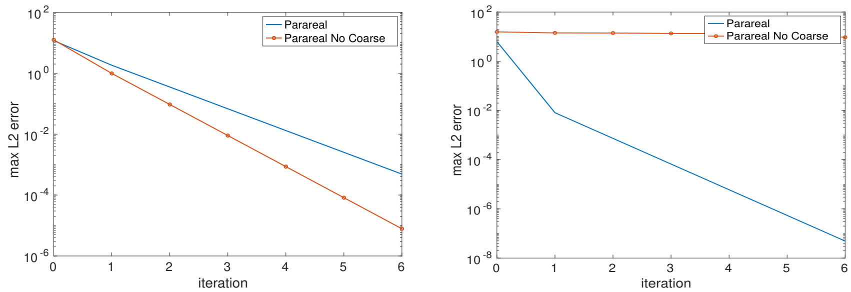

Fig. 2 Convergence of the Parareal algorithm for the 1D heat equation with Dirichlet boundary conditions on the left, and Neumann boundary conditions on the right, with and without coarse solver.

We show in Figure 2 a numerical experiment for the 1D heat equation (1) discretized with centered finite differences in space and Backward Euler in time, for a solution as shown in Figure 1 on the two left plots using twelve time subintervals and one Backward Euler step for each coarse solver. We see on the left in Figure 2 that Parareal for this configuration with Dirichlet boundary conditions converges linearly, but using Parareal without coarse correction, i.e.

$$\mathbf{U}_{n+1}^{k+1} := \mathbf{F}(T_{n+1}, T_n, \mathbf{U}_n^k), \tag{4}$$

converges even faster. On the right, we see that when the Dirichlet boundary conditions are replaced by Neumann boundary conditions, Parareal only converges with the coarse solver now, without coarse solver it stagnates. This we can easily understand from the solution picture we have seen in Figure 1: with Dirichlet boundary conditions, the solution is completely local in time, one does not have to propagate any information over long time using a coarse propagator, whereas on the right the constant needs to be propagated using a coarse mechanism in the algorithm over long time. To illustrate these components and how they propagate in time, we study the singular value decomposition of the evolution map given by the fine propagator, $\mathbf{F}(T, 0, \mathbf{e}_j)$, where the $\mathbf{e}_j$ are the canonical basis vectors with entry 1 at location j and zero everywhere else for our in space discretized solver. Applying the coarse propagator with right hand side zero gives us a matrix[1], whose singular values are shown in Figure 3 in the top row, on the left again for Dirichlet boundary conditions and on the right for Neumann boundary conditions. We see that the singular values decay, and the decay is faster the longer the time interval T in $\mathbf{F}(T, 0, \mathbf{e}_j)$ is. For long $T = 1/2$, one mode suffices to approximate the fine evolution map to an accuracy of $10e-3$, and with two modes the accuracy is $10e-7$, and with three modes $10e-13$! The corresponding singular vectors (modes) are shown in the plot in the bottom row in Figure 3, and we recognize the sine modes from the separation of variables

[1] It is only the fine map with zero right hand side and boundary data that must be approximated by the coarse propagator, since only the difference of the coarse propagator appears in the Parareal algorithm (3).

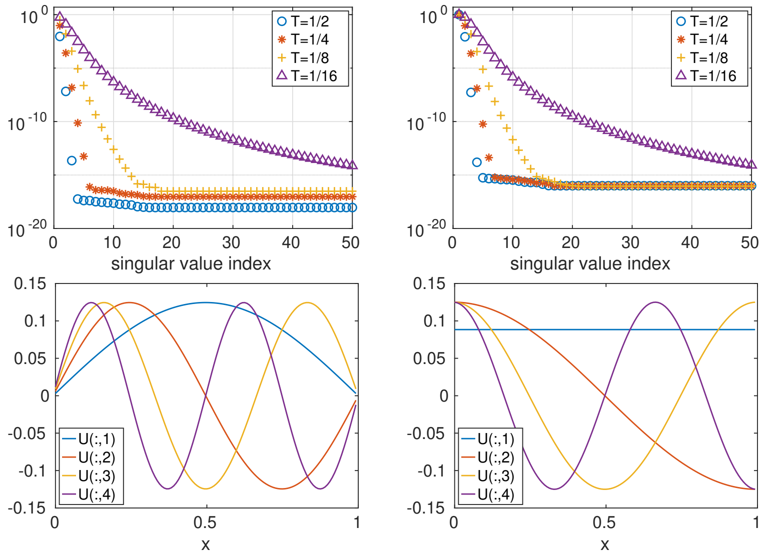

Fig. 3 First singular values (top) and singular vectors (bottom) of the fine propagator with Dirichlet boundary conditions (left) and Neumann boundary conditions (right).

representation of the solution $u(x,t) = \sum_{k=1}^{\infty} \hat{u}_k e^{-k\pi t} \sin(k\pi x)$. This is the main motivation for using a reduced basis approximation of $\mathbf{F}$ for the coarse propagator $\mathbf{G}$, see [9], it would be hardly possible to design a better such low dimensional $\mathbf{G}$ propagator! On the right in Figure 3 we show the corresponding results for the case of Neumann boundary conditions. We see that the singular values decay as fast as in the Dirichlet case, but now there is an extra singular value equal to 1, and the corresponding singular vector is the constant mode, as we see in the bottom right panel, and the modes are now the cosine modes for the Neumann boundary condition, as in the separation of variables solution $u(x,t) = \sum_{0=1}^{\infty} \hat{u}_k e^{-k\pi t} \cos(k\pi x)$.

Performing a similar numerical experiment for the case of the wave equation, discretized with the standard wave equation scheme, i.e. centered finite differences in space and time[2], we get the Parareal convergence shown in Figure 4. We see on the left that Parareal for this configuration with Dirichlet boundary conditions converges only once the algorithm has propagated the fine solution sequentially through, i.e. after 6 iterations for 6 time subintervals, $\Delta T = T_{n+1} - T_n = 1$ on the time interval $(0, T = 6]$, and 12 iterations for $\Delta T = 1/2$. As expected, with Dirichlet boundary conditions, the coarse propagator, which we run here on a coarser mesh in space and time for stability, has no effect. On the right in Figure 4, we see that with transparent boundary conditions, Parareal converges already at the second iteration for $\Delta T = 1$, whereas no convergence is observed with $\Delta T = 1/2$ until iteration 12 again as before. This can be easily understood by looking at the fourth panel in Figure 1: we see that

[2] This is the Yee scheme for Maxwell's equations, with all good properties for approximating the hyperbolic problem, like energy conservation, conservation of the Gauss laws etc.

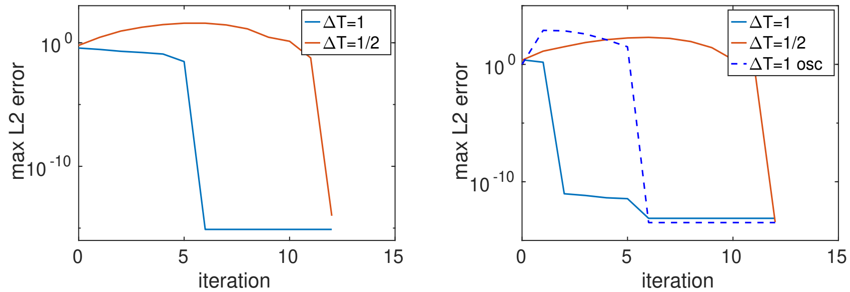

Fig. 4 Convergence of the Parareal algorithm for the 1D wave equation with Dirichlet boundary conditions on the left, and transparent boundary conditions on the right, with two different length of the time subintervals ΔT. The dashed line on the right indicates using a highly oscillatory source, see later in the text.

the time interval must be $\Delta T = 1$ or larger for all information to leave the boundary[3], otherwise the transparent boundary condition does not eliminate all information, and some must propagate to the next time interval in Parareal which then completely hampers convergence again.

We show the corresponding singular values and singular vectors for the wave equation fine evolution map $\mathbf{F}(T, 0, \mathbf{e}_j)$ in Figure 5. We see that the singular values for the wave equation with Dirichlet conditions on the left do not decay for this classical energy conserving wave equation discretization, and on the right they decay rapidly for time $T = 1, 2$, and not at all for smaller T. The corresponding singular vectors are shown in the bottom row, corresponding to the non decaying (all) modes in the separation of variables solution $u(x,t) = \sum_{k=1}^{\infty} \hat{u}_k e^{-ik\pi t} sin(k\pi x)$, and they all would need to be propagated by the coarse solver $\mathbf{G}$, which is very difficult to do on a coarser mesh, and explains why in the MGRIT efforts coarsening in space was recognized to not be able to succeed. On the right we see the constant mode, but also a highly oscillatory one, since the discretization of the transparent boundary condition is not exact any more[4]. We see indeed two large singular values in the plot above, not just one. Starting with a random initial guess, or using a highly oscillatory source term, convergence is then also only obtained after 6 iterations, as we indicate with the dashed line in Figure 4 on the right.

We show in Figure 6 the singular value decay and corresponding modes when the wave equation is discretized with SDIRK-1 in time instead, as in Figure 1 (right). Now the singular values decay, as for the heat equation, and the dominant sine modes are again like for the heat equation. With enough numerical damping, Parareal and MGRIT could work [14, 15], but with important loss in high frequency accuracy of the computed solution over time, as we saw comparing panels 3 and 5 in Figure 1.

[3] See e.g. the front in the fourth panel of Figure 1 that starts in the center at $x = 0.5$ and $t = 0$ and travels to the left and right and leaves the domain at $t = 1/2$, so to travel across the entire domain $(0, 1)$ it would need $t = 1$.

[4] We used a centered discretization for the transparent boundary condition like for the interior wave equation, and run the scheme at the CFL limit.

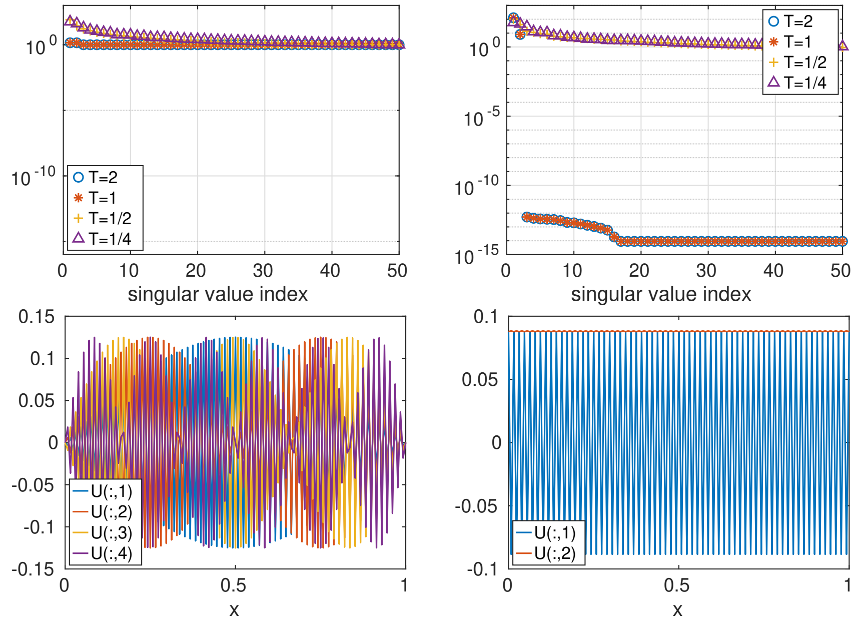

Fig. 5 First singular values (top) and singular vectors (bottom) of the fine propagator of the wave equation with Dirichlet boundary conditions (left) and transparent boundary conditions (right).

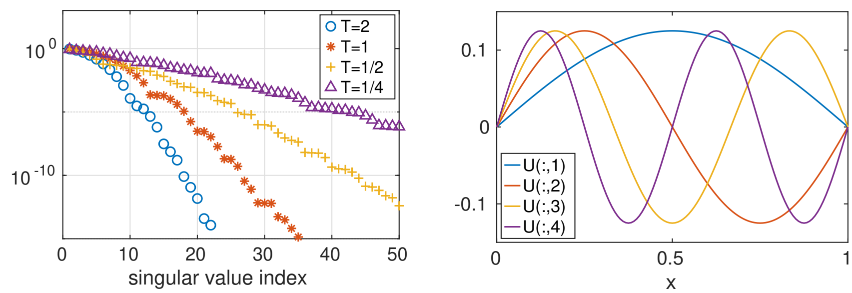

Fig. 6 First singular values and singular vectors of the fine propagator of the wave equation with Dirichlet boundary conditions discretized with SDIRK-1.

In conclusion, separation of variables shows that Fourier modes of the 1D heat equation decay rapidly, which enables highly effective coarse propagators based on reduced-order models, and in some cases eliminates the need for a coarse propagator altogether. In contrast, for the 1D second-order wave equation, no frequency component decays, which is characteristic of hyperbolic problems such as pure inviscid conservation laws. On an unbounded domain with transparent boundary conditions, as in scattering problems, energy leaves the computational domain and Parareal may converge even at the continuous level for such problems. Otherwise, long-time amplitude accuracy requires nearly unitary discretizations, whose singular values are close to one, inherently limiting rapid convergence of time-parallel methods such as

Parareal. My answer to the question posed in the title is therefore: it does not work. For rapid convergence of any Parareal-type algorithm applied to such problems, each mode either requires the coarse propagator to approximate it with essentially the same fidelity as the fine propagator [1, 2, 3], or its error becomes large over time due to the numerical damping required for fast convergence, see e.g. Figure 1 comparing panel 3 with 5 and [14, Fig. 2] for the wave equation, or [15, Fig. 2] for inviscid Burger's and the Buckley-Leverett equations, and [12]. Fortunately, alternative time-parallel methods exist and can be highly effective for hyperbolic problems; see, for example, the Maxwell solver in [13] and the overview in [12].

References

1. De Sterck, H., Falgout, R.D., Friedhoff, S., Krzysik, O.A., MacLachlan, S.P.: Optimizing multigrid reduction-in-time and parareal coarse-grid operators for linear advection. Numer. Linear Algebra Appl. **28**(4), e2367 (2021)
2. De Sterck, H., Falgout, R.D., Krzysik, O.A.: Fast multigrid reduction-in-time for advection via modified semi-Lagrangian coarse-grid operators. SIAM J. Sci. Comput. **45**(4), A1890–A1916 (2023)
3. De Sterck, H., Falgout, R.D., Krzysik, O.A., Schroder, J.B.: Efficient multigrid reduction-in-time for method-of-lines discretizations of linear advection. J. Sci. Comput. **96**(1), 1 (2023)
4. Falgout, R.D., Friedhoff, S., Kolev, T.V., MacLachlan, S.P., Schroder, J.B.: Parallel time integration with multigrid. SIAM J. Sci. Comput. **36**(6), C635–C661 (2014)
5. Gander, M.J.: Analysis of the parareal algorithm applied to hyperbolic problems using characteristics. SeMA J. **42**, 21–36 (2008)
6. Gander, M.J.: 50 years of time parallel time integration. In: Multiple Shooting and Time Domain Decomposition Methods, pp. 69–113. Springer (2015)
7. Gander, M.J., Hairer, E.: Nonlinear convergence analysis for the parareal algorithm. In: Domain decomposition methods in science and engineering XVII, pp. 45–56. Springer (2008)
8. Gander, M.J., Kwok, F., Zhang, H.: Multigrid interpretations of the parareal algorithm leading to an overlapping variant and MGRIT. Comput. Vis. Sci. **19**(3), 59–74 (2018)
9. Gander, M.J., Ohlberger, M., Rave, S.: A parareal algorithm with spectral coarse solver. arXiv preprint arXiv:2508.08873 (2025)
10. Gander, M.J., Ohlberger, M., Rave, S.: A parareal algorithm without coarse propagator? In: Proc. 28th Int. Conf. Domain Decomp. Methods. Springer (2025)
11. Gander, M.J., Vandewalle, S.: Analysis of the parareal time-parallel time-integration method. SIAM J. Sci. Comput. **29**(2), 556–578 (2007)
12. Gander, M.J., Wu, S.L., Zhou, T.: Time parallelization for hyperbolic and parabolic problems. Acta Numerica **34**, 385–489 (2025)
13. Gopalakrishnan, J., Hochsteger, M., Schöberl, J., Wintersteiger, C.: An explicit mapped tent pitching scheme for Maxwell equations. In: Spectral and high order methods for partial differential equations—ICOSAHOM 2018, pp. 359–369 (2020)
14. Krzysik, O.A., De Sterck, H., Falgout, R.D., Schroder, J.B.: Parallel-in-time solution of hyperbolic PDE systems via characteristic-variable block preconditioning. SIAM J. Sci. Comput. pp. S337–S363 (2025)
15. Krzysik, O.A., De Sterck, H., Falgout, R.D., Schroder, J.B.: Parallel-in-time solution of scalar nonlinear conservation laws. SIAM J. Sci. Comput. **47**(6), A3134–A3160 (2025)
16. Lions, J.L., Maday, Y., Turinici, G.: Résolution d'EDP par un schéma en temps «pararéel». C. R. Acad. Sci. Paris Sér. I Math. **332**(7), 661–668 (2001)